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Papers published in the *Proceedings*, pp. 1-548.

Corrigenda, p. 549.

Index of Papers published in this Volume arranged according to the names of Authors, p. 550.

Index according to Subjects, p. 551.

PAPERS

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PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

ON THE OSCILLATIONS OF A ROTATING MECHANICAL SYSTEM OF INFINITE FREEDOM

By J. PROUDMAN and F. EDITH MERCER.

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Introduction.

In this paper we consider a mechanical system whose configuration is specified with reference to a rotating frame. We suppose that the specification requires an infinite number of Lagrangian coordinates, and that the frame of reference rotates with constant angular speed about a fixed axis. The motion is then determined by the kinetic energy, the potential energy, and the moment of momentum about the axis of rotation, regarded as functions of the coordinates and their speeds, together with the external forces, which may be infinite in number. Since we consider only small motion we take the kinetic and potential energies to be homogeneous quadratic functions of the speeds and coordinates respectively, and the moment of momentum to be a bilinear function of the speeds and coordinates, all three functions having constant coefficients.

In order to secure the existence of certain limits, we place certain restrictions on these three functions; but we leave them sufficiently general to cover important problems in dynamics. Our restrictions, in fact, are not sufficiently stringent to make the resulting linear equations amenable to the theory of infinite determinants as at present developed.

For free motion we prove the existence of an infinite sequence of fundamental harmonic oscillations. For forced motion we prove that the general case of harmonic oscillations can be expressed in terms of the fundamental modes, by formulae of exactly the same type as for systems of finite freedom. Our method*, which we may perhaps call the "method of sections", is as follows. We form a subsidiary mechanical system by neglecting all but a finite number N of the coordinates and their speeds, thus using what we may call "sections"† of the system. On making N tend to infinity we find that our restrictive assumptions are sufficient to secure that all the types of the motion of the sectional system of finite freedom tend to definite limiting types, and that these limiting types actually apply to the system of infinite freedom.

We also establish a transformation for any motion of the system and use it to prove the uniqueness of the limiting processes of the method of sections, as well as to show that the most general free motion can be expressed as a convergent series of multiples of the fundamental oscillations.

When the angular speed of the frame of reference vanishes, the moment of momentum function does not enter into the consideration, and the discussion becomes practically identical with the simultaneous reduction of the kinetic and potential energies to sums of squares. This is a problem which has been treated recently by J. Hyslop‡ with such generality that there is a continuum of possible free periods (continuous spectrum) in addition to the discrete sequence (discontinuous spectrum) as obtained in the present paper. The special case in which the potential energy is already a sum of squares was that originally considered by D. Hilbert§, who showed that when the kinetic energy is a completely continuous function of the speeds the continuous spectrum vanishes.

The more purely mathematical part of the reasoning of the present paper is given separately in an Appendix.

1. Preliminary relations.

Let $a_{r,s}, \beta_{r,s}, c_{r,s} \quad (r, s = 1, 2, 3, \dots)$

* "Principe des réduites", F. Riesz, *Les systèmes d'équations linéaires* . . . (Paris, 1913).

† "Abschnitte" is the term used by D. Hilbert for the truncated quadratic and bilinear forms, *Göttinger Nachrichten* (1906), 157-226.

‡ *Proc. London Math. Soc.* (2), 24 (1924), 264-304.

§ *Loc. cit.*

denote real constants, such that

$$\left. \begin{aligned} a_{s,r} &= a_{r,s} \\ \beta_{s,r} &= -\beta_{r,s}, \quad \beta_{s,s} = 0 \\ c_{s,r} &= c_{r,s} \end{aligned} \right\}, \quad (1.11)$$

and let (x) , (y) denote respectively the infinite sequences

$$x_1, x_2, \dots, x_s, \dots,$$

$$y_1, y_2, \dots, y_s, \dots$$

of real or complex variables. Take

$$\left. \begin{aligned} 2T_N(x, y) &= \sum_{r,s=1}^N a_{r,s} x_s y_r \\ 2M_N(x, y) &= \sum_{r,s=1}^N \beta_{r,s} x_s y_r \\ 2V_N(x, y) &= \sum_{r,s=1}^N c_{r,s} x_s y_r \end{aligned} \right\}, \quad (1.12)$$

so that

$$\left. \begin{aligned} \frac{2\partial T_N(x, y)}{\partial y_r} &= \sum_{s=1}^N a_{r,s} x_s \\ \frac{2\partial M_N(x, y)}{\partial y_r} &= \sum_{s=1}^N \beta_{r,s} x_s \\ \frac{2\partial V_N(x, y)}{\partial y_r} &= \sum_{s=1}^N c_{r,s} x_s \end{aligned} \right\}, \quad (1.13)$$

for $r = 1, 2, \dots, N$, and

$$\left. \begin{aligned} T_N(x, y) &= \sum_{r=1}^N y_r \frac{\partial T_N(x, y)}{\partial y_r} \\ M_N(x, y) &= \sum_{r=1}^N y_r \frac{\partial M_N(x, y)}{\partial y_r} \\ V_N(x, y) &= \sum_{r=1}^N y_r \frac{\partial V_N(x, y)}{\partial y_r} \end{aligned} \right\}. \quad (1.14)$$

From (1.11) we have

$$\left. \begin{aligned} T_N(y, x) &= T_N(x, y) \\ M_N(y, x) &= -M_N(x, y) \\ V_N(y, x) &= V_N(x, y) \end{aligned} \right\}, \quad (1.15)$$

and we shall write

$$T_N(x, x) = T_N(x), \quad V_N(x, x) = V_N(x), \quad (1.16)$$

$$\text{so that} \quad \frac{\partial T_N(x)}{\partial x_r} = \sum_{s=1}^N a_{r,s} x_s, \quad \frac{\partial V_N(x)}{\partial x_r} = \sum_{s=1}^N c_{r,s} x_s, \quad (1.17)$$

for $r = 1, 2, \dots, N$, and

$$2T_N(x) = \sum_{r=1}^N x_r \frac{\partial T_N(x)}{\partial x_r}, \quad 2V_N(x) = \sum_{r=1}^N x_r \frac{\partial V_N(x)}{\partial x_r}. \quad (1.18)$$

If $(\bar{x})$ denotes the sequence conjugate to (x) we know, from Theorem 2 of the Appendix, that $T_N(x, \bar{x})$ and $V_N(x, \bar{x})$ are real. We shall suppose that, for all positive integral values of N , and for all values of (x) and (y) ,

$$\left. \begin{aligned} a \sum_{s=1}^N \frac{|x_s|^2}{\nu_s^2} &\leq 2T_N(x, \bar{x}) \leq A \sum_{s=1}^N \frac{|x_s|^2}{\nu_s^2} \\ |2M_N(x, y)|^2 &\leq B^2 \sum_{s=1}^N \frac{|x_s|^2}{\nu_s^2} \sum_{s=1}^N |y_s|^2 \\ c \sum_{s=1}^N |x_s|^2 &\leq 2V_N(x, \bar{x}) \leq C \sum_{s=1}^N |x_s|^2 \end{aligned} \right\}, \quad (1.21)$$

where A, B, C, a, c are positive constants, and

$$\nu_1, \nu_2, \dots, \nu_s, \dots$$

is an infinite sequence of real constants, independent of N , bounded below by a positive constant, but not necessarily bounded above. We shall also suppose that

$$\sum_{s=1}^{\infty} \nu_s^2 c_{r,s}^2 \leq M_r^2 \quad (r = 1, 2, 3, \dots), \quad (1.22)$$

where M_r is a finite constant. At the end of § 3 we shall make an

assumption (3.4) which places a further restriction on the functions (1.12).

It follows, from (1.21), that $a_{s,s}$, $c_{s,s}$ and, more generally, the determinants of the quadratic forms $T_N(x)$, $V_N(x)$ are always positive. By Theorems 3 (ii) and 4 of the Appendix we deduce, from (1.21),

$$\left. \begin{aligned} \sum_{r=1}^N \nu_r^2 \left| \sum_{s=1}^N a_{r,s} x_s \right|^2 &\leq A^2 \sum_{s=1}^N \frac{|x_s|^2}{\nu_s^2} \\ \sum_{r=1}^N \left| \sum_{s=1}^N \beta_{r,s} x_s \right|^2 &\leq B^2 \sum_{s=1}^N \frac{|x_s|^2}{\nu_s^2} \\ \sum_{r=1}^N \left| \sum_{s=1}^N c_{r,s} x_s \right|^2 &\leq C^2 \sum_{s=1}^N |x_s|^2 \end{aligned} \right\}, \quad (1.23)$$

and then, from (1.13), (1.17), and (1.21), we have

$$\left. \begin{aligned} \sum_{r=1}^N \nu_r^2 \left| \frac{\partial T_N(x)}{\partial x_r} \right|^2 &\leq \frac{2A^2}{a} T_N(x, \bar{x}) \\ \sum_{r=1}^N \left| \frac{\partial M_N(x, y)}{\partial y_r} \right|^2 &\leq \frac{B^2}{2a} T_N(x, \bar{x}) \\ \sum_{r=1}^N \left| \frac{\partial V_N(x)}{\partial x_r} \right|^2 &\leq \frac{2C^2}{c} V_N(x, \bar{x}) \end{aligned} \right\}. \quad (1.24)$$

By Theorem 3 (i) of the Appendix we also have, from (1.21),

$$\left. \begin{aligned} \sum_{s=1}^N \nu_s^2 a_{r,s}^2 &\leq \frac{A^2}{\nu_r^2} \\ \sum_{s=1}^N \nu_s^2 \beta_{r,s}^2 &\leq B^2 \\ \sum_{s=1}^N c_{r,s}^2 &\leq C^2 \end{aligned} \right\} \quad (r = 1, 2, \dots, N), \quad (1.25)$$

and we then deduce the convergence of

$$\sum_{s=1}^{\infty} a_{r,s}^2, \quad \sum_{s=1}^{\infty} \beta_{r,s}^2, \quad \sum_{s=1}^{\infty} c_{r,s}^2,$$

for $r = 1, 2, 3, \dots$

2. *Dynamical equations.*

Let $q_1, q_2, \dots, q_s, \dots,$

or (q) , denote the Lagrangian coordinates of the mechanical system, and let

$$\dot{q}_1, \dot{q}_2, \dots, \dot{q}_s, \dots,$$

or $(\dot{q})$, denote their speeds.

When all coordinates q_s for which $s > N$ are kept zero, so that their speeds are also zero, we shall refer to the system so obtained as the sectional system of N degrees of freedom. For this system we shall suppose that

$$T_N(\dot{q}), \quad V_N(q)$$

represent respectively the kinetic and potential energies, and we shall consider only such motions as make

$$T_N(\dot{q}, \bar{q}) + V_N(q, \bar{q}) \leq E, \quad (2.11)$$

for all values of N , E being a constant independent of N . We shall also suppose that

$$M_N(q, \dot{q})$$

represents the moment of momentum about the axis of rotation multiplied by twice the angular speed of rotation.

We shall denote the Lagrangian components of external forces by

$$Q_1, Q_2, \dots, Q_s, \dots,$$

or (Q) , and we shall suppose them to be such that the motion which they generate satisfies (2.11). The rate of doing work by these forces on the sectional system is given by

$$\sum_{s=1}^N Q_s \dot{q}_s,$$

and, by Theorem 1 of the Appendix, the modulus of this is less than

$$\left\{ \sum_{s=1}^N \nu_s^2 |Q_s|^2 \sum_{s=1}^N \frac{|\dot{q}_s|^2}{\nu_s^2} \right\}^{\frac{1}{2}}.$$

In order to secure that this rate of doing work shall always be finite we

shall suppose that
$$\sum_{s=1}^{\infty} \nu_s^2 |Q_s|^2 \quad (2.12)$$

is convergent.

For the sectional system Lagrange's equations of motion may be written

$$\frac{d}{dt} \left(\frac{\partial R_N}{\partial \dot{q}_r} \right) - \frac{\partial R_N}{\partial q_r} = Q_r, \quad (2.21)$$

for $r = 1, 2, \dots, N$, where t denotes the time and

$$R_N = T_N(\dot{q}) + M_N(q, \dot{q}) - V_N(q). \quad (2.22)$$

For *free* motion we take $Q_r = 0$ ($r = 1, 2, \dots, N$).

On substituting from (2.22) into (2.21), and using (1.13) and (1.17), we obtain*

$$\sum_{s=1}^N (a_{r,s} \ddot{q}_s + \beta_{r,s} \dot{q}_s + c_{r,s} q_s) = Q_r, \quad (2.23)$$

for $r = 1, 2, \dots, N$.

Now suppose that $(\dot{q}')$ represents a second possible motion of this system; on multiplying (2.23) by $\dot{q}'_r$ and adding for $r = 1, 2, \dots, N$, we obtain

$$2T_N(\ddot{q}, \dot{q}') + 2M_N(\dot{q}, \dot{q}') + 2V_N(q, \dot{q}') = \sum_{r=1}^N Q_r \dot{q}'_r. \quad (2.31)$$

If (Q') represents the forces corresponding to the motion $(\dot{q}')$ we may obtain similarly

$$2T_N(\ddot{q}', \dot{q}) + 2M_N(\dot{q}', \dot{q}) + 2V_N(q', \dot{q}) = \sum_{r=1}^N Q'_r \dot{q}_r. \quad (2.32)$$

On adding (2.31) and (2.32) and using (1.15) we obtain

$$2T_N(\ddot{q}, \dot{q}') + 2T_N(\ddot{q}', \dot{q}) + 2V_N(q, \dot{q}') + 2V_N(q', \dot{q}) = \sum_{r=1}^N (Q_r \dot{q}'_r + Q'_r \dot{q}_r), \quad (2.33)$$

which we shall call the *energy relation*.

From (1.21) and (2.11) we have

$$\sum_{s=1}^N \frac{|\dot{q}_s|^2}{\nu_s^2} \leq \frac{2E}{a}, \quad \sum_{s=1}^N |q_s|^2 \leq \frac{2E}{c}, \quad (2.41)$$

* Cf. H. Lamb, *Higher mechanics*, Art. 81.

and it follows, from (1.23), that

$$\left. \begin{aligned} \sum_{r=1}^N \nu_r^2 \left| \sum_{s=1}^N a_{r,s} \dot{q}_s \right|^2 &\leq \frac{2A^2 E}{a} \\ \sum_{r=1}^N \left| \sum_{s=1}^N \beta_{r,s} \dot{q}_s \right|^2 &\leq \frac{2B^2 E}{a} \\ \sum_{r=1}^N \left| \sum_{s=1}^N c_{r,s} q_s \right|^2 &\leq \frac{2C^2 E}{c} \end{aligned} \right\} \quad (2.42)$$

By Theorem 1 of the Appendix, (1.22), and (2.41), we have

$$\begin{aligned} \left| \sum_{s=1}^N c_{r,s} \dot{q}_s \right|^2 &\leq \sum_{s=1}^N \nu_s^2 c_{r,s}^2 \sum_{s=1}^N \frac{|\dot{q}_s|^2}{\nu_s^2} \\ &\leq \frac{2E}{a} M_r^2 \quad (r = 1, 2, \dots, N). \end{aligned} \quad (2.43)$$

Also, for free motion of the sectional system, we deduce from Lagrange's equations (2.23) with $Q_r = 0$, and (2.42) that

$$\begin{aligned} \left| \sum_{s=1}^N a_{r,s} \ddot{q}_s \right| &\leq \left| \sum_{s=1}^N \beta_{r,s} \dot{q}_s \right| + \left| \sum_{s=1}^N c_{r,s} q_s \right| \\ &\leq B \left(\frac{2E}{a} \right)^{\frac{1}{2}} + C \left(\frac{2E}{c} \right)^{\frac{1}{2}}, \end{aligned} \quad (2.44)$$

for $r = 1, 2, \dots, N$.

From (2.11), by Theorems 4 and 3 (iii) of the Appendix, we know that the double series

$$T(\dot{q}), \quad V(q),$$

obtained from $T_N(\dot{q})$ and $V_N(q)$ by making $N \rightarrow \infty$, are convergent, and these we take to be the kinetic and potential energies of the system with infinite freedom. Also, from (1.21), we see that the double series

$$M(q, \dot{q})$$

similarly obtained from $M_N(q, \dot{q})$ is convergent, and this we take to represent the moment of momentum multiplied by twice the angular speed of rotation.

The equations of motion of the system with infinite freedom we take to be

$$\sum_{s=1}^{\infty} (a_{r,s} \ddot{q}_s + \beta_{r,s} \dot{q}_s + c_{r,s} q_s) = Q_r, \quad (2.5)$$

for $r = 1, 2, 3, \dots$, and we suppose that, when all the coordinates and their speeds are prescribed at any instant, the motion is uniquely determinate at every subsequent instant.

3. Fundamental periods.

The equations of free motion for the sectional system of N degrees of freedom are

$$\sum_{s=1}^N (a_{r,s} \ddot{q}_s + \beta_{r,s} \dot{q}_s + c_{r,s} q_s) = 0, \quad (3.11)$$

for $r = 1, 2, \dots, N$. The solutions of these equations are known to take the form

$$q_s = \alpha_s e^{i\sigma t} \quad (s = 1, 2, \dots, N), \quad (3.12)$$

$$\sigma \text{ being a root of} \quad \Delta_N(\sigma) = 0, \quad (3.13)$$

where $\Delta_N(\sigma)$ is the determinant

$$\begin{vmatrix} -\sigma^2 a_{1,1} + c_{1,1} & \dots & \dots & \dots & \dots & -\sigma^2 a_{1,N} + i\sigma \beta_{1,N} + c_{1,N} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & -\sigma^2 a_{r,s} + i\sigma \beta_{r,s} + c_{r,s} & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -\sigma^2 a_{N,1} + i\sigma \beta_{N,1} + c_{N,1} & \dots & \dots & \dots & \dots & -\sigma^2 a_{N,N} + c_{N,N} \end{vmatrix}.$$

From (3.11) it follows that

$$\Delta_N(-\sigma) = \Delta_N(\sigma), \quad (3.14)$$

so that $\Delta_N(\sigma)$ is a polynomial of degree N in σ^2 , and the equation (3.13) has its roots of the form

$$\pm \sigma_1(N), \quad \pm \sigma_2(N), \quad \dots, \quad \pm \sigma_n(N), \quad \dots, \quad \pm \sigma_N(N),$$

these quantities being, of course, functions of N . The fundamental periods are given by $2\pi/\sigma_n(N)$.

We next proceed to establish the existence of real quantities

$$\sigma_1, \sigma_2, \dots, \sigma_n, \dots,$$

depending only on $\alpha_{r,s}$, $\beta_{r,s}$, $c_{r,s}$, and such that

$$\sigma_n(N) \rightarrow \sigma_n \text{ as } N \rightarrow \infty, \quad (3.21)$$

for all the values of n .

If $D_{r,s}(N)$ denote the determinant obtained from $\Delta_N(\sigma)$ by the omission of the r -th row and s -th column, we see that $D_{r,s}(N)$ and $D_{s,r}(N)$ are conjugate complexes. By a known theorem* we have

$$\Delta_N(\sigma) \Delta_{N-2}(\sigma) = \Delta_{N-1}(\sigma) D_{2,2}(N) - D_{1,2}(N) D_{2,1}(N), \quad (3.22)$$

and, since $D_{1,2}(N) D_{2,1}(N)$ is real and never negative, it follows that, when $\Delta_{N-1}(\sigma) = 0$, $\Delta_N(\sigma) \Delta_{N-2}(\sigma)$ is real and never positive.

Also, it follows from § 1, that $\Delta_N(0) > 0$, and that, for sufficiently large real values of σ , $\Delta_N(\sigma)$ has the sign of $(-1)^N$. When $\Delta_1(\sigma) = 0$, $\sigma^2 = c_{1,1}/a_{1,1}$, which is real and positive, and so we may take $\sigma_1(1)$ positive. When $\sigma = \sigma_1(1)$ it is easily seen that $\Delta_2(\sigma) = -D_{1,2}(2) D_{2,1}(2)$ and is negative or zero, whence it follows that there is a real root of $\Delta_2(\sigma) = 0$, say $\sigma_1(2)$, such that

$$0 < \sigma_1(2) \leq \sigma_1(1),$$

and another real root, say $\sigma_2(2)$, such that

$$\sigma_1(2) \leq \sigma_2(2).$$

When $\sigma = \sigma_1(2)$ it follows, from (3.22), that $\Delta_3(\sigma)$ is negative or zero, and therefore that there is a real root of $\Delta_3(\sigma) = 0$, say $\sigma_1(3)$, such that

$$0 < \sigma_1(3) \leq \sigma_1(2).$$

Also, when $\sigma = \sigma_2(2)$, $\Delta_1(\sigma)$ is negative or zero, and it follows from (3.22) that $\Delta_3(\sigma)$ is positive or zero, so that we deduce the existence of real roots $\sigma_2(3)$, $\sigma_3(3)$ such that

$$0 < \sigma_1(3) \leq \sigma_2(3) \leq \sigma_3(3),$$

$$\sigma_2(3) \leq \sigma_2(2).$$

By continuing this argument we prove that the roots $\sigma_n(N)$ ($n = 1, 2, \dots, N$)

* The argument which follows is an extension of that used by E. J. Routh to prove the reality of $\sigma_n(N)$; cf. *Advanced rigid dynamics*, art. 58, 123, 124.

are real and satisfy the relations

$$0 < \sigma_n(N) \leq \sigma_n(N-1), \quad (3.23)$$

if they are arranged so that

$$0 < \sigma_1(N) \leq \sigma_2(N) \dots \leq \sigma_N(N). \quad (3.24)$$

It follows, from (3.23), that for any value of n there exists a quantity σ_n greater than or equal to 0 such that

$$\sigma_n(N) \rightarrow \sigma_n \text{ as } N \rightarrow \infty, \quad (3.31)$$

and, from (3.24), we have

$$0 \leq \sigma_1 \leq \sigma_2 \dots \leq \sigma_n \dots \quad (3.32)$$

We shall assume that

$$\sigma_n \rightarrow \infty \text{ as } n \rightarrow \infty. \quad (3.4)$$

4. Fundamental modes.

When $\sigma = \sigma_n(N)$ in the equations (3.12), let us write

$$\alpha_s = \alpha_{n,s}(N) \quad (s = 1, 2, \dots, N),$$

so that the equations (3.11) become

$$\sum_{s=1}^N \{ -\sigma_n^2(N) a_{r,s} + i\sigma_n(N) \beta_{r,s} + c_{r,s} \} \alpha_{n,s}(N) = 0, \quad (4.11)$$

for $r = 1, 2, \dots, N$. Then a set

$$\alpha_{n,1}(N), \alpha_{n,2}(N), \dots, \alpha_{n,N}(N), \quad (4.12)$$

satisfying (4.11) certainly exists, though clearly these quantities are as yet undetermined to the extent of a common factor, and they may possibly be incompletely determinate in other ways. If

$$\alpha_{-n,1}(N), \alpha_{-n,2}(N), \dots, \alpha_{-n,N}(N) \quad (4.13)$$

denote respectively the conjugate complexes of (4.12) we see that, when $\sigma = -\sigma_n(N)$, the equations (4.11) are satisfied by

$$\alpha_s = \alpha_{-n,s}(N) \quad (s = 1, 2, \dots, N).$$

We thus have $2N$ solutions of Lagrange's equations for free motion given by

$$q_s = a_{n,s}(N) e^{i\sigma_n(N)t} \quad (s = 1, 2, \dots, N),$$

for $n = -N, -N+1, \dots, -1, 1, \dots, N$, if we write

$$\sigma_{-n}(N) = -\sigma_n(N). \quad (4.14)$$

From § 1 we know that

$$T_N\{a_n(N), a_{-n}(N)\}, \quad V_N\{a_n(N), a_{-n}(N)\}$$

are real and positive, so that we may suppose (4.12) and (4.13) to satisfy the equation

$$\sigma_n^2(N) T_N\{a_n(N), a_{-n}(N)\} + V_N\{a_n(N), a_{-n}(N)\} = 1. \quad (4.2)$$

From (1.21) and (4.2) it follows that

$$\sum_{s=1}^N |\alpha_{n,s}(N)|^2 < \frac{2}{c}. \quad (4.31)$$

Now when once we have agreed upon the quantities $\alpha_{n,s}(N)$ for all the values of N, n , and s , it follows, from the Principle of Choice (Theorem 10 of the Appendix), in view of (4.31), that a sequence of positive integers

$$N_1, N_2, \dots, N_m, \dots \quad (4.32)$$

can be found such that, as N increases indefinitely through this sequence, there exists limits $\alpha_{n,s}$ such that

$$a_{n,s}(N) \rightarrow \alpha_{n,s}, \quad (4.33)$$

for all the values of n and s . It also follows, from (4.11), by the corollary to Theorem 7 of the Appendix in view of (4.31), that the limits σ_n and $\alpha_{n,s}$ satisfy the equations

$$\sum_{s=1}^{\infty} (-\sigma_n^2 a_{r,s} + i\sigma_n \beta_{r,s} + c_{r,s}) \alpha_{n,s} = 0, \quad (4.34)$$

for all the values of n and r . In other words, the quantities q_s , given by

$$q_s = \alpha_{n,s} e^{i\sigma_n t} \quad (s = 1, 2, 3, \dots), \quad (4.35)$$

satisfy Lagrange's equations of free motion for the system of infinite freedom, and they do so for all positive and negative integral values of n .

5. Transformation for sectional system.

In the sequel, when $\sigma_n(N)$, $a_{n,s}(N)$ occur in an expression and σ_n , a_n , do not, we shall write the latter instead of the former, and indicate the contracted notation by placing the symbol (N) before the expression or equation.

If in the energy relation (2.33) we write

$$(Q) = 0, \quad (Q') = 0,$$

$$(N) \quad (q) = (a_n e^{i\sigma_n t}), \quad (q') = (a_m e^{i\sigma_m t}),$$

we obtain

$$(N) \quad -i\sigma_m \sigma_n (\sigma_m + \sigma_n) T(a_m, a_n) + i(\sigma_m + \sigma_n) V(a_m, a_n) = 0,$$

from which we deduce that

$$(N) \quad -\sigma_m \sigma_n T(a_m, a_n) + V(a_m, a_n) = 0, \quad (5.1)$$

provided that $(N) \sigma_m + \sigma_n \neq 0$. We shall next show that we may always secure (5.1) when $m+n \neq 0$, even though $(N) \sigma_m + \sigma_n = 0$. When $m+n=0$ we have, of course, the equation (4.2).

The possibility of having simultaneously

$$(N) \quad \sigma_m + \sigma_n = 0, \quad m+n \neq 0,$$

means that the equation $\Delta_N(\sigma) = 0$ possesses a multiple root. Suppose now that there is a μ -fold root of this equation, so that

$$(N) \quad \sigma_{n_1} = \sigma_{n_2} = \dots = \sigma_{n_\mu}.$$

Then it is known that, instead of any one of the sequences

$$(N) \quad (a_{n_1}), (a_{n_2}), \dots, (a_{n_\mu}),$$

we may take any linear combination of them all, so that, so far as satisfying the equations of motion is concerned, we have μ^2 arbitrary constants. Satisfying the μ equations (4.2) determines μ of these constants, and we are thus left with $\mu(\mu-1)$ disposable constants to satisfy the $\frac{1}{2}\mu(\mu-1)$ relations of the type (5.1).

Now let us write

$$(N) \quad \xi_n = V_N(a_{-n}, q) - i\sigma_n T_N(a_{-n}, \dot{q}), \quad (5.21)$$

for $n = -N, \dots, -1, 1, \dots, N$, so that the $2N$ linear equations (5.21)

connect the $2N$ quantities

$$q_1, \dots, q_N, \dot{q}_1, \dots, \dot{q}_N$$

with the $2N$ quantities

$$(N) \quad \xi_{-N}, \dots, \xi_{-1}, \xi_1, \dots, \xi_N.$$

The $2N$ fundamental modes of oscillation furnish $2N$ particular solutions of the equations (5.21), viz.

$$(N) \quad q_s = a_{n,s} e^{i\sigma_m t}, \quad \dot{q}_s = i\sigma_m a_{n,s} e^{i\sigma_m t} \quad (s = 1, 2, \dots, N)$$

corresponds to

$$(N) \quad \begin{aligned} \xi_n &= e^{i\sigma_m t} \quad (n = m), \\ &= 0 \quad (n \neq m), \end{aligned}$$

for each of the $2N$ values of m . Now these $2N$ particular cases also satisfy the $2N$ equations

$$(N) \quad \left\{ \begin{aligned} q_s &= \sum_{n=-N}^N a_{n,s} \xi_n \\ \dot{q}_s &= \sum_{n=-N}^N i\sigma_n a_{n,s} \xi_n \end{aligned} \right\}, \quad (5.22)$$

for $s = 1, 2, \dots, N$. It follows that the equations (5.21) and (5.22) are completely equivalent, so that any instantaneous state and motion of the system may be specified by the quantities $\xi_n(N)$.

In the equations (5.22) let us substitute from (5.21), so as to eliminate the quantities $\xi_n(N)$. The result must consist of $2N$ identities, and, since $q_s, \dot{q}_s$ ($s = 1, 2, \dots, N$) may be regarded as $2N$ arbitrary quantities, we may equate coefficients. In this way we obtain

$$(N) \quad \left\{ \begin{aligned} \frac{1}{2} \sum_{n=-N}^N a_{n,s} \frac{\partial V(a_{-n})}{\partial a_{-n,r}} &= 1 \quad (r = s), \quad = 0 \quad (r \neq s) \\ \frac{1}{2} \sum_{n=-N}^N \sigma_n a_{n,s} \frac{\partial V(a_{-n})}{\partial a_{-n,r}} &= 0 \\ \frac{1}{2} \sum_{n=-N}^N \sigma_n a_{n,s} \frac{\partial T(a_{-n})}{\partial a_{-n,r}} &= 0 \\ \frac{1}{2} \sum_{n=-N}^N \sigma_n^2 a_{n,s} \frac{\partial T(a_{-n})}{\partial a_{-n,r}} &= 1 \quad (r = s), \quad = 0 \quad (r \neq s) \end{aligned} \right\}. \quad (5.31)$$

From (1.17) we have

$$(N) \quad \sum_{\rho=1}^N c_{r,\rho} a_{n,\rho} = \frac{\partial V(a_n)}{\partial a_{n,r}}, \quad \sum_{\rho=1}^N a_{r,\rho} a_{n,\rho} = \frac{\partial T(a_n)}{\partial a_{n,r}};$$

in the equations (5.31) we now replace s, r by ρ, s respectively. Then, on multiplying the first and third of these equations by $c_{r,\rho}$, the second and fourth by $a_{r,\rho}$, and adding for $\rho = 1, 2, \dots, N$ in each case, we obtain

$$(N) \quad \left\{ \begin{array}{l} \frac{1}{2} \sum_{n=-N}^N \frac{\partial V(a_n)}{\partial a_{n,r}} \frac{\partial V(a_{-n})}{\partial a_{-n,s}} = c_{r,s} \\ \frac{1}{2} \sum_{n=-N}^N \sigma_n \frac{\partial V(a_n)}{\partial a_{n,r}} \frac{\partial T(a_{-n})}{\partial a_{-n,s}} = 0 \\ \frac{1}{2} \sum_{n=-N}^N \sigma_n^2 \frac{\partial T(a_n)}{\partial a_{n,r}} \frac{\partial T(a_{-n})}{\partial a_{-n,s}} = a_{r,s} \end{array} \right\}, \quad (5.32)$$

for $r, s = 1, 2, \dots, N$, the second of these equations arising in two ways.

Now let us consider a second motion of the system and distinguish the corresponding variables by accents, as in § 2. From (5.21), (1.17), and (5.32) in turn we deduce that

$$\begin{aligned} (N) \quad & \sum_{n=-N}^N \xi_n \xi'_{-n} \\ &= \sum_{n=-N}^N \{ V(a_{-n}, q) - i\sigma_n T(a_{-n}, \dot{q}) \} \{ V(a_n, q') + i\sigma_n T(a_n, \dot{q}') \} \\ &= \frac{1}{4} \sum_{n=-N}^N \sum_{r=1}^N \left\{ \frac{\partial V(a_{-n})}{\partial a_{-n,r}} q_r - i\sigma_n \frac{\partial T(a_{-n})}{\partial a_{-n,r}} \dot{q}_r \right\} \\ & \quad \times \sum_{s=1}^N \left\{ \frac{\partial V(a_n)}{\partial a_{n,s}} q'_s + i\sigma_n \frac{\partial T(a_n)}{\partial a_{n,s}} \dot{q}'_s \right\} \\ &= \frac{1}{2} \sum_{r,s=1}^N (c_{s,r} q_r q_s + a_{s,r} \dot{q}_r \dot{q}'_s), \end{aligned}$$

and hence

$$(N) \quad \sum_{n=-N}^N \xi_n \xi'_{-n} = T_N(\dot{q}, \dot{q}') + V_N(q, q'). \quad (5.41)$$

If in (5.41) we write

$$(q') = (q), \quad (\dot{q}') = (\dot{q}),$$

we deduce that*

$$(N) \quad 2 \sum_{n=1}^N \xi_n \xi_{-n} = T_N(\dot{q}) + V_N(q), \quad (5.42)$$

while, if we write $(q') = (\bar{q}), \quad (\dot{q}') = (\dot{\bar{q}}),$

we deduce that

$$(N) \quad \sum_{n=-N}^N |\xi_n|^2 = T_N(\dot{q}, \dot{\bar{q}}) + V_N(q, \bar{q}), \quad (5.43)$$

since $\xi_n(N)$ and $\xi'_{-n}(N)$ are then conjugate. From (2.11) and (5.43) we have

$$(N) \quad \sum_{n=-N}^N |\xi_n|^2 \leq E. \quad (5.44)$$

For *general free motion* we have

$$(N) \quad \xi_n = k_n e^{i\sigma_n t} \quad (n = -N, \dots, -1, 1, \dots, N), \quad (5.51)$$

where $k_n(N)$ is a constant, and then from (5.22) we deduce

$$(N) \quad \left\{ \begin{aligned} q_s &= \sum_{n=-N}^N k_n a_{n,s} e^{i\sigma_n t} \\ \dot{q}_s &= i \sum_{n=-N}^N \sigma_n k_n a_{n,s} e^{i\sigma_n t} \\ \ddot{q}_s &= - \sum_{n=-N}^N \sigma_n^2 k_n a_{n,s} e^{i\sigma_n t} \end{aligned} \right\}, \quad (5.52)$$

for $s = 1, 2, \dots, N$. From (5.42) and (5.44) we have respectively

$$(N) \quad 2 \sum_{n=1}^N k_n k_{-n} = T_N(\dot{q}) + V_N(q), \quad (5.53)$$

$$(N) \quad \sum_{n=-N}^N |k_n|^2 \leq E. \quad (5.54)$$

* Cf. H. Poincaré, *Leçons de mécanique céleste*, 3 (Paris, 1910), 28.

6. *Boundedness and convergence.*

In this section and in the next two sections we shall always suppose that N is a member of the sequence of positive integers (4.32) associated with the Principle of Choice.

In § 4 we have, in effect, proved that

$$\left. \begin{aligned} \lim_{N \rightarrow \infty} \frac{\partial V\{a_n(N)\}}{\partial a_{n,r}(N)} &= \frac{\partial V(a_n)}{\partial a_{n,r}} \\ \lim_{N \rightarrow \infty} \frac{\partial T\{a_n(N)\}}{\partial a_{n,r}(N)} &= \frac{\partial T(a_n)}{\partial a_{n,r}} \end{aligned} \right\} \quad (6.11)$$

for all the values of n and r . We have also, from (1.24) and (4.2),

$$(N) \quad \left\{ \begin{aligned} \sum_{s=1}^N \left| \frac{\partial V(a_n)}{\partial a_{n,s}} \right|^2 &\leq \frac{2C^2}{c} V(a_n, a_{-n}) \leq \frac{2C^2}{c} \\ \sum_{s=1}^N \nu_s^2 \left| \frac{\partial T(a_n)}{\partial a_{n,s}} \right|^2 &\leq \frac{2A^2}{a} T(a_n, a_{-n}) \leq \frac{2A^2}{a\sigma_n^2} \end{aligned} \right\} \quad (6.12)$$

for all the values of n , and hence, from (3.23) and (3.31), it follows that

$$\sum_{s=1}^N \nu_s^2 \left| \frac{\partial T\{a_n(N)\}}{\partial a_{n,s}(N)} \right|^2 \leq \frac{2A^2}{a\sigma_n^2}, \quad (6.13)$$

for all the values of n .

We shall next prove some inequalities relating to summations extending over the fundamental modes of the sectional system, and we shall do so from a consideration of the general free motion of this system. Let us take any complex constants

$$k_{-N}(N), \dots, k_{-1}(N), k_1(N), \dots, k_N(N) \quad (6.21)$$

subject to the condition

$$(N) \quad \sum_{n=-N}^N |k_n|^2 \leq E, \quad (6.22)$$

and to the existence of limits k_n such that

$$k_n(N) \rightarrow k_n \quad \text{as} \quad N \rightarrow \infty,$$

for all the values of n . Then we may regard equations (5.52) as giving

$$q_s, \dot{q}_s, \ddot{q}_s \quad (s = 1, 2, \dots, N)$$

and these quantities will be functions of N satisfying the fundamental assumption (2.11). For our present purpose it is necessary to consider only the speeds and accelerations at the instant $t = 0$. From (2.43) and (2.44) we have

$$(N) \quad \left\{ \begin{array}{l} \left| \sum_{s=1}^N c_{r,s} \dot{q}_s \right| \leq M_r \left(\frac{2E}{a} \right)^{\frac{1}{2}} \\ \left| \sum_{s=1}^N a_{r,s} \ddot{q}_s \right| \leq B \left(\frac{2E}{a} \right)^{\frac{1}{2}} + C \left(\frac{2E}{c} \right)^{\frac{1}{2}} \end{array} \right\}, \quad (6.23)$$

for all the values of r . But, from (5.52) and (1.17) at $t = 0$, we have

$$(N) \quad \begin{aligned} \sum_{s=1}^N c_{r,s} \dot{q}_s &= i \sum_{s=1}^N c_{r,s} \sum_{n=-N}^N \sigma_n \alpha_{n,s} k_n \\ &= i \sum_{n=-N}^N \sigma_n \frac{\partial V(\alpha_n)}{\partial \alpha_{n,r}} k_n, \end{aligned} \quad (6.24)$$

$$(N) \quad \begin{aligned} \sum_{s=1}^N a_{r,s} \ddot{q}_s &= - \sum_{s=1}^N a_{r,s} \sum_{n=-N}^N \sigma_n^2 \alpha_{n,s} k_n \\ &= - \sum_{n=-N}^N \sigma_n^2 \frac{\partial T(\alpha_n)}{\partial \alpha_{n,r}} k_n, \end{aligned} \quad (6.25)$$

so that, from (6.23), we deduce that

$$(N) \quad \left\{ \begin{array}{l} \left| \sum_{n=-N}^N \sigma_n \frac{\partial V(\alpha_n)}{\partial \alpha_{n,r}} k_n \right| \leq M_r \left(\frac{2E}{a} \right)^{\frac{1}{2}} \\ \left| \sum_{n=-N}^N \sigma_n^2 \frac{\partial T(\alpha_n)}{\partial \alpha_{n,r}} k_n \right| \leq B \left(\frac{2E}{a} \right)^{\frac{1}{2}} + C \left(\frac{2E}{c} \right)^{\frac{1}{2}} \end{array} \right\}, \quad (6.26)$$

for all the values of r . Now the only restriction on the quantities $k_n(N)$ is (6.22), and hence, from (6.26), by Theorem 9 of the Appendix it follows that

$$(N) \quad \left\{ \begin{array}{l} \sum_{n=-N}^N \sigma_n^2 \left| \frac{\partial V(\alpha_n)}{\partial \alpha_{n,r}} \right|^2 < H_r \\ \sum_{n=-N}^N \sigma_n^4 \left| \frac{\partial T(\alpha_n)}{\partial \alpha_{n,r}} \right|^2 < K_r \end{array} \right\}, \quad (6.27)$$

for all the values of r , where H_r and K_r are independent of N .

From (6.11), and in view of (6.27), Theorem 8 of the Appendix gives

$$\left. \begin{aligned} \lim_{N \rightarrow \infty} \sum_{n=-N}^N \left| \frac{\partial V(a_n(N))}{\partial a_{n,r}(N)} \right|^2 &= \sum_{n=-\infty}^{\infty} \left| \frac{\partial V(a_n)}{\partial a_{n,r}} \right|^2 \\ \lim_{N \rightarrow \infty} \sum_{n=-N}^N \sigma_n^2(N) \left| \frac{\partial T(a_n(N))}{\partial a_{n,r}(N)} \right|^2 &= \sum_{n=-\infty}^{\infty} \sigma_n^2 \left| \frac{\partial T(a_n)}{\partial a_{n,r}} \right|^2 \end{aligned} \right\}, \quad (6.31)$$

for $r = 1, 2, 3, \dots$. From (5.32), and in view of (6.31), Theorem 7 of the Appendix gives

$$\left. \begin{aligned} \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{\partial V(a_n)}{\partial a_{n,r}} \frac{\partial V(a_{-n})}{\partial a_{-n,s}} &= c_{r,s} \\ \frac{1}{2} \sum_{n=-\infty}^{\infty} \sigma_n \frac{\partial V(a_n)}{\partial a_{n,r}} \frac{\partial T(a_{-n})}{\partial a_{-n,s}} &= 0 \\ \frac{1}{2} \sum_{n=-\infty}^{\infty} \sigma_n^2 \frac{\partial T(a_n)}{\partial a_{n,r}} \frac{\partial T(a_{-n})}{\partial a_{-n,s}} &= a_{r,s} \end{aligned} \right\}, \quad (6.32)$$

for $r, s = 1, 2, 3, \dots$

If we write

$$(N) \quad \left\{ \begin{aligned} A_{r,s} &= \frac{1}{2} \sum_{n=-N}^N a_{n,r} a_{-n,s} \\ A'_{r,s} &= \frac{1}{2} \sum_{n=-N}^N \sigma_n a_{n,r} a_{-n,s} \\ A''_{r,s} &= \frac{1}{2} \sum_{n=-N}^N \sigma_n^2 a_{n,r} a_{-n,s} \end{aligned} \right\}, \quad (6.41)$$

the relations (5.31) take the form

$$(N) \quad \left\{ \begin{aligned} c_{r,1} A_{s,1} + \dots + c_{r,N} A_{s,N} &= 1 \quad (r=s), \quad = 0 \quad (r \neq s) \\ c_{r,1} A'_{s,1} + \dots + c_{r,N} A'_{s,N} &= 0 \\ a_{r,1} A'_{s,1} + \dots + a_{r,N} A'_{s,N} &= 0 \\ a_{r,1} A''_{s,1} + \dots + a_{r,N} A''_{s,N} &= 1 \quad (r=s), \quad = 0 \quad (r \neq s) \end{aligned} \right\}. \quad (6.42)$$

For each of the N values of s there are N sets of equations (6.42) corresponding to $r = 1, 2, \dots, N$. From the N equations corresponding to the first of (6.42) we see that

$$A_{s,1}, A_{s,2}, \dots, A_{s,N}$$

are determined completely and that $A_{r,s}(N)$ ($r, s = 1, 2, \dots, N$) depend only on the coefficients of $V_N(x)$. Similarly, from the fourth of (6.42), we see that $A''_{r,s}(N)$ are determined completely and depend only on the coefficients of $T_N(x)$. From either the second or third of (6.42) we deduce that

$$A'_{r,s}(N) = 0 \quad (r, s = 1, 2, \dots, N), \quad (6.43)$$

since the determinants of the coefficients of $V_N(x)$ and $T_N(x)$ cannot vanish. On multiplying the first and fourth equations of (6.42) by $A_{s,r}(N)$ and $A''_{s,r}(N)$ respectively, and adding in each case for $r = 1, 2, \dots, N$, we deduce that

$$(N) \quad \left\{ \begin{array}{l} 2V(A_s) = A_{s,s} \\ 2T(A''_s) = A''_{s,s} \end{array} \right\}. \quad (6.44)$$

But we know, from (1.21), that

$$(N) \quad \left\{ \begin{array}{l} 2V(A_s) \geq c A_{s,s}^2 \\ 2T(A''_s) \geq a \frac{A''_{s,s}{}^2}{\nu_s^2} \end{array} \right\}, \quad (6.45)$$

and therefore we have

$$(N) \quad \left\{ \begin{array}{l} \sum_{n=1}^N |a_{n,s}|^2 \leq \frac{1}{c} \\ \sum_{n=1}^N \sigma_n^2 |a_{n,s}|^2 \leq \frac{\nu_s^2}{a} \end{array} \right\}, \quad (6.46)$$

for $s = 1, 2, \dots, N$.

On applying Theorem 7 of the Appendix to (5.31), in view of (6.31) and (6.46), we find that

$$\left. \begin{array}{l} \frac{1}{2} \sum_{n=-\infty}^{\infty} a_{n,s} \frac{\partial V(a_{-n})}{\partial a_{-n,r}} = 1 \quad (r=s), \quad = 0 \quad (r \neq s) \\ \frac{1}{2} \sum_{n=-\infty}^{\infty} \sigma_n a_{n,s} \frac{\partial V(a_{-n})}{\partial a_{-n,r}} = 0 \\ \frac{1}{2} \sum_{n=-\infty}^{\infty} \sigma_n a_{n,s} \frac{\partial T(a_{-n})}{\partial a_{-n,r}} = 0 \\ \frac{1}{2} \sum_{n=-\infty}^{\infty} \sigma_n^2 a_{n,s} \frac{\partial T(a_{-n})}{\partial a_{-n,r}} = 1 \quad (r=s), \quad = 0 \quad (r \neq s) \end{array} \right\}. \quad (6.5)$$

7. *Forced periodic motion.*

For complex periodic forced motion the disturbing forces, the co-ordinates and their speeds will contain the time through a factor of the form $e^{i\sigma t}$, so that

$$(N) \quad (\dot{q}) = i\sigma(q), \quad (\ddot{q}) = i\sigma(\dot{q}).$$

On substituting into the energy equation (2.33) with accented letters corresponding to the complex fundamental mode of time-factor $e^{-i\sigma_n(N)t}$, we obtain

$$(N) \quad 2\sigma_n(\sigma - \sigma_n) T_N(a_{-n}, \dot{q}) + 2i(\sigma - \sigma_n) V_N(a_{-n}, q) = -i\sigma_n \sum_{r=1}^N a_{-n,r} Q_r,$$

so that, by comparison with (5.21), we see that*

$$(N) \quad \xi_n = \frac{-\sigma_n}{2(\sigma - \sigma_n)} \sum_{r=1}^N a_{-n,r} Q_r, \quad (7.1)$$

for all the values of n . We notice incidentally, from (5.44), that either

$$\sigma \neq \sigma_n(N),$$

or else

$$(N) \quad \sum_{r=1}^N a_{-n,r} Q_r = 0,$$

for any values of N and n .

By the corollary to Theorem 7 of the Appendix, in view of (4.31) and (2.12), we deduce that

$$\xi_n(N) \rightarrow \xi_n \quad \text{as } N \rightarrow \infty, \quad (7.21)$$

where

$$\xi_n = \frac{-\sigma_n}{2(\sigma - \sigma_n)} \sum_{r=1}^{\infty} a_{-n,r} Q_r, \quad (7.22)$$

for all the values of n . From (5.22), and in view of (5.44) and (6.46), it follows, by Theorems 7 and 8 of the Appendix, that

$$q_s(N) \rightarrow q_s \quad \text{as } N \rightarrow \infty, \quad (7.23)$$

where

$$q_s = \sum_{n=-\infty}^{\infty} a_{n,s} \xi_n, \quad (7.24)$$

* Cf. H. Poincaré, *loc. cit.*, 26.

for $s = 1, 2, 3, \dots$. On applying again the corollary to Theorem 7 of the Appendix, in view of (2.41), we see that (7.24) and their time-derivatives satisfy the equations of motion. We have thus shown that the method of sections may be applied to periodic forced motion, and we have found an expression for the general solution.

If we now substitute from (7.22) into (7.24) we have

$$q_s = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{-\sigma_n}{\sigma - \sigma_n} \alpha_{n,s} \sum_{r=1}^{\infty} \alpha_{-n,r} Q_r, \quad (7.25)$$

for $s = 1, 2, 3, \dots$; and we must either have

$$\sigma \neq \sigma_n,$$

or else

$$\sum_{r=1}^{\infty} \alpha_{-n,r} Q_r = 0,$$

for all values of n . The equations (7.25) give expressions for the general forced periodic motion in terms of the forces and the coefficients of the fundamental modes. Their form shows explicitly the effect of an approach of σ to one of the values of σ_n , and on this account they may be called the *resonance equations*.

8. Transformation for general system.

We now suppose that the coordinates (q), regarded as functions of time, represent any motion of the system satisfying the fundamental assumptions of § 2. Since these coordinates are independent of N we are not now using the method of sections. In view of (2.41), (6.12), and (6.13) the corollary to Theorem 7 of the Appendix gives

$$\lim_{N \rightarrow \infty} \sum_{r=1}^N \frac{\partial V\{a_n(N)\}}{\partial a_{n,r}(N)} q_r = \sum_{r=1}^{\infty} \frac{\partial V(a_n)}{\partial a_{n,r}} q_r,$$

$$\lim_{N \rightarrow \infty} \sum_{r=1}^N \frac{\partial T\{a_n(N)\}}{\partial a_{n,r}(N)} \dot{q}_r = \sum_{r=1}^{\infty} \frac{\partial T(a_n)}{\partial a_{n,r}} \dot{q}_r,$$

or

$$\left. \begin{aligned} \lim_{N \rightarrow \infty} V\{a_n(N), q\} &= V(a_n, q) \\ \lim_{N \rightarrow \infty} T\{a_n(N), \dot{q}\} &= T(a_n, \dot{q}) \end{aligned} \right\}, \quad (8.11)$$

for all the values of n . If $\xi_n(N)$ is given by the formula (5.21) we see, from (8.11), that

$$\xi_n(N) \rightarrow \xi_n \quad \text{as } N \rightarrow \infty, \quad (8.12)$$

where

$$\xi_n = V(a_{-n}, q) - i\sigma_n T(a_{-n}, \dot{q}), \quad (8.13)$$

for all the values of n .

For each of the values of n we now define a function of time $\eta_n(\nu)$ by

$$\eta_n(\nu) = \frac{1}{2} \sum_{r=1}^{\nu} \left\{ \frac{\partial V(a_{-n})}{\partial a_{-n,r}} q_r - i\sigma_n \frac{\partial T(a_{-n})}{\partial a_{-n,r}} \dot{q}_r \right\}, \quad (8.21)$$

so that

$$\eta_n(\nu) \rightarrow \xi_n \quad \text{as } \nu \rightarrow \infty, \quad (8.22)$$

for all the values of n .

Next let us consider a second motion of the system, as in §§ 2, 5, satisfying the fundamental assumptions in § 2, and let us distinguish the corresponding values of the coordinates by accents. From (8.21), (6.32), and (1.12), in turn, we find that

$$\begin{aligned} & \sum_{\nu=-\infty}^{\infty} \eta_n(\nu) \eta'_{-n}(\nu) \\ &= \frac{1}{4} \sum_{n=-\infty}^{\infty} \sum_{r=1}^{\nu} \left\{ \frac{\partial V(a_{-n})}{\partial a_{-n,r}} q_r - i\sigma_n \frac{\partial T(a_{-n})}{\partial a_{-n,r}} \dot{q}_r \right\} \sum_{s=1}^{\nu} \left\{ \frac{\partial V(a_n)}{\partial a_{n,s}} q'_s + i\sigma_n \frac{\partial T'(a_n)}{\partial a_{n,s}} \dot{q}'_s \right\} \\ &= \frac{1}{2} \sum_{r,s=1}^{\nu} (c_{s,r} q_r q'_s + a_{s,r} \dot{q}_r \dot{q}'_s) \\ &= V_{\nu}(q, q') + T_{\nu}(\dot{q}, \dot{q}'). \end{aligned} \quad (8.23)$$

As particular cases of (8.23) we have

$$2 \sum_{n=1}^{\infty} \eta_n(\nu) \eta_{-n}(\nu) = V_{\nu}(q) + T_{\nu}(\dot{q}), \quad (8.24)$$

and

$$\sum_{n=-\infty}^{\infty} |\eta_n(\nu)|^2 = V_{\nu}(q, \bar{q}) + T_{\nu}(\dot{q}, \bar{\dot{q}}), \quad (8.25)$$

and then we deduce from (1.21) that

$$\sum_{n=-\infty}^{\infty} |\eta_n(\nu)|^2 \leq \frac{1}{2} C \sum_{s=1}^{\infty} |q_s|^2 + \frac{1}{2} A \sum_{s=1}^{\infty} \frac{|\dot{q}_s|^2}{\nu^2}. \quad (8.26)$$

Since (8.26) is true for all values of ν , we have

$$\sum_{n=-\infty}^{\infty} |\xi_n|^2 \leq \frac{1}{2} C \sum_{s=1}^{\infty} |q_s|^2 + \frac{1}{2} A \sum_{s=1}^{\infty} \frac{|\dot{q}_s|^2}{\nu_s^2}. \quad (8.27)$$

From (8.13) and (8.21) it follows that

$$\xi_n - \eta_n(\nu) = \frac{1}{2} \sum_{s=\nu+1}^{\infty} \left\{ \frac{\partial V(a_{-n})}{\partial a_{-n,s}} q_s - i \sigma_n \frac{\partial T(a_{-n})}{\partial a_{-n,s}} \dot{q}_s \right\},$$

and since, in (8.27), (q) and $(\dot{q})$ are quite arbitrary, we see that

$$\sum_{n=-\infty}^{\infty} |\xi_n - \eta_n(\nu)|^2 \leq \frac{1}{2} C \sum_{s=\nu+1}^{\infty} |q_s|^2 + \frac{1}{2} A \sum_{s=\nu+1}^{\infty} \frac{|\dot{q}_s|^2}{\nu_s^2}.$$

For any definite prescribed motion we therefore have

$$\lim_{\nu \rightarrow \infty} \sum_{n=-\infty}^{\infty} |\xi_n - \eta_n(\nu)|^2 = 0. \quad (8.28)$$

From (8.24) and (8.28) it now follows, by Theorems 5 and 7 of the Appendix, that

$$2 \sum_{n=1}^{\infty} \xi_n \xi_{-n} = V(q) + T(\dot{q}), \quad (8.31)$$

and, from (8.25), that

$$\sum_{n=-\infty}^{\infty} |\xi_n|^2 = V(q, \bar{q}) + T(\dot{q}, \dot{\bar{q}}). \quad (8.32)$$

But
$$V(q, \bar{q}) + T(\dot{q}, \dot{\bar{q}}) = \lim_{N \rightarrow \infty} \{ V_N(q, \bar{q}) + T_N(\dot{q}, \dot{\bar{q}}) \},$$

and therefore
$$\sum_{n=-\infty}^{\infty} |\xi_n|^2 = \lim_{N \rightarrow \infty} \sum_{n=-N}^N |\xi_n(N)|^2. \quad (8.33)$$

In view of (8.33) and (6.46) we can apply Theorem 7 of the Appendix to the equations (5.22) and thus obtain

$$\left. \begin{aligned} q_s &= \sum_{n=-\infty}^{\infty} \alpha_{n,s} \xi_n \\ \dot{q}_s &= i \sum_{n=-\infty}^{\infty} \sigma_n \alpha_{n,s} \xi_n \end{aligned} \right\} \quad (s = 1, 2, 3, \dots). \quad (8.34)$$

We have therefore established the transformation given by (8.13) and (8.34) for any motion of the system satisfying our fundamental assumptions.

On comparing the method of this section with that of the preceding section we notice that, on account of the second equation of (8.34), we require (8.33), whereas for (7.24) or the first of equations (8.34) the condition (5.44) is a sufficient restriction on ξ_n .

9. Deductions from the transformation.

(i) Energy relations.

If in equations (8.34) we write

$$\xi_n = 0 \quad (n \neq m), \quad \xi_m = e^{i\sigma_m t},$$

we obtain

$$q_s = a_{m,s} e^{i\sigma_m t},$$

$$\dot{q}_s = i\sigma_m a_{m,s} e^{i\sigma_m t},$$

for $s = 1, 2, 3, \dots$, corresponding, in fact, to a fundamental mode. On substituting these values into (8.13) we deduce that

$$\left. \begin{aligned} V(a_{-n}, a_m) + \sigma_n \sigma_m T(a_{-n}, a_m) &= 0 & (n \neq m) \\ &= 1 & (n = m) \end{aligned} \right\}, \quad (9.1)$$

for all the values of n and m . These equations are the extensions of (4.2) and (5.1) to the case of infinite freedom.

(ii) General free motion.

If in (8.34) we write

$$\xi_n = k_n e^{i\sigma_n t},$$

for all the values of n , where k_n is any constant, we obtain the most general superposition of the fundamental modes of oscillation established in § 4, viz.

$$q_s = \sum_{n=-\infty}^{\infty} k_n a_{n,s} e^{i\sigma_n t}, \quad (9.21)$$

for $s = 1, 2, 3, \dots$. Equations (8.13) then become

$$k_n e^{i\sigma_n t} = V(a_{-n}, q) - i\sigma_n T(a_{-n}, \dot{q}), \quad (9.22)$$

and, if (q^0) , $(\dot{q}^0)$ denote respectively the values of (q) , $(\dot{q})$ at the initial

instant $t = 0$, we have

$$k_n = V(a_{-n}, q^0) - i\sigma_n T(a_{-n}, \dot{q}^0), \quad (9.23)$$

for all the values of n . The equations (9.23) give all the constants k_n of (9.21) in terms of the initial displacements and velocities. Since these initial conditions may be the most general possible, and since a free motion is completely determined by a prescription of all initial displacements and velocities, it follows that (9.21) represents the most general possible free motion of the system.

(iii) *Uniqueness of limits.*

We now proceed to show that the limits $a_{n,s}$ do not depend on the particular sequence (4.32) associated with the Principle of Choice. This means that, in calculating the limits, N need not be restricted to the sequence (4.32), as has been done in §§ 4, 6, 7, 8, but may be allowed to tend to infinity through all positive integers. From a practical point of view this is very important.

We have proved in § 3 that the sequence of values of σ_n is unique, and then, in § 9(ii), that (9.21) represents the most general possible free motion. Let us now suppose that a free motion exists in which

$$q_s = a'_{m,s} e^{i\sigma_m t} \quad (s = 1, 2, 3, \dots), \quad (9.31)$$

so that, on substituting into (9.21), and then dividing by $e^{i\sigma_m t}$, we obtain

$$a'_{m,s} - k_{m_1} a_{m_1,s} - k_{m_2} a_{m_2,s} \dots - k_{m_\mu} a_{m_\mu,s} = \sum'_{n=-\infty}^{\infty} k_n a_{n,s} e^{i(\sigma_n - \sigma_m)t}, \quad (9.32)$$

where $m_1 m_2 \dots m_\mu$ are all the values of n for which $\sigma_n = \sigma_m$, and the terms corresponding to these values are omitted from the summation Σ' . On taking the time-mean of each side of (9.32), from $t = 0$ to $t = \tau$, the left-hand side is unaffected, but the right-hand side becomes

$$\sum'_{n=-\infty}^{\infty} k_n a_{n,s} \frac{e^{i(\sigma_n - \sigma_m)\tau} - 1}{i(\sigma_n - \sigma_m)\tau}. \quad (9.33)$$

Now the series

$$\sum'_{n=-\infty}^{\infty} k_n a_{n,s}$$

is supposed absolutely convergent, in view of (6.22), (6.46), and Theorem 1 of the Appendix. The value of (9.33) may therefore be

made as small as we please by taking τ sufficiently great, and it follows that

$$a'_{m,s} = k_{m_1} a_{m_1,s} + \dots + k_{m_{\mu}} a_{m_{\mu},s}. \quad (9.34)$$

Hence the only indeterminacy in the value of $a_{m,s}$ is of exactly the same type as that which obtains for a system of finite freedom, and no indeterminacy is introduced by the limiting process.

10. Oscillations about equilibrium.

As stated in the Introduction, the theory of the oscillations of a mechanical system of infinite freedom about a configuration of equilibrium has already been worked out. It is not without interest, however, to derive the results of this theory from those established for the more general system in the foregoing sections, and this we proceed to do.

We have now a zero angular speed for our rotating system, so that

$$M(q, \dot{q}) = 0,$$

for all values of (q) , $(\dot{q})$, and therefore

$$\beta_{r,s} = 0 \quad (r, s = 1, 2, 3, \dots). \quad (10.11)$$

Lagrange's equations of free motion give

$$\sigma_n^2 \frac{\partial T(a_n)}{\partial a_{n,r}} = \frac{\partial V(a_n)}{\partial a_{n,r}}, \quad (10.12)$$

for all the values of n and r , so that

$$\sigma_n^2 T(a_n, q) = V(a_n, q). \quad (10.13)$$

We notice that $a_{n,s}$ may now be taken to be real, so that

$$a_{-n,s} = a_{n,s} \quad (s = 1, 2, 3, \dots), \quad (10.14)$$

though, of course, the equation

$$\sigma_{-n} = -\sigma_n,$$

still holds. The first equation of (9.1) may now be written

$$V(a_n, a_m) + \sigma_n \sigma_m T(a_n, a_m) = 0,$$

provided that $m \neq n$, and, on changing the sign of m , we immediately deduce that

$$V(a_n, a_m) - \sigma_n \sigma_m T(a_n, a_m) = 0,$$

provided that $m \neq -n$. It follows that

$$V(a_n, a_m) = 0, \quad T(a_n, a_m) = 0, \quad (10.15)$$

provided that $m^2 \neq n^2$. From the second equation of (9.1) and (10.13) with $(q) = (a_n)$, we deduce

$$2V(a_n) = 2\sigma_n^2 T(a_n) = 1. \quad (10.16)$$

Now let us take a linear combination of the coordinates given by

$$p_n = 2V(a_n, q) = 2\sigma_n^2 T(a_n, q), \quad (10.21)$$

for $n = 1, 2, 3, \dots$. We deduce that

$$\dot{p}_n = 2V(a_n, \dot{q}) = 2\sigma_n^2 T(a_n, \dot{q}), \quad (10.22)$$

so that the formulae (8.13) become

$$\left. \begin{aligned} \xi_n &= \frac{1}{2} \left(p_n - \frac{i\dot{p}_n}{\sigma_n} \right) \\ \xi_{-n} &= \frac{1}{2} \left(p_n + \frac{i\dot{p}_n}{\sigma_n} \right) \end{aligned} \right\}, \quad (10.23)$$

and we have

$$\left. \begin{aligned} p_n &= \xi_n + \xi_{-n} \\ \frac{1}{4} \left(p_n^2 + \frac{\dot{p}_n^2}{\sigma_n^2} \right) &= \xi_n \xi_{-n} \end{aligned} \right\}, \quad (10.24)$$

for $n = 1, 2, 3, \dots$. It then follows, from (8.34), that

$$q_s = \sum_{n=1}^{\infty} a_{n,s} p_n \quad (s = 1, 2, 3, \dots), \quad (10.25)$$

and, from (8.31), that

$$\sum_{n=1}^{\infty} \left(p_n^2 + \frac{\dot{p}_n^2}{\sigma_n^2} \right) = 2V(q) + 2T(\dot{q}).$$

Since (q) and $(\dot{q})$ are quite independent we see that

$$\left. \begin{aligned} 2V(q) &= \sum_{n=1}^{\infty} p_n^2 \\ 2T(\dot{q}) &= \sum_{n=1}^{\infty} \frac{\dot{p}_n^2}{\sigma_n^2} \end{aligned} \right\}, \quad (10.26)$$

and it appears that the sequence (p) , given by (10.21), constitutes a set of *normal coordinates*.

The resonance equation (7.25) may be written

$$q_s = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{-\sigma_n}{\sigma - \sigma_n} + \frac{\sigma_n}{\sigma + \sigma_n} \right) a_{n,s} \sum_{r=1}^{\infty} a_{n,r} Q_r,$$

or

$$q_s = \sum_{n=1}^{\infty} \frac{-\sigma_n^2}{\sigma^2 - \sigma_n^2} a_{n,s} \sum_{r=1}^{\infty} a_{n,r} Q_r, \quad (10.3)$$

for $s = 1, 2, 3, \dots$

Appendix.

The object of this Appendix is to supply the more purely mathematical part of the reasoning of the paper. We proceed to enunciate a number of theorems which have been used in the foregoing sections. All these theorems, or variations of them, are of frequent occurrence in the literature on infinite systems of linear equations, but we shall now enunciate them exactly in the form in which we have used them. In this form, with minor exceptions, it has proved impossible to quote references to complete separate proofs, and we shall therefore supply proofs, except for the well known inequalities.

In the following, A, C denote positive constants independent of the positive integers N, n, v, r , and s ; a_s, b_s denote any numbers; $a_{r,s}, b_{r,s}$ real numbers, with

$$a_{s,r} = a_{r,s} \quad (r, s = 1, 2, 3, \dots); \quad (A.1)$$

while $x_s, \bar{x}_s$ are conjugate complexes. Also, we suppose that

$$\sum_{s=1}^N |x_s(N)|^2 \leq C, \quad \sum_{s=1}^N |y_s(N)|^2 \leq C, \quad (A.2)$$

and

$$\left. \begin{array}{l} x_s(N) \rightarrow x_s \\ y_s(N) \rightarrow y_s \end{array} \right\} \text{ as } N \rightarrow \infty \quad (s = 1, 2, 3, \dots). \quad (A.3)$$

$$\text{THEOREM 1.} \quad \left| \sum_{s=1}^N a_s b_s \right|^2 \leq \sum_{s=1}^N |a_s|^2 \sum_{s=1}^N |b_s|^2.$$

$$\text{THEOREM 2.} \quad \sum_{r,s=1}^N a_{r,s} x_s \bar{x}_r \text{ is real.}$$

To prove this we write

$$x_s = u_s + iv_s \quad (s = 1, 2, 3, \dots),$$

where u_s and v_s are real. Then

$$a_{s,s} x_s \bar{x}_s = a_{s,s} (u_s^2 + v_s^2),$$

$$a_{r,s} x_s \bar{x}_r + a_{s,r} x_r \bar{x}_s = 2a_{r,s} (u_r u_s + v_r v_s),$$

and the theorem follows immediately.

THEOREM 3. *If*

$$\left| \sum_{r,s=1}^N b_{r,s} x_s y_r \right|^2 \leq A^2 \sum_{s=1}^N |x_s|^2 \sum_{r=1}^N |y_r|^2, \quad (\text{A. 31})$$

then

$$(i) \quad \sum_{s=1}^N b_{r,s}^2 \leq A^2, \quad (\text{A. 32})$$

$$(ii) \quad \sum_{r=1}^N \left| \sum_{s=1}^N b_{r,s} x_s \right|^2 \leq A^2 \sum_{s=1}^N |x_s|^2, \quad (\text{A. 33})$$

$$(iii) \text{ the double series } \sum_{r,s=1}^{\infty} b_{r,s} x_s y_r$$

is convergent, provided that

$$\sum_{s=1}^N |x_s|^2 \leq C, \quad \sum_{s=1}^N |y_s|^2 \leq C. \quad (\text{A. 34})$$

To prove (i) we take in (A. 31) one particular value of y_r to be unity and all other values to be zero, while we write

$$x_s = b_{r,s} \quad (s = 1, 2, 3, \dots).$$

Then

$$\left| \sum_{s=1}^N b_{r,s}^2 \right| \leq A^2 \sum_{s=1}^N b_{r,s}^2,$$

from which (A. 32) follows immediately.

To prove (ii) we write*

$$y_r = \sum_{s=1}^N b_{r,s} \bar{x}_s,$$

* Cf. F. Riesz, *loc. cit.*, 127.

so that (A . 31) becomes

$$\left| \sum_{r=1}^N \left| \sum_{s=1}^N b_{r,s} x_s \right|^2 \right|^2 \leq A^2 \sum_{s=1}^N |x_s|^2 \sum_{r=1}^N \left| \sum_{s=1}^N b_{r,s} x_s \right|^2,$$

from which (A . 33) follows immediately.

To prove (iii) we have*

$$\begin{aligned} & \left| \sum_{r=1}^{m+p} \sum_{s=1}^{n+q} b_{r,s} x_s y_r - \sum_{r=1}^m \sum_{s=1}^n b_{r,s} x_s y_r \right| \\ & \leq \left| \sum_{r=1}^m \sum_{s=n+1}^{n+q} b_{r,s} x_s y_r \right| + \left| \sum_{r=m+1}^{m+p} \sum_{s=1}^{n+q} b_{r,s} x_s y_r \right| \\ & \leq AC^{\frac{1}{2}} \left\{ \sum_{s=n+1}^{n+q} |x_s|^2 \right\}^{\frac{1}{2}} + AC^{\frac{1}{2}} \left\{ \sum_{r=m+1}^{m+p} |y_r|^2 \right\}^{\frac{1}{2}} \\ & \leq 2\epsilon, \end{aligned}$$

ϵ being any positive number, if m and n be so chosen that

$$\sum_{r=n+1}^{\infty} |x_s|^2 \leq \frac{\epsilon^2}{A^2 C}, \quad \sum_{r=m+1}^{\infty} |y_r|^2 \leq \frac{\epsilon^2}{A^2 C}.$$

THEOREM 4. If

$$\left| \sum_{r,s=1}^N a_{r,s} x_s \bar{x}_r \right| \leq A \sum_{s=1}^N |x_s|^2, \quad (\text{A . 41})$$

$$\text{then} \quad \left| \sum_{r,s=1}^N a_{r,s} x_s y_r \right|^2 \leq A^2 \sum_{s=1}^N |x_s|^2 \sum_{r=1}^N |y_r|^2. \quad (\text{A . 42})$$

Let us first prove this for the case in which (x) and (y) are real†. We have then

$$\begin{aligned} 4 \left| \sum_{r,s=1}^N a_{r,s} x_s y_r \right| &= \left| \sum_{r,s=1}^N a_{r,s} \{ (x_s + y_s)(x_r + y_r) - (x_s - y_s)(x_r - y_r) \} \right| \\ &\leq A \sum_{s=1}^N \{ (x_s + y_s)^2 + (x_s - y_s)^2 \} \\ &\leq 2A \sum_{s=1}^N (x_s^2 + y_s^2), \end{aligned}$$

* F. Riesz, *loc. cit.*, 84.

† *Loc. cit.*, 126.

and the theorem is consequently true if

$$\sum_{s=1}^N |x_s|^2 = \sum_{s=1}^N |y_s|^2 = 1.$$

On account of the homogeneity the theorem is therefore true for all real values of (x) and (y) , and we deduce, from Theorem 3 (ii), that

$$\sum_{r=1}^N \left(\sum_{s=1}^N a_{r,s} x_s \right)^2 \leq A^2 \sum_{s=1}^N x_s^2,$$

for any real (x) . On next writing

$$x_s = u_s + iv_s,$$

we have

$$\begin{aligned} \sum_{r=1}^N \left| \sum_{s=1}^N a_{r,s} x_s \right|^2 &= \sum_{r=1}^N \left\{ \left(\sum_{s=1}^N a_{r,s} u_s \right)^2 + \left(\sum_{s=1}^N a_{r,s} v_s \right)^2 \right\} \\ &\leq A^2 \sum_{s=1}^N (u_s^2 + v_s^2). \end{aligned}$$

It now follows from Theorem 1 that

$$\begin{aligned} \left| \sum_{r=1}^N \left(\sum_{s=1}^N a_{r,s} x_s \right) y_r \right|^2 &\leq \sum_{r=1}^N \left| \sum_{s=1}^N a_{r,s} x_s \right|^2 \sum_{r=1}^N |y_r|^2 \\ &\leq A^2 \sum_{s=1}^N (u_s^2 + v_s^2) \sum_{r=1}^N |y_r|^2, \end{aligned}$$

which is equivalent to (A. 42).

$$\text{THEOREM 5. If } \sum_{s=1}^{\infty} |x_s(\nu)|^2 \leq C, \quad (\text{A. 51})$$

$$\text{and } \lim_{\nu \rightarrow \infty} \sum_{s=1}^{\infty} |x_s(\nu) - x_s|^2 = 0, \quad (\text{A. 52})$$

$$\text{then* } \lim_{\nu \rightarrow \infty} \sum_{s=1}^{\infty} |x_s(\nu)|^2 = \sum_{s=1}^{\infty} |x_s|^2. \quad (\text{A. 53})$$

To prove this we first deduce, from (A. 52), that

$$x_s(\nu) \rightarrow x_s \quad \text{as } \nu \rightarrow \infty \quad (s = 1, 2, 3, \dots),$$

and then, from (A. 51), that

$$\sum_{s=1}^{\infty} |x_s|^2 \leq C. \quad (\text{A. 54})$$

* Cf. F. Riesz, *loc. cit.*, 59, foot-note.

We next make use of the known inequality

$$\left\{ \sum_{s=1}^N |a_s - b_s|^2 \right\}^{\frac{1}{2}} \geq \left\{ \sum_{s=1}^N |a_s|^2 \right\}^{\frac{1}{2}} - \left\{ \sum_{s=1}^N |b_s|^2 \right\}^{\frac{1}{2}},$$

and hence deduce, in view of (A. 51) and (A. 54),

$$\left\{ \sum_{s=1}^{\infty} |x_s(\nu) - x_s|^2 \right\}^{\frac{1}{2}} \geq \left\{ \sum_{s=1}^{\infty} |x_s(\nu)|^2 \right\}^{\frac{1}{2}} - \left\{ \sum_{s=1}^{\infty} |x_s|^2 \right\}^{\frac{1}{2}},$$

$$\left\{ \sum_{s=1}^{\infty} |x_s - x_s(\nu)|^2 \right\}^{\frac{1}{2}} \geq \left\{ \sum_{s=1}^{\infty} |x_s|^2 \right\}^{\frac{1}{2}} - \left\{ \sum_{s=1}^{\infty} |x_s(\nu)|^2 \right\}^{\frac{1}{2}}.$$

On using (A. 52) we therefore have

$$\overline{\lim}_{\nu \rightarrow \infty} \sum_{s=1}^{\infty} |x_s(\nu)|^2 \leq \sum_{s=1}^{\infty} |x_s|^2,$$

$$\underline{\lim}_{\nu \rightarrow \infty} \sum_{s=1}^{\infty} |x_s(\nu)|^2 \geq \sum_{s=1}^{\infty} |x_s|^2,$$

from which (A. 53) follows.

THEOREM 6. If
$$\sum_{s=1}^N |x_s(N)|^2 \leq C, \quad (\text{A. 61})$$

then
$$\sum_{s=1}^{\infty} |x_s|^2 \leq C. \quad (\text{A. 62})$$

To prove this we notice that, if $N_1 < N$, it follows, from (A. 61), that

$$\sum_{s=1}^{N_1} |x_s(N)|^2 \leq C,$$

and consequently
$$\sum_{s=1}^{N_1} |x_s|^2 \leq C,$$

which, in turn, leads to (A. 62).

THEOREM 7. If
$$\lim_{N \rightarrow \infty} \sum_{s=1}^N |x_s(N)|^2 = \sum_{s=1}^{\infty} |x_s|^2, \quad (\text{A. 71})$$

then*
$$\lim_{N \rightarrow \infty} \sum_{s=1}^N x_s(N) y_s(N) = \sum_{s=1}^{\infty} x_s y_s, \quad (\text{A. 72})$$

no restrictions on $(y(N))$ except (A. 2) and (A. 3) being necessary.

* Cf. F. Riesz, *loc. cit.*, 56, foot-note.

To prove this we have, from (A.2), (A.3), and Theorem 6,

$$\sum_{s=1}^{\infty} |x_s|^2 \leq C,$$

and then, on using (A.71),

$$\lim_{N \rightarrow \infty} \sum_{s=m}^N |x_s(N)|^2 = \sum_{s=m}^{\infty} |x_s|^2 \leq \frac{\epsilon^2}{C}, \quad (\text{A.73})$$

ϵ being any positive number, and m (independent of N) being chosen accordingly. We have also, from (A.2), (A.3), and for $N > m$,

$$\overline{\lim}_{N \rightarrow \infty} \sum_{s=m}^N |y_s(N)|^2 \leq C, \quad (\text{A.74})$$

and, on using Theorem 6,

$$\sum_{s=m}^{\infty} |y_s|^2 \leq C. \quad (\text{A.75})$$

Now

$$\left| \sum_{s=1}^N x_s(N) y_s(N) - \sum_{s=1}^{\infty} x_s y_s \right|$$

$$\leq \left| \sum_{s=1}^{m-1} \{x_s(N) y_s(N) - x_s y_s\} \right| + \left| \sum_{s=m}^N x_s(N) y_s(N) \right| + \left| \sum_{s=m}^{\infty} x_s y_s \right|,$$

and, since m is fixed by ϵ ,

$$\lim_{N \rightarrow \infty} \left| \sum_{s=1}^{m-1} \{x_s(N) y_s(N) - x_s y_s\} \right| = 0,$$

while

$$\overline{\lim}_{N \rightarrow \infty} \left| \sum_{s=m}^N x_s(N) y_s(N) \right| \leq \overline{\lim}_{N \rightarrow \infty} \left\{ \sum_{s=m}^N |x_s(N)|^2 \right\}^{\frac{1}{2}} \left\{ \sum_{s=m}^N |y_s(N)|^2 \right\}^{\frac{1}{2}}$$

$$\leq \epsilon,$$

$$\left| \sum_{s=m}^{\infty} x_s y_s \right| \leq \left\{ \sum_{s=m}^{\infty} |x_s|^2 \right\}^{\frac{1}{2}} \left\{ \sum_{s=m}^{\infty} |y_s|^2 \right\}^{\frac{1}{2}}$$

$$\leq \epsilon,$$

by Theorem 1, (A.73), (A.74), and (A.75). We therefore have

$$\overline{\lim}_{N \rightarrow \infty} \left| \sum_{s=1}^N x_s(N) y_s(N) - \sum_{s=1}^{\infty} x_s y_s \right| \leq 2\epsilon,$$

from which (A.72) follows immediately.

Corollary to Theorem 7. If

$$\sum_{s=1}^{\infty} |a_s|^2 < A,$$

where a_s is independent of N , then*

$$\lim_{N \rightarrow \infty} \sum_{s=1}^N a_s x_s(N) = \sum_{s=1}^{\infty} a_s x_s,$$

for here $a_s, x_s(N)$ take the place respectively of $x_s(N), y_s(N)$ of Theorem 7.

THEOREM 8. If

$$\sigma_1(N), \sigma_2(N), \dots, \sigma_N(N)$$

are functions of N which satisfy the conditions

$$0 < \sigma_n(N) \leq \sigma_{n+1}(N), \quad \sigma_n(N+1) \leq \sigma_n(N), \quad (\text{A. 81})$$

$$\lim_{N \rightarrow \infty} \sigma_n(N) = \sigma_n, \quad \lim_{n \rightarrow \infty} \sigma_n = \infty; \quad (\text{A. 82})$$

and if
$$\sum_{n=1}^N \sigma_n(N) |x_n(N)|^2 \leq A, \quad (\text{A. 83})$$

then
$$\lim_{N \rightarrow \infty} \sum_{n=1}^N |x_n(N)|^2 = \sum_{n=1}^{\infty} |x_n|^2. \quad (\text{A. 84})$$

To prove this we first notice that, if ϵ is any positive number, we can, from (A. 82), find m (independent of N) so that

$$\frac{1}{\sigma_m} \leq \frac{\epsilon}{A}, \quad (\text{A. 85})$$

and we then have, from (A. 81) and (A. 82), for $N > m$,

$$\begin{aligned} \sum_{n=m}^N |x_n(N)|^2 &\leq \frac{1}{\sigma_m(N)} \sum_{n=m}^N \sigma_n(N) |x_n(N)|^2 \\ &\leq \frac{1}{\sigma_m} \sum_{n=m}^N \sigma_n(N) |x_n(N)|^2 \\ &\leq \epsilon. \end{aligned} \quad (\text{A. 86})$$

Now
$$\left| \sum_{n=1}^N |x_n(N)|^2 - \sum_{n=1}^{\infty} |x_n|^2 \right|$$

$$\leq \left| \sum_{n=1}^{m-1} (|x_n(N)|^2 - |x_n|^2) \right| + \sum_{n=m}^N |x_n(N)|^2 + \sum_{n=m}^{\infty} |x_n|^2,$$

* F. Riesz, *loc. cit.*, 56.

and since m is fixed by ϵ ,

$$\lim_{N \rightarrow \infty} \sum_{n=1}^{m-1} \{ |x_n(N)|^2 - |x_n|^2 \} = 0,$$

while

$$\overline{\lim}_{N \rightarrow \infty} \sum_{n=m}^N |x_n(N)|^2 \leq \epsilon,$$

$$\sum_{n=m}^{\infty} |x_n|^2 \leq \epsilon,$$

by (A. 86) and Theorem 6. We therefore have

$$\overline{\lim}_{N \rightarrow \infty} \left| \sum_{n=1}^N |x_n(N)|^2 - \sum_{n=1}^{\infty} |x_n|^2 \right| \leq 2\epsilon,$$

from which (A. 84) follows immediately.

$$\text{THEOREM 9. If } \left| \sum_{s=1}^N a_s(N) x_s(N) \right| \quad (\text{A. 91})$$

is bounded for all sequences $(x(N))$, as defined above, then a value A can be found such that*

$$\sum_{s=1}^N |a_s(N)|^2 < A. \quad (\text{A. 92})$$

To prove this, let us suppose that no such constant A exists. Then, whatever positive constant K may be, we can always find N so that

$$\sum_{s=1}^N |a_s(N)|^2 > K^2. \quad (\text{A. 93})$$

Now take

$$x_s(N) = \frac{KC^{\frac{1}{2}} \bar{a}_s(N)}{\sum_{s=1}^N |a_s(N)|^2},$$

and we have

$$\sum_{s=1}^N |x_s(N)|^2 = \frac{K^2 C}{\sum_{s=1}^N |a_s(N)|^2} < C,$$

on using (A. 93), while

$$\left| \sum_{s=1}^N a_s(N) x_s(N) \right| = KC^{\frac{1}{2}}.$$

Since K may be taken as large as we please, it follows that (A. 91) is unbounded for a certain sequence satisfying (A. 2). Consequently, the

* Cf. F. Riesz, *loc. cit.*, 47.

supposition that no constant A exists satisfying (A. 92) is impossible, and the theorem follows.

THEOREM 10. *Principle of Choice**. If N , n , and s assume all positive integral values, and if for each triad we have a number $x_{n,s}(N)$ satisfying

$$|x_{n,s}(N)| \leq C, \quad (\text{A. 10, 1})$$

we can find a sequence of positive integers

$$N_1, N_2, \dots, N_m, \dots, \quad (\text{A. 10, 2})$$

such that, as N increases indefinitely through this sequence, there exist limits $x_{n,s}$ so that

$$x_{n,s}(N) \rightarrow x_{n,s} \quad (n, s = 1, 2, 3, \dots).$$

To prove this we first consider one value of n and remark that if we are given a sequence of integers

$$N_1(n-1), N_2(n-1), \dots, N_m(n-1), \dots,$$

or S_{n-1} , we know, by a theorem due to Weierstrass, that we can select, from S_{n-1} , a subsequence S'_{n-1} , say

$$N'_1, N'_2, \dots, N'_m, \dots,$$

and a limit $x_{n,1}$, such that, as $N \rightarrow \infty$ through S'_{n-1} , we have

$$x_{n,1}(N) \rightarrow x_{n,1};$$

similarly from S'_{n-1} we can select a subsequence S''_{n-1} , say

$$N''_1, N''_2, \dots, N''_m, \dots,$$

and a limit $x_{n,2}$, such that, as $N \rightarrow \infty$ through S''_{n-1} , we have

$$x_{n,2}(N) \rightarrow x_{n,2}.$$

Continuing this process we arrive at a sequence of sequences

$$S'_{n-1}, S''_{n-1}, S'''_{n-1}, \dots,$$

each member being contained in all the preceding members. If, then, we write

$$N_1(n) = N'_1, \quad N_2(n) = N''_2, \quad N_3(n) = N'''_3, \quad \dots,$$

* Cf. F. Riesz, *loc. cit.*, 57.

we see that the sequence

$$N_1(n), N_2(n), \dots, N_m(n), \dots,$$

or S_n , thus defined, is contained, except for a finite number of terms, in each of the sequences S'_{n-1} , S''_{n-1} , S'''_{n-1} , ... It follows that, as $N \rightarrow \infty$ through S_n ,

$$x_{n,s}(N) \rightarrow x_{n,s} \quad (s = 1, 2, 3, \dots).^*$$

Taking now the case given by $n = 1$, with S_0 as the sequence

$$1, 2, 3, \dots, m, \dots,$$

we arrive, by the above process, at a sequence

$$N_1(1), N_2(1), \dots, N_m(1), \dots,$$

or S_1 , and a set of limits $x_{1,s} \quad (s = 1, 2, 3, \dots)$,

such that, as $N \rightarrow \infty$ through S_1 ,

$$x_{1,s}(N) \rightarrow x_{1,s} \quad (s = 1, 2, 3, \dots).$$

Taking next the case given by $n = 2$ we arrive at a sequence

$$N_1(2), N_2(2), \dots, N_m(2), \dots,$$

or S_2 , and a set of limits $x_{2,s} \quad (s = 1, 2, 3, \dots)$,

such that, as $N \rightarrow \infty$ through S_2 ,

$$x_{2,s}(N) \rightarrow x_{2,s} \quad (s = 1, 2, 3, \dots).$$

Proceeding in this way we arrive at a sequence of sequences $S_1, S_2, S_3, \dots$, each member being contained in all preceding members. If then we write

$$N_1 = N_1(1), N_2 = N_2(2), N_3 = N_3(3), \dots,$$

we see that the sequence

$$N_1, N_2, \dots, N_m, \dots, \quad (\text{A. 10, 2})$$

thus defined, is contained, except for a finite number of terms, in each of the sequences $S_1, S_2, S_3, \dots$. It follows that, as $N \rightarrow \infty$ through (A. 10, 2),

$$x_{n,s}(N) \rightarrow x_{n,s} \quad (n, s = 1, 2, 3, \dots),$$

and the theorem is proved.

* Up to this point the argument is the same as that given by Riesz.

COMPARISON THEOREMS IN THE THEORY OF CESÀRO SUMMABILITY

By A. F. ANDERSEN.

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Introduction.

Let Σu_ν and Σv_ν be two such series that there exists a connexion between the terms of one and the terms of the other. I shall call any theorem which specifies the nature of the summability of one series in terms of the nature of the summability of the other a *comparison theorem*; and in this memoir, I shall confine myself to summability by the method of Cesàro. The theorems which I shall establish are of two distinct types, and I shall discuss them separately, though there are certain resemblances between them.

We shall employ the following notation :

$$A_n^{(r)} = \binom{n+r}{n} = \frac{\Gamma(n+r+1)}{\Gamma(n+1)\Gamma(r+1)},$$

$$S_n^{(r)}(u) = \sum_{\nu=0}^n u_\nu A_{n-\nu}^{(r)},$$

$$C_n^{(r)} = \frac{S_n^{(r)}}{A_n^{(r)}},$$

and we shall adopt the usual definition of summability, namely :

The series $\sum_{\nu=0}^{\infty} u_\nu$ is said to be summable (C, r) with the sum U if

$$C_n^{(r)} = \frac{\sum_{\nu=0}^n u_\nu A_{n-\nu}^{(r)}}{A_n^{(r)}} \rightarrow U,$$

as $n \rightarrow \infty$.

For $r=0$, we have ordinary convergence. This definition is applicable to all real numbers r (negative integers excepted), but we shall confine ourselves to values of r such that $r \geq -1$. We shall say that the series Σu_n is summable $(C, -1)$ when it is convergent and satisfies the further condition that $nu_n \rightarrow 0$ as $n \rightarrow \infty$.

1. The first problem.

1. The starting point of our problem is to be found in an important memoir by Hardy and Littlewood[†]; these writers consider two series Σu_n and Σv_n for which the relation connecting corresponding terms is given by the equations

$$(1) \quad u_n = b_n, \quad v_n = (n+1)(b_n - b_{n+1}) \quad (n = 0, 1, 2, \dots);$$

this relation is such that the sequence u_n determines the sequence v_n uniquely, while the converse is untrue; in fact, when the sequence v_n is given the terms of the sequence u_n are determined save for an additive constant. Once b_0 is fixed, the other numbers b_n are completely determined by the relations

$$(2) \quad v_n = (n+1)(b_n - b_{n+1}) \quad (n = 0, 1, 2, \dots),$$

but b_0 may be chosen arbitrarily, and the effect of altering b_0 by the addition of a constant c is to increase *all* the numbers b_n by c .

I shall say, with reference to the two series under consideration,

$$(3) \quad \sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} b_n, \quad \sum_{n=0}^{\infty} v_n = \sum_{n=0}^{\infty} (n+1)(b_n - b_{n+1}),$$

that the second is obtained from the first by an *ordinary Abelian transformation*, since Abel's well known formula

$$\sum_{n=0}^n 1 \cdot b_n = \sum_{n=0}^{n-1} (n+1)(b_n - b_{n+1}) + (n+1)b_n$$

transforms a partial sum of the first series into a partial sum of the second with the addition of a single term. It is this very term which gives rise to the present problem. In the theory of convergent series, it is an elementary result that the convergence of the second series (3) is not in

[†] "Solution of the Cesàro summability problem for power series and Fourier series" *Math. Zeitschrift*, 19 (1924), 67-96. See Theorem A, 69.

general deducible from the convergence of the first. It is consequently natural to consider the series from the more general standpoint afforded by Cesàro's method of summability, and to inquire whether there does not exist some relation between the nature of the summabilities of the series. To this question an affirmative answer has been given by Hardy and Littlewood in their proof of the two following theorems:

THEOREM 1A. *If the series $\Sigma u_\nu = \Sigma b_\nu$ is summable (C, r) where r is an integer (≥ -1), then the transformed series $\Sigma v_\nu = \Sigma(\nu+1)(b_\nu - b_{\nu+1})$ is summable $(C, r+1)$, and the two series have the same sum.*

THEOREM 1B. *If the series Σv_ν is summable $(C, r+1)$, then there exists, among all the possible corresponding series Σu_ν [determined by the equations (2)], one and only one series $\Sigma u_\nu = \Sigma b_\nu^*$ which is summable (C, r) , and this series has the same sum as Σv_ν .*

The actual formal determination of the sequence b_ν^* has also been investigated by Hardy and Littlewood.

Then, by repeated applications by Abel's transformation, they arrive at the following result. Let the relation between corresponding terms $u_\nu = b_\nu$ and v_ν be determined by the set of equations†:

$$\begin{aligned} b_\nu^{(1)} &= (\nu+1) \Delta b_\nu, \\ b_\nu^{(2)} &= (\nu+1) \Delta b_\nu^{(1)}, \\ &\dots \quad \dots \quad \dots \\ b_\nu^{(s-1)} &= (\nu+1) \Delta b_\nu^{(s-2)}, \\ v_\nu &= (\nu+1) \Delta b_\nu^{(s-1)}. \end{aligned}$$

If now the series Σu_ν is summable (C, r) ($r \geq -1$), with sum U , then the series Σv_ν is summable $(C, r+s)$ with the same sum; and, conversely, provided that, starting from the known sequence v_ν , appropriate solutions

$$b_\nu^{*(s-1)}, b_\nu^{*(s-2)}, \dots, b_\nu^{*(1)}, b_\nu^* = u_\nu$$

are selected.

† The explicit expression of the terms v_ν is somewhat complicated; for instance, if we take $s = 2$, so that the Abelian transformation is applied only twice, we have

$$v_\nu = (\nu+1)[(\nu+1) \Delta b_\nu - (\nu+2) \Delta b_{\nu+1}] = (\nu+1)^2 \Delta^2 b_\nu - (\nu+1) \Delta b_{\nu+1}.$$

Hardy and Littlewood have given a number of applications of this theorem, and have shown how several known theorems can be easily deduced from it. But what best exhibits the importance of the theorem is the fact that they have been led by means of it to determine necessary and sufficient conditions to be satisfied by an analytic function $f(x)$ in order that the power series representing the function may be summable by the method of Cesàro at a given point on its circle of convergence.

2. In the theory of Cesàro summability, one frequently needs to apply a transformation which may be regarded as a generalization of the Abelian transformation referred to above. This generalized transformation expresses the sum $\Sigma a_\nu b_\nu$ in terms of a second sum, which (apart from certain additional terms) consists only of products of sums of order s of the factors a_ν and of differences of order s of the factors b_ν . Furthermore, it is not difficult to construct a corresponding transformation when s is not an integer. In my Thesis† I developed this transformation, but since I shall have occasion to make use of it with some frequency in this memoir, I shall explain it precisely later.

After these explanatory remarks, I propose to follow up the researches of Hardy and Littlewood, already referred to, by examining the effect of an Abelian transformation of order s ; however, since I have not yet succeeded in surmounting the difficulties presented by the general case in which s denotes any real number, I shall confine myself to the case *when s is a positive integer*. We are consequently going to compare the summabilities of the two series

$$(4) \quad \sum_{\nu=0}^{\infty} u_\nu \quad \text{and} \quad \sum_{\nu=0}^{\infty} v_\nu \equiv \sum_{\nu=0}^{\infty} A_\nu^{(s)} \Delta^s u_\nu.$$

It is hardly necessary to point out that this transformation is by no means identical with the transformation which is obtained by repetition of the transformation used by Hardy and Littlewood (*cf.* note†, page 41). However, it can also be obtained by repetition of an ordinary Abelian transformation, but this must be applied in a different manner. When the series

$$\sum_{\nu=0}^{\infty} (\nu+1)(u_\nu - u_{\nu+1}) = \sum_{\nu=0}^{\infty} A_\nu^{(1)} \Delta u_\nu$$

has been obtained by means of the first transformation, the second trans-

† *Studier over Cesàro's Summabilitetsmetode* (Copenhagen, 1921), 16.

formation will consist of forming the sums of the factors $A_\nu^{(1)}$ and the differences of the factors Δu_ν , and it consequently will lead to the series

$$\sum_{\nu=0}^{\infty} A_\nu^{(2)} \Delta^2 u_\nu;$$

accordingly, if we apply the subsequent transformations in the corresponding manner, after successively obtaining the series

$$(5) \quad \sum_{\nu=0}^{\infty} A_\nu^{(p)} \Delta^p u_\nu \quad (p = 3, 4, \dots, s-1),$$

we shall, after s transformations, reach the series

$$\sum_{\nu=0}^{\infty} A_\nu^{(s)} \Delta^s u_\nu.$$

It is consequently sufficient for our purpose to investigate the effect of an ordinary Abelian transformation, carried out in the manner indicated for the series (5) for any integral value of p (zero included). For $p = 0$, the problem coincides with the problem of Hardy and Littlewood. But we shall make a generalization by abandoning the restriction that p must be an integer, and by considering not only integral indices of summability but also non-integral indices. However, we shall restrict ourselves, as is usual, to indices $r \geq -1$.

The result of our investigations is given by the two following theorems:

THEOREM 2A. *If the series*

$$\sum_{\nu=0}^{\infty} u_\nu = \sum_{\nu=0}^{\infty} A_\nu^{(\rho-1)} b_\nu \quad (\rho > 0)$$

is summable (C, r) with sum U , r being any real number (≥ -1) , then the transformed series

$$\sum_{\nu=0}^{\infty} v_\nu = \sum_{\nu=0}^{\infty} A_\nu^{(\rho)} \Delta b_\nu$$

is summable $(C, r+1)$, with the same sum U .

THEOREM 2B. *If the series Σv_ν is summable $(C, r+1)$, then there exists, among all the possible corresponding series Σu_ν which have the series Σv_ν as their transformed series, one and only one series Σu_ν^* which is summable (C, r) , and this series has the same sum as the series Σv_ν .*

Starting from these two theorems, we see at once that the solution of the comparison problem for the two series (4) is given by the following theorem :

THEOREM 3. *If the series Σu_ν is summable (C, r) , ($r \geq -1$), then the series $\Sigma v_\nu = \Sigma A_\nu^{(s)} \Delta^s u_\nu$ is summable $(C, r+s)$; and, if the series $\Sigma v_\nu = \Sigma A_\nu^{(s)} \Delta^s u_\nu$ is summable $(C, r+s)$, then there exists a unique series Σu_ν^* which has the series Σv_ν as its transformed series of order s and which is summable (C, r) . Moreover, the series under consideration have the same sum.*

Finally we shall investigate whether Theorems 2A and 2B still hold for $\rho < 0$, with the exception, of course, of negative integral values of ρ . It turns out that the first of these theorems remains valid as it stands, while the second must be modified so as to read that all the series Σu_ν , corresponding to a given series Σv_ν , are summable (C, r) with the same sum as Σv_ν .

2. Generalization of the Abelian transformation.

Let $\epsilon_0, \epsilon_1, \epsilon_2, \dots, \epsilon_\nu, \dots$ be any sequence of real numbers. In accordance with the definition usually adopted in problems of summability, we denote by the differences of orders 1, 2, ..., s of these numbers the following expressions :

$$\Delta^1 \epsilon_\nu = \epsilon_\nu - \epsilon_{\nu+1}, \quad \Delta^2 \epsilon_\nu = \Delta^1 \epsilon_\nu - \Delta^1 \epsilon_{\nu+1} = \epsilon_\nu - 2\epsilon_{\nu+1} + \epsilon_{\nu+2},$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$\Delta^s \epsilon_\nu = \epsilon_\nu - \binom{s}{1} \epsilon_{\nu+1} + \binom{s}{2} \epsilon_{\nu+2} - \dots + (-1)^s \binom{s}{s} \epsilon_{\nu+s}$$

$$(\nu = 0, 1, 2, \dots).$$

In the notation introduced earlier we write

$$\Delta^s \epsilon_\nu = A_0^{(-s-1)} \epsilon_\nu + A_1^{(-s-1)} \epsilon_{\nu+1} + \dots + A_s^{(-s-1)} \epsilon_{\nu+s} = \sum_{p=0}^{\infty} A_p^{(-s-1)} \epsilon_{\nu+p},$$

since all the coefficients $A_p^{(-s-1)}$ are zero when $p > s$. Taking this expression as the basis of the definition of differences of order r , where r denotes any real number, we naturally write

$$(6) \quad \Delta^r \epsilon_\nu = \sum_{p=0}^{\infty} A_p^{(-r-1)} \epsilon_{\nu+p} \quad (\nu = 0, 1, 2, \dots),$$

and this is the expression which we adopt in this memoir as the definition of the difference $\Delta^r \epsilon_n$, whenever this series converges in the ordinary sense of the word; we leave every other case outside our consideration.

We introduce also the notation

$$\Delta_n^r \epsilon_\nu = \sum_{p=0}^{n-\nu} A_p^{(-r-1)} \epsilon_{\nu+p} \quad (n > \nu);$$

this expression constitutes a partial sum of the series (6); it ends with the term containing ϵ_n . We shall call it a *broken difference*. If we make $n \rightarrow \infty$, the broken difference tends to the complete difference; for positive integral values of s we have

$$(7) \quad \Delta_n^s \epsilon_\nu = \Delta^s \epsilon_\nu$$

for $n \geq \nu + s$.

In the sequel it will be useful to recall the fundamental formula associated with the binomial coefficients, namely

$$(8) \quad \sum_{\nu=0}^n A_\nu^{(r)} A_{n-\nu}^{(\rho)} = A_n^{(r+\rho+1)},$$

a formula which, for $\rho = 0$, reduces to

$$(9) \quad \sum_{\nu=0}^n A_\nu^{(r)} = A_n^{(r+1)}.$$

By the *sums of orders* 1, 2, ..., s of the terms of the sequence $(\epsilon_\nu)_r$ starting from the initial term, we mean the expressions

$$S_\nu^{(0)}(\epsilon) = \epsilon_0 + \epsilon_1 + \dots + \epsilon_\nu, \quad S_\nu^{(1)}(\epsilon) = S_0^{(0)} + S_1^{(0)} + \dots + S_\nu^{(0)} = \sum_{p=0}^{\nu} \epsilon_p A_{\nu-p}^{(1)},$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$S_\nu^{(s-1)}(\epsilon) = S_0^{(s-2)} + S_1^{(s-2)} + \dots + S_\nu^{(s-2)} = \sum_{p=0}^{\nu} \epsilon_p A_{\nu-p}^{(s-1)}$$

$$(\nu = 0, 1, 2, \dots)_r$$

and taking this last expression as the basis of the definition of sums of order $r+1$, where r denotes any real number, it is convenient to define†

$$S_\nu^{(r)}(\epsilon) = \sum_{p=0}^{\nu} \epsilon_p A_{\nu-p}^{(r)} \quad (\nu = 0, 1, 2, \dots).$$

† It is convenient to denote sums of order $r+1$ by the symbol $S_\nu^{(r)}$, and not $S_\nu^{(r+1)}$, since these sums occur in the definition of summability of order r of a series Σu_ν .

When no ambiguity can arise, we shall suppress the ϵ in the symbol $S_\nu^{(\epsilon)}$.

With this understanding we shall consider the ordinary Abelian transformation

$$\sum_{\nu=0}^n a_\nu b_\nu = \sum_{\nu=0}^{n-1} (a_0 + a_1 + \dots + a_\nu)(b_\nu - b_{\nu+1}) + (a_0 + a_1 + \dots + a_n) b_n.$$

Making use of the notation which has just been explained, we may write this formula

$$\sum_{\nu=0}^n a_\nu b_\nu = \sum_{\nu=0}^n S_\nu^{(0)} \Delta_n^1 b_\nu.$$

The generalization of this formula, to which reference has already been made in the preceding section, is constituted by the formula

$$(10) \quad \sum_{\nu=0}^n a_\nu b_\nu = \sum_{\nu=0}^n S_\nu^{(r-1)} \Delta_n^r b_\nu \quad (r \text{ any real number}),$$

which we shall call the Abelian transformation of order r .

To establish (10) we shall select from the expression

$$\sum_{\nu=0}^n S_\nu^{(r-1)} \Delta_n^r b_\nu,$$

the coefficient $K_{p,q}$ of the product

$$a_p \cdot b_q \quad (0 \leq p \leq n, \quad 0 \leq q \leq n).$$

Observing that $S_\nu^{(r-1)}$ contains the factor a_p only when $0 \leq p \leq \nu$ and that $\Delta_n^r b_\nu$ contains the other factor b_q only when $\nu \leq q \leq n$, we easily deduce that the necessary and sufficient condition for the product $a_p \cdot b_q$ to occur in the expression

$$\sum_{\nu=0}^n S_\nu^{(r-1)} \Delta_n^r b_\nu$$

is that $p \leq q$; moreover, the desired coefficient is obtained by making ν take the values $p, p+1, \dots, q$. We thus find that

$$\begin{aligned} K_{p,q} &= A_0^{(r-1)} A_{q-p}^{(-r-1)} + A_1^{(r-1)} A_{q-p-1}^{(-r-1)} + \dots + A_{q-p}^{(r-1)} A_0^{(-r-1)} \\ &= A_{q-p}^{(-1)} = \begin{cases} 0 & (q > p), \\ 1 & (q = p), \end{cases} \end{aligned}$$

and this establishes the formula.

If s is a positive integer, we have, by reason of (7),

$$(11) \quad \sum_{\nu=0}^n a_{\nu} b_{\nu} = \sum_{\nu=0}^{n-s} S_{\nu}^{(s-1)} \Delta^s b_{\nu} + \sum_{\nu=n-s+1}^n S_{\nu}^{(s-1)} \Delta_n^s b_{\nu},$$

a formula, which, in view of the form of the supplementary terms

$$\sum_{\nu=n-s+1}^n S_{\nu}^{(s-1)} \Delta_n^s b_{\nu},$$

is in no wise identical with the formula obtained by s successive applications of the ordinary Abelian transformation. I lay some stress on this very simple point, since these formulae, closely related as they are, are by no means of equal use in researches in the theory of summability with any indices; formula (11) frequently proves to be superior to the other.

3. Proof of Theorem 2A.

1. Applying the ordinary Abelian transformation, we have

$$\begin{aligned} S_n^{(0)}(u) &= \sum_{\nu=0}^n A_{\nu}^{(\rho-1)} b_{\nu} = \sum_{\nu=0}^{n-1} A_{\nu}^{(\rho)} \Delta b_{\nu} + A_n^{(\rho)} b_n \\ &= \sum_{\nu=0}^n A_{\nu}^{(\rho)} \Delta b_{\nu} + A_n^{(\rho)} b_{n+1}, \end{aligned}$$

whence

$$(12) \quad S_n^{(0)}(v) = S_n^{(0)}(u) - \frac{n+1}{\rho} u_{n+1} \quad (n = 0, 1, 2, \dots)$$

by observing that

$$A_n^{(\rho)} b_{n+1} = A_{n+1}^{(\rho-1)} b_{n+1} \cdot \frac{A_n^{(\rho)}}{A_{n+1}^{(\rho-1)}} = u_{n+1} \frac{n+1}{\rho}.$$

Formula (12) establishes the theorem at once for $r = -1$; and for $r > -1$, it is merely necessary to establish the corresponding relation between Cesàro's sums of order $r+1$, $C_n^{(r+1)}(v)$ and $C_n^{(r+1)}(u)$. We have, with the help of (12),

$$\sum_{\nu=0}^n S_{\nu}^{(0)}(v) A_{n-\nu}^{(r)} = \sum_{\nu=0}^n S_{\nu}^{(0)}(u) A_{n-\nu}^{(r)} - \frac{1}{\rho} \sum_{\nu=0}^n (\nu+1) u_{\nu+1} A_{n-\nu}^{(r)},$$

that is to say

$$(13) \quad S_n^{(r+1)}(v) = S_n^{(r+1)}(u) - \frac{1}{\rho} \sum_{\nu=0}^{n+1} u_{\nu} \cdot \nu A_{n-\nu+1}^{(r)},$$

and, by applying to the last expression the Abelian transformation of order $r+1$ [see formula (10)], in order to introduce into it the sums $S_\nu^{(r)}(u)$, we obtain the result

$$\sum_{\nu=0}^{n+1} u_\nu (\nu A_{n+1-\nu}^{(r)}) = \sum_{\nu=0}^{n+1} S_\nu^{(r)}(u) \Delta_{n+1}^{r+1} (\nu A_{n+1-\nu}^{(r)}) = \sum_{\nu=0}^{n+1} S_\nu^{(r)} B_{n+1, \nu}$$

where

$$B_{n+1, \nu} = \nu A_0^{(-r-2)} A_{n+1-\nu}^{(r)} + (\nu+1) A_1^{(-r-2)} A_{n-\nu}^{(r)} + \dots + (n+1) A_{n+1-\nu}^{(-r-2)} A_0^{(r)}.$$

Now we find much more simple expressions for these quantities $B_{n+1, \nu}$ by observing that they can be obtained by multiplying two power series. It is easy to verify that, when $|x| < 1$,

$$\begin{aligned} \sum_{n=\nu-1}^{\infty} B_{n+1, \nu} x^n &= \left(\sum_{n=0}^{\infty} A_n^{(r)} x^n \right) \{ \nu x^{\nu-1} A_0^{(-r-2)} + (\nu+1) x^\nu A_1^{(-r-2)} + \dots \} \\ &= \frac{1}{(1-x)^{r+1}} \frac{d}{dx} (x^\nu A_0^{(-r-2)} + x^{\nu+1} A_1^{(-r-2)} + \dots) \\ &= \frac{1}{(1-x)^{r+1}} \frac{d}{dx} \{ x^\nu (1-x)^{r+1} \} \\ &= \nu x^{\nu-1} - (r+1) \frac{x^\nu}{1-x} \\ &= \nu x^{\nu-1} - (r+1) \sum_{n=\nu}^{\infty} x^n. \end{aligned}$$

And this equation makes it obvious that

$$B_{\nu, \nu} = \nu, \quad B_{n+1, \nu} = -(r+1) \quad (n \geq \nu),$$

that is to say

$$B_{n+1, \nu} = -(r+1) \quad (0 \leq \nu \leq n), \quad B_{n+1, n+1} = n+1.$$

Accordingly

$$\sum_{\nu=0}^{n+1} u_\nu \cdot \nu A_{n+1-\nu}^{(r)} = -(r+1) \sum_{\nu=0}^n S_\nu^{(r)}(u) + (n+1) S_{n+1}^{(r)}(u),$$

and, if we substitute this expression in (13), we have

$$S_n^{(r+1)}(v) = S_n^{(r+1)}(u) + \frac{r+1}{\rho} S_{n+1}^{(r+1)}(u) - \frac{n+1}{\rho} S_{n+1}^{(r)}(u).$$

Dividing both sides of this equation by

$$A_n^{(r+1)} = \frac{n+1}{r+1} A_{n+1}^{(r)},$$

we obtain at last the fundamental formula

$$(14) \quad \underline{C_n^{(r+1)}(v) = \left(1 + \frac{r+1}{\rho}\right) C_n^{(r+1)}(u) - \frac{r+1}{\rho} C_{n+1}^{(r)}(u) \quad (r > -1),}$$

and this establishes the theorem under consideration; for this formula makes it obvious that the hypothesis

$$C_n^{(r)}(u) \rightarrow U$$

entails the result

$$C_n^{(r+1)}(v) \rightarrow \left(1 + \frac{r+1}{\rho}\right) U - \frac{r+1}{\rho} U = U,$$

since every series which is summable (C, r) , $(r > -1)$, is also summable $(C, r+1)$ with the same sum.

2. A glance at this proof is sufficient to convince one that it was unnecessary to suppose that $\rho > 0$; the proof is effected in precisely the same manner if ρ is less than zero, and not a negative integer. The theorem is evidently trivial when ρ is a negative integer, since the coefficients $A_v^{(\rho-1)}$ and $A_v^{(\rho)}$ then all vanish after a particular value of v .

It would be relevant, for this reason, to investigate whether the theorem remains valid when the coefficients $A_v^{(\rho)}$ are replaced by v^ρ , but we shall not concern ourselves with the investigation of this problem

4. Proof of Theorem 2B.

1. We start with a series

$$\sum_{v=0}^{\infty} v_v = \sum_{v=0}^{\infty} A_v^{(\rho)} \Delta b_v \quad (\rho > 0)$$

summable $(C, r+1)$, $(r \geq -1)$, with sum V . All the quantities Δb_v are known, but not the numbers b_v themselves. We shall consider all the series Σu_v whose terms are determined by the equations

$$(15) \quad v_v = A_v^{(\rho)} \Delta b_v, \quad u_v = A_v^{(\rho-1)} b_v.$$

Let

$$b'_\nu \quad (\nu = 0, 1, 2, \dots)$$

be any solution of the equations

$$v_\nu = A_\nu^{(\rho)} \Delta b'_\nu;$$

then every other set of solutions is determined by the equations

$$b_\nu = b'_\nu + c,$$

where c is some constant. That is to say, all the numbers u_ν which correspond to the given numbers v_ν are obtained by means of the equations

$$(16) \quad u_\nu = u'_\nu + c A_\nu^{(\rho-1)} \quad (\nu = 0, 1, 2, \dots),$$

where

$$u'_\nu = A_\nu^{(\rho-1)} b'_\nu.$$

Now, since the series $\sum_{\nu=0}^{\infty} A_\nu^{(\rho-1)}$ ($\rho > 0$) cannot be summable (C, r) for any value of r not less than -1 , it follows that *there exists, among all the series under consideration, at most one series Σu_ν which can be summable with an order of summability not less than -1 .*

Since we shall subsequently have to consider sums $S_n^{(r+1)}$ belonging to different series Σu_ν , it is convenient to notice now that these sums are connected by the relation

$$S_n^{(r+1)}(u) = S_n^{(r+1)}(u') + c \sum_{\nu=0}^n A_\nu^{(\rho-1)} A_{n-\nu}^{(r+1)} = S_n^{(r+1)}(u') + c A_n^{(r+\rho+1)},$$

or, preferably, by the relation

$$(17) \quad C_n^{(r+1)}(u) = C_n^{(r+1)}(u') + c \frac{A_n^{(r+\rho+1)}}{A_n^{(r+1)}}.$$

2. We shall now prove that there actually is one series Σu_ν which is summable (C, r) with the same sum as the series Σv_ν . It is convenient to suppose that the sum of the series Σv_ν is zero, and we see without difficulty that this supposition in no way restricts the generality of the proof.

We make use of formula (14), which has just been proved, namely

$$(14) \quad C_n^{(r+1)}(v) = \left(1 + \frac{r+1}{\rho}\right) C_n^{(r+1)}(u) - \frac{r+1}{\rho} C_{n+1}^{(r)}(u) \quad (r > -1);$$

it is valid for *all the series* Σu_v which correspond to the given series Σv_v ; let us select a particular one of these series, say $\Sigma u'_v$.

If we notice that

$$\begin{aligned} C_{n+1}^{(r)}(u') &= \frac{S_{n+1}^{(r)}(u')}{A_{n+1}^{(r)}} = \frac{S_{n+1}^{(r+1)}(u') - S_n^{(r+1)}(u')}{A_{n+1}^{(r)}} \\ &= \frac{S_{n+1}^{(r+1)}(u')}{A_{n+1}^{(r+1)}} \cdot \frac{A_{n+1}^{(r+1)}}{A_{n+1}^{(r)}} - \frac{S_n^{(r+1)}(u')}{A_n^{(r+1)}} \cdot \frac{A_n^{(r+1)}}{A_{n+1}^{(r)}} \\ &= C_{n+1}^{(r+1)}(u') \frac{n+r+2}{r+1} - C_n^{(r+1)}(u') \frac{n+1}{r+1}, \end{aligned}$$

we can write formula (14) in the form

$$(18) \quad \underline{C_n^{(r+1)}(v) = \frac{n+r+\rho+2}{\rho} C_n^{(r+1)}(u') - \frac{n+r+2}{\rho} C_{n+1}^{(r+1)}(u').}$$

This formula reduces to (12) if $r = -1$, and accordingly it still remains valid for this value of r .

Consequently, the Cesàro sums of the series $\Sigma u'_v$ are such that

$$(n+r+\rho+2) C_n^{(r+1)}(u') - (n+r+2) C_{n+1}^{(r+1)}(u') = o(1),$$

since, by hypothesis, $C_n^{(r+1)}(v) \rightarrow 0$ as $n \rightarrow \infty$. If now we multiply by

$$\frac{\Gamma(n+r+2)}{\Gamma(n+r+\rho+3)} = O(n^{-\rho-1}),$$

we find that

$$\frac{\Gamma(n+r+2)}{\Gamma(n+r+\rho+2)} C_n^{(r+1)}(u') - \frac{\Gamma(n+r+3)}{\Gamma(n+r+\rho+3)} C_{n+1}^{(r+1)}(u') = o(n^{-\rho-1}),$$

and in this form the relation is susceptible to the application of an artifice used by Hardy and Littlewood in their proof of the special Theorem 1B. The essential feature of this form is that the second term is obtained by replacing n by $n+1$ in the first term; accordingly, replacing n in succession by $n+1$, $n+2$, ..., m , and adding, we have

$$\begin{aligned} &\frac{\Gamma(n+r+2)}{\Gamma(n+r+\rho+2)} C_n^{(r+1)}(u') - \frac{\Gamma(m+r+3)}{\Gamma(m+r+\rho+3)} C_{m+1}^{(r+1)}(u') \\ &= o\{n^{-\rho-1} + (n+1)^{-\rho-1} + \dots + m^{-\rho-1}\}, \end{aligned}$$

which shows that, since $\rho > 0$,

$$\frac{\Gamma(m+r+3)}{\Gamma(m+r+\rho+3)} C_{m+1}^{(r+1)}(u') \rightarrow K = K(u')$$

as $m \rightarrow \infty$, and consequently that

$$\frac{\Gamma(n+r+2)}{\Gamma(n+r+\rho+2)} C_n^{(r+1)}(u') = K + o(n^{-\rho}),$$

that is to say

$$(19) \quad C_n^{(r+1)}(u') = K \frac{\Gamma(r+\rho+2)}{\Gamma(r+2)} \frac{A_n^{(r+\rho+1)}}{A_n^{(r+1)}} + o(1).$$

If $K \neq 0$, we abandon the series so far considered, in order to select the series Σu_ν^* , whose terms are given by the formula

$$u_\nu^* = u'_\nu - K \frac{\Gamma(r+\rho+2)}{\Gamma(r+2)} A_\nu^{(\rho-1)} \quad [\text{Cf. (16)}];$$

for, by reason of (17), we have, for this series,

$$C_n^{(r+1)}(u^*) = C_n^{(r+1)}(u') - K \frac{\Gamma(r+\rho+2)}{\Gamma(r+2)} \frac{A_n^{(r+\rho+1)}}{A_n^{(r+1)}} = o(1).$$

If $r > -1$, we deduce further, from (14), that

$$C_n^{(r)}(u^*) \rightarrow 0,$$

as $n \rightarrow \infty$, which means that the series Σu_ν^* is summable (C, r) with sum zero; and, if $r = -1$, we have $C_n^{(r+1)}(u^*) = S_n^{(0)}(u^*)$, and, by using (12), we see that

$$(n+1) u_n^* \rightarrow 0$$

as $n \rightarrow \infty$, so that the series Σu_ν^* is not merely convergent, but even summable $(C, -1)$. The theorem is now completely proved.

3. This proof fails when $\rho < 0$. Also, in this case, which we are now going to examine, the theorem no longer retains the same form. In fact, it turns out that, as was enunciated above, *ALL the series Σu_ν , which correspond to the given series Σv_ν are summable (C, r) , with a sum which is equal to that of the series Σv_ν , which, by hypothesis, is summable $(C, r+1)$.* To see this, it is necessary to modify only slightly

the reasoning applied above. The starting point is, as before, the fundamental formula (18), which is valid, when $r \geq -1$, for all the series Σu_n under consideration.

Supposing that the series Σv_n has the sum zero, we have, as just above,

$$a_n C_n^{(r+1)}(u) - a_{n+1} C_{n+1}^{(r+1)}(u) = \epsilon_n (n+1)^{-\rho-1} \quad (\epsilon_n \rightarrow 0),$$

where, for brevity, we have written a_n in place of

$$\frac{\Gamma(n+r+2)}{\Gamma(n+r+\rho+2)}.$$

We replace n by $N, N+1, \dots, n-1$ in succession in this formula, and we deduce by addition that

$$\begin{aligned} a_N C_N^{(r+1)}(u) - a_n C_n^{(r+1)}(u) \\ = \epsilon_N (N+1)^{-\rho-1} + \epsilon_{N+1} (N+2)^{-\rho-1} + \dots + \epsilon_{n-1} n^{-\rho-1}, \end{aligned}$$

$$\text{and hence} \quad a_n |C_n^{(r+1)}(u)| \leq |a_N C_N^{(r+1)}(u)| + \bar{\epsilon}_N n^{-\rho}$$

$$\text{or} \quad |C_n^{(r+1)}(u)| \leq |a_N C_N^{(r+1)}(u)| O(n^\rho) + \bar{\epsilon}_N O(1),$$

where $\bar{\epsilon}_N$ denotes the upper bound of the sequence $|\epsilon_N|, |\epsilon_{N+1}|, |\epsilon_{N+2}|, \dots$; now, corresponding to any $\epsilon > 0$, an integer N may be selected so that

$$\bar{\epsilon}_N < \epsilon,$$

and, observing that $O(n^\rho) = o(1)$ as $n \rightarrow \infty$, since $\rho < 0$, we obtain

$$|C_n^{(r+1)}(u)| \leq \epsilon + \epsilon \cdot O(1)$$

for $n > P > N = N(\epsilon)$, that is to say

$$C_n^{(r+1)}(u) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This step reached, the remainder of the proof follows from arguments already given.

4. The proof of Theorem 2B, which has just been expounded, is a proof of existence. We see, with the hypothesis $\rho \geq 0$, that there really exists one series Σu_n^* , and only one, which is summable (C, r) with sum zero, but we have not given a definite means of constructing the numbers

u_ν^* in terms of the given numbers v_ν . This is the problem which we shall now attack.

It is evidently formula (19) which enables us to distinguish the series Σu_ν^* from all other possible series Σu_ν . The series Σu_ν^* is that one which yields the value zero for K . Now instead of considering the formula (19) itself, it is convenient to replace the order of summability $r+1$ by an *integer* s , exceeding $r+1$; this is permissible, since a series which is summable $(C, r+1)$ is also summable (C, s) with the same sum. We have then

$$(20) \quad C_n^{(s)}(u^*) = \frac{\sum_{\nu=0}^n A_\nu^{(\rho-1)} b_\nu^* A_{n-\nu}^{(s)}}{A_n^{(s)}} = o(1) = o(n^\rho),$$

$$\text{while} \quad C_n^{(s)}(u) = \frac{\sum_{\nu=0}^n A_\nu^{(\rho-1)} b_\nu A_{n-\nu}^{(s)}}{A_n^{(s)}} = Kn^\rho + o(n^\rho) \quad [K = K(u) \neq 0].$$

for every other possible series Σu_ν .

We next observe that the series

$$\sum_{\nu=0}^{\infty} \frac{v_\nu}{A_\nu^{(\rho)}}$$

is summable $(C, r+1)$ when the series Σv_ν is summable in this manner (compare Theorem 4 below), and, for this reason, it is also summable (C, s) . Let us denote its sum by

$$\sum_{\nu=0}^{\infty} \frac{v_\nu}{A_\nu^{(\rho)}} (C, s),$$

adopting a notation due to Hardy and Littlewood. Now, since every solution b_ν ($\nu = 0, 1, 2, \dots$) of the equations

$$v_\nu = A_\nu^{(\rho)} \Delta b_\nu,$$

satisfies the relations

$$b_0 - b_{n+1} = \sum_{\nu=0}^n \frac{v_\nu}{A_\nu^{(\rho)}},$$

they will consequently be all summable (C, s) with the sum

$$b_0 - \sum_{\nu=0}^{\infty} \frac{v_\nu}{A_\nu^{(\rho)}} (C, s).$$

We select the solution for which the sum is zero, so that b'_0 is given by the formula

$$b'_0 = \sum_{\nu=0}^{\infty} \frac{v_{\nu}}{A_{\nu}^{(\rho)}} (C, s),$$

and consequently

$$(21) \quad b'_n = \sum_{\nu=n}^{\infty} \frac{v_{\nu}}{A_{\nu}^{(\rho)}} (C, s) \quad (n = 0, 1, 2, \dots).$$

We have, accordingly, for this solution b'_ν ,

$$(22) \quad \sum_{\nu=0}^n b'_\nu A_{n-\nu}^{(s-1)} = o(n^s).$$

We shall now prove that this relation entails† the following relation

$$(23) \quad \sum_{\nu=0}^n A_{\nu}^{(\rho-1)} b'_\nu A_{n-\nu}^{(s)} = o(n^{s+\rho}).$$

From this it will follow that the numbers b'_n defined by (21) are in reality the numbers b_n^* . The explicit determination of the terms u_n^* is then given by the formula‡

$$u_n^* = A_n^{(\rho-1)} \sum_{\nu=n}^{\infty} \frac{v_{\nu}}{A_{\nu}^{(\rho)}} (C, s).$$

If we use the general Abelian transformation (11) we have

$$\sum_{\nu=0}^n b'_\nu A_{\nu}^{(\rho-1)} A_{n-\nu}^{(s)} = \sum_{\nu=0}^{n-s} S_{\nu}^{(s-1)}(b') \cdot \Delta^s(A_{\nu}^{(\rho-1)} A_{n-\nu}^{(s)}) + o(n^s) \cdot O(n^{\rho-1}).$$

In order to estimate the order of magnitude of the sum thus obtained, we employ a known formula, namely

$$(24) \quad \Delta^s(a_\nu b_\nu) = \sum_{p=0}^s \binom{s}{p} \Delta^p a_\nu \cdot \Delta^{s-p} b_{\nu+p},$$

† For this result, we could have recourse to a general theorem which I have enunciated in a note "Om nogle nyere Sætninger vedrørende Cesàro-og Hölder-Summabilitet", *Mat. Tidsskrift*, B (1925), 93. My note "Om Ækvivalensen af Cesàros og Hölders Summabilitetsmetoder", *Mat. Tidsskrift*, B (1923), 1-6, and a memoir by K. Knopp "Zur Theorie der C- und H-Summierbarkeit", *Math. Zeitschrift*, 19 (1924), 97-113, should also be consulted. Since, however, I have never published the complete proof, I felt that I ought not to omit the following analysis; and it was with a view to this that it was found convenient to make the change from the unrestricted order of summability $r+1$ to the integral order s .

‡ Here we could obviously change from s to $r+1$ again, or even to $r-\rho+1$, but I lay no stress on this point.

and we have

$$\Delta^s(A_\nu^{(\rho-1)}A_{n-\nu}^{(s)}) = \sum_{\nu'=0}^s (-1)^{\nu'} \binom{s}{\nu'} A_{\nu+\nu'}^{(\rho-\nu'-1)} A_{n-\nu-\nu'}^{(\nu')},$$

so that it will be adequate to consider the $s+1$ sums

$$\sum_{\nu=0}^{n-s} S_\nu^{(s-1)}(b') A_{\nu+p}^{(\rho-p-1)} A_{n-\nu-p}^{(p)} \quad (0 \leq p \leq s),$$

whose order of magnitude can be written down at once. Thus

$$\sum_{\nu=0}^n o(\nu^s) \cdot O(\nu^{\rho-p-1}) A_{n-\nu}^{(p)} = o\left(\sum_{\nu=0}^n A_\nu^{(s+\rho-p-1)} A_{n-\nu}^{(p)}\right) = o(n^{s+\rho}),$$

when $0 \leq p \leq s$, and this establishes the result which had to be proved.

5. The second problem.

Let $\sum_{\nu=0}^{\infty} u_\nu$ be a series which is summable (C, r) , $r \geq 0$. We construct from this another series $\sum_{\nu=0}^{\infty} u_\nu \epsilon_\nu$ where the factors which are introduced form a sequence (ϵ_ν) which exhibits a certain amount of regularity. In fact, we assume that the sequence (ϵ_ν) is convergent, and, further, that the series

$$\sum_{\nu=0}^{\infty} A_\nu^{(r)} \Delta^{r+1} \epsilon_\nu$$

is absolutely convergent. We shall see presently that the regularity prescribed for the sequence (ϵ_ν) by these two hypotheses necessitates the result that

$$\Delta^\rho \epsilon_\nu = o(\nu^{-\rho}) \quad (0 < \rho \leq r).$$

We are primarily concerned with the determination of the nature of the summability of the series $\sum u_\nu \epsilon_\nu$. For the special case in which only positive integral values of the order of summability r are considered, the problem was solved nearly twenty years ago by G. H. Hardy and by H. Bohr independently. It is well known that they found that the series in question, $\sum u_\nu \epsilon_\nu$, is also summable (C, r) with a sum which is the same as that of the absolutely convergent series $\sum_{\nu=0}^{\infty} S_\nu^{(r)}(u) \Delta^{r+1} \epsilon_\nu$, increased by a term $U\epsilon$ where U is the sum of the series $\sum u_\nu$ and ϵ is the limit of

the sequence (ϵ_ν) . I have proved in my Thesis that this result remains true in the general case in which r denotes any positive number provided that differences of any order are defined in the manner indicated in § 2. I propose to give here a new proof of this theorem, which, while it adheres to the principles of the previous proof, is distinctly less complicated. By applying this method, based on the general Abelian transformation, also to the special case studied by Hardy and Bohr, I hope that I shall succeed in making clear the difficulties which arise on account of the transition from integral to non-integral values of r .

In the second place, I propose to determine, by the same procedure, the nature of the summability of the set of series

$$\sum_{\nu=0}^{\infty} S_{\nu}^{(\rho-1)}(u) \Delta^{\rho} \epsilon_{\nu} \quad (0 \leq \rho \leq r),$$

which are derived from the series $\sum u_{\nu} \epsilon_{\nu}$ by the application of a general Abelian transformation.

In fact, I shall prove the following theorem :

THEOREM 4. *If the series $\sum_{\nu=0}^{\infty} u_{\nu}$ is summable (C, r) with sum U , and if the sequence (ϵ_{ν}) satisfies the two conditions :*

$$(1) \quad \epsilon_{\nu} \rightarrow \epsilon \text{ as } \nu \rightarrow \infty,$$

$$(2) \quad \sum_{\nu=0}^{\infty} A_{\nu}^{(r)} |\Delta^{r+1} \epsilon_{\nu}| \text{ is convergent,}$$

then the series

$$\sum_{\nu=0}^{\infty} S_{\nu}^{(\rho-1)}(u) \Delta^{\rho} \epsilon_{\nu}$$

is summable $(C, r-\rho)$ for $0 \leq \rho < r$, and convergent for $r \leq \rho \leq r+1$.

Further, the sum of all these series is equal to the sum of the absolutely convergent series $\sum_{\nu=0}^{\infty} S_{\nu}^{(r)}(u) \Delta^{r+1} \epsilon_{\nu}$ ($\rho = r+1$), with the sole exception of the series $\sum_{\nu=0}^{\infty} u_{\nu} \epsilon_{\nu}$ ($\rho = 0$) which has the sum

$$U\epsilon + \sum_{\nu=0}^{\infty} S_{\nu}^{(r)}(u) \Delta^{r+1} \epsilon_{\nu}.$$

In this theorem the only restriction laid on the real number r is that $r \geq 0$.

For the proof I shall need several lemmas. I must state that these auxiliary theorems are, indeed, proved in my Thesis; but since my aim is to give in this memoir an exposition which is not lacking in completeness, and in view of the fact that I have succeeded in considerably simplifying the original proofs, I venture to develop them here.

6. Some auxiliary theorems.

1. The formula $\Delta^r(\Delta^\rho \epsilon_\nu) = \Delta^{r+\rho} \epsilon_\nu$

is obvious when r and ρ are positive integers (zero included). Does it remain true when r and ρ are unrestricted real numbers? This is the problem which must be investigated first. However, we shall not need to investigate the problem in all its bearings; it will be adequate to prove the following lemma:

LEMMA 1. If $\epsilon_\nu = o(1)$,

then $\Delta^r(\Delta^\rho \epsilon_\nu) = \Delta^{r+\rho} \epsilon_\nu$

whenever $r \geq -1$, $\rho \geq 0$, and $r+\rho \geq 0$.

Let us first examine the case in which $r \geq 0$. It is hardly necessary to remark that the differences $\Delta^\rho \epsilon_\nu$ and $\Delta^{r+\rho} \epsilon_\nu$ exist for every ν with the conditions postulated. And since $\Delta^\rho \epsilon_\nu \rightarrow 0$ as $\nu \rightarrow \infty$, so also do the differences $\Delta^r(\Delta^\rho \epsilon_\nu)$ exist, because $r \geq 0$. Consequently we merely have to prove that the two convergent series

$$\sum_{p=0}^{\infty} A_p^{(-r-1)} \sum_{q=0}^{\infty} A_q^{(-\rho-1)} \epsilon_{\nu+p+q}, \quad \sum_{p=0}^{\infty} A_p^{(-r-\rho-1)} \epsilon_{\nu+p},$$

have the same sum. Now this results immediately from the consideration that the double series is not merely convergent, but is actually *absolutely convergent* for the specified values of r and ρ . For, by reason of the absolute convergence of the double series, the sum cannot be altered by any derangement of its terms; and we readily see that we obtain the second series by suitably arranging the terms of the double series. In fact, the coefficients of $\epsilon_{\nu+n}$ ($n = 0, 1, 2, \dots$) in the two series are equal, their values being

$$A_0^{(-r-1)} A_n^{(-\rho-1)} + A_1^{(-r-1)} A_{n-1}^{(-\rho-1)} + \dots + A_n^{(-r-1)} A_0^{(-\rho-1)} = A_n^{(-r-\rho-1)}.$$

Lastly, we consider the case $-1 \leq r < 0$; this needs more delicate reasoning, since we are now unable to assert that the double series is absolutely convergent, or even that it converges at all.

We put $r = -\delta$ ($0 < \delta \leq 1$), and we shall first prove that

$$(25) \quad \Delta^{-\delta}(\Delta^{\delta}\epsilon_{\nu}) = \Delta^0\epsilon_{\nu} = \epsilon_{\nu}.$$

To exhibit more clearly the idea which is guiding us, we take separately the most simple case when δ is *unity*. We may now write

$$\epsilon_{\nu} = \Delta^1\epsilon_{\nu} + \Delta^1\epsilon_{\nu+1} + \dots + \Delta^1\epsilon_{\nu+n-1} + \epsilon_{\nu+n},$$

and from this it follows at once that $\sum_{p=0}^{\infty} \Delta^1\epsilon_{\nu+p}$ converges to the sum ϵ_{ν} , that is to say

$$\Delta^{-1}(\Delta^1\epsilon_{\nu}) = \epsilon_{\nu},$$

in view of the fact that $\epsilon_{\nu+n} \rightarrow 0$ as $n \rightarrow \infty$.

In the general case, by applying the general Abelian transformation (10), we have

$$\begin{aligned} \epsilon_{\nu} &= 1 \cdot \epsilon_{\nu} + 0 \cdot \epsilon_{\nu+1} + 0 \cdot \epsilon_{\nu+2} + \dots + 0 \cdot \epsilon_{\nu+n} \\ &= \sum_{p=0}^n A_p^{(\delta-1)} \Delta_n^{\delta} \epsilon_{\nu+p}, \\ &= \sum_{p=0}^n A_p^{(\delta-1)} \Delta^{\delta} \epsilon_{\nu+p} + \sum_{p=0}^n A_p^{(\delta-1)} a_{n,p}, \end{aligned}$$

where
$$a_{n,p} = -A_{n-p+1}^{(-\delta-1)} \epsilon_{\nu+n+1} - A_{n-p+2}^{(-\delta-1)} \epsilon_{\nu+n+2} - \dots$$

Let us denote by $\bar{\epsilon}_n$ the upper bound of the sequence

$$|\epsilon_{n+1}|, |\epsilon_{n+2}|, \dots$$

If we observe that $A_q^{(-\delta-1)} \leq 0$ for all $q \geq 1$, we have

$$|a_{n,p}| \leq \bar{\epsilon}_n \sum_{q=n-p+1}^{\infty} (-A_q^{(-\delta-1)}) = \bar{\epsilon}_n \left(\sum_{q=0}^{n-p} A_q^{(-\delta-1)} - \sum_{q=0}^{\infty} A_q^{(-\delta-1)} \right) = \bar{\epsilon}_n A_{n-p}^{(-\delta)}$$

since
$$\sum_{q=0}^{\infty} A_q^{(-\delta-1)} = \lim_{x \rightarrow 1} \sum_{q=0}^{\infty} A_q^{(-\delta-1)} x^q = \lim_{x \rightarrow 1} (1-x)^{\delta} = 0.$$

Accordingly
$$\epsilon_{\nu} = \sum_{p=0}^n A_p^{(\delta-1)} \Delta^{\delta} \epsilon_{\nu+p} + o(1),$$

and this is equivalent to the statement that the series

$$\sum_{p=0}^{\infty} A_p^{(\delta-1)} \Delta^{\delta} \epsilon_{\nu+p}$$

is convergent and has the sum ϵ_{ν} , that is to say

$$\epsilon_{\nu} = \Delta^{-\delta} (\Delta^{\delta} \epsilon_{\nu}).$$

It remains for us to prove that

$$\Delta^{-\delta} (\Delta^s \epsilon_{\nu}) = \Delta^{s-\delta} \epsilon_{\nu} \quad (s-\delta > 0).$$

But this is deducible from the two cases considered. For the first result gives

$$\Delta^s \epsilon_{\nu} = \Delta^{\delta} (\Delta^{s-\delta} \epsilon_{\nu}) = \Delta^{\delta} b_{\nu},$$

say, where $b_{\nu} \rightarrow 0$ as $\nu \rightarrow \infty$; and then, with the help of what has just been proved, we have

$$\Delta^{s-\delta} \epsilon_{\nu} = b_{\nu} = \Delta^{-\delta} (\Delta^{\delta} b_{\nu}) = \Delta^{-\delta} (\Delta^s \epsilon_{\nu}).$$

We shall now derive some consequences of the conditions of regularity which have been prescribed for the sequence (ϵ_{ν}) .

LEMMA 2. *If the sequence (ϵ_{ν}) satisfies the following conditions :*

$$(26) \quad \left\{ \begin{array}{l} (1) \quad \epsilon_{\nu} \rightarrow 0 \text{ as } \nu \rightarrow \infty, \\ (2) \quad \sum_{\nu=0}^{\infty} A_{\nu}^{(r)} | \Delta^{r+1} \epsilon_{\nu} | \text{ is convergent,} \end{array} \right.$$

then all the series

$$(27) \quad \sum_{\nu=0}^{\infty} A_{\nu}^{(\rho-1)} \Delta^{\rho} \epsilon_{\nu} \quad (0 \leq \rho \leq r+1)$$

are absolutely convergent, and they all have the sum† ϵ_0 .

If we look back at Lemma 1, we see that

$$\Delta^{r-\delta+1} \epsilon_{\nu} = \Delta^{-\delta} (\Delta^{r+1} \epsilon_{\nu}) \quad (0 < \delta \leq 1).$$

† This result has been given by Bromwich, *Math. Annalen*, 65 (1908), 361, in the case when r and ρ are positive integers. This hypothesis considerably simplifies the proof, particularly because the use of Lemma 1 is not then required.

With this starting-point, a formal calculation gives

$$\begin{aligned}\sum_{\nu=0}^{\infty} A_{\nu}^{(r-\delta)} \Delta^{r-\delta+1} \epsilon_{\nu} &= \sum_{\nu=0}^{\infty} A_{\nu}^{(r-\delta)} \Delta^{-\delta} (\Delta^{r+1} \epsilon_{\nu}) \\ &= \sum_{\nu=0}^{\infty} A_{\nu}^{(r-\delta)} \sum_{p=0}^{\infty} A_p^{(\delta-1)} \Delta^{r+1} \epsilon_{\nu+p} \\ &= \sum_{\nu=0}^{\infty} A_{\nu}^{(r)} \Delta^{r+1} \epsilon_{\nu},\end{aligned}$$

when we use the fact that the coefficient of $\Delta^{r+1} \epsilon_q$ ($q = 0, 1, 2, \dots$) in the double series is

$$\sum_{\nu=0}^q A_{\nu}^{(r-\delta)} A_{q-\nu}^{(\delta-1)} = A_q^{(r)}.$$

To justify this formal calculation, it is sufficient to prove that the double series which occurs is absolutely convergent, that is to say that the series of positive terms

$$\sum_{\nu=0}^{\infty} A_{\nu}^{(r-\delta)} \sum_{p=0}^{\infty} A_p^{(\delta-1)} |\Delta^{r+1} \epsilon_{\nu+p}|$$

is convergent. But this follows immediately from the consideration that the transformation which we have used just above leads to the series

$$\sum_{\nu=0}^{\infty} A_{\nu}^{(r)} |\Delta^{r+1} \epsilon_{\nu}|,$$

and this is convergent by hypothesis.

Now that we have proved that the series (27) is absolutely convergent for $r \leq \rho < r+1$, we are able to extend yet further the domain of absolute convergence of this series by starting from the series

$$(28) \quad \sum_{\nu=0}^{\infty} A_{\nu}^{(r-1)} \Delta^r \epsilon_{\nu},$$

and treating it by the method already employed. We deduce that the series

$$\sum_{\nu=0}^{\infty} A_{\nu}^{(r-\delta-1)} \Delta^{r-\delta} \epsilon_{\nu}$$

is absolutely convergent for

$$0 < \delta \leq \begin{cases} r, \\ 1, \end{cases}$$

and that its sum is equal to the sum of the series (28). If $r > 1$, we must repeat the procedure, but, no matter how large r may be, the proof is attained by continuing in this manner.

Since the series (27) all have the same sum, that sum must be ϵ_0 , which is the sum of the series (27) when $\rho = 0$, since

$$\sum_{\nu=0}^{\infty} A_{\nu}^{(-1)} \epsilon_{\nu} = \epsilon_0.$$

LEMMA 3. *If the sequence (ϵ_{ν}) satisfies the conditions (26), then*

$$\Delta^{\rho} \epsilon_{\nu} = o(\nu^{-\rho}) \quad (0 \leq \rho \leq r).$$

For the ordinary Abelian transformation gives

$$\sum_{\nu=0}^n A_{\nu}^{(\rho-1)} \Delta^{\rho} \epsilon_{\nu} = \sum_{\nu=0}^{n-1} A_{\nu}^{(\rho)} \Delta^{\rho+1} \epsilon_{\nu} + A_n^{(\rho)} \Delta^{\rho} \epsilon_n,$$

whence, if we make use of Lemma 2, we see at once that

$$A_n^{(\rho)} \Delta^{\rho} \epsilon_n \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and this is equivalent to the result stated.

LEMMA 4. *If the conditions (26) are satisfied, then*

$$\phi_{\nu} = \sum_{p=0}^{\infty} A_p^{(-\delta)} |\Delta^{\sigma+1} \epsilon_{\nu+p}| = o(\nu^{-\sigma-\delta})$$

as $\nu \rightarrow \infty$, whenever $0 \leq \delta < 1$ and $0 \leq \sigma \leq r - \delta$.

The (absolutely convergent) series $\sum_{\nu=0}^{\infty} A_{\nu}^{(-\delta)} \Delta^{\sigma+1} \epsilon_{\nu+p}$ is the representation of $\Delta^{\delta-1}(\Delta^{\sigma+1} \epsilon_{\nu}) = \Delta^{\sigma+\delta} \epsilon_{\nu}$ and, for this reason, its sum is $o(\nu^{-\sigma-\delta})$. The point of the lemma is consequently that the regularity of the sequence (ϵ_{ν}) is such that this order of magnitude is unaffected by replacing the differences $\Delta^{\sigma+1} \epsilon_{\nu+p}$ which occur in the series by their moduli.

The proof is quite easy. By using the expansion

$$\Delta^{\sigma+1} \epsilon_{\nu+p} = \Delta^{-\delta}(\Delta^{\sigma+1+\delta} \epsilon_{\nu+p}) = \sum_{q=0}^{\infty} A_q^{(\delta-1)} \Delta^{\sigma+1+\delta} \epsilon_{\nu+p+q},$$

we have

$$\phi_{\nu} \leq \sum_{p=0}^{\infty} A_p^{(-\delta)} \sum_{q=0}^{\infty} A_q^{(\delta-1)} |\Delta^{\sigma+1+\delta} \epsilon_{\nu+p+q}| = \sum_{n=0}^{\infty} |\Delta^{\sigma+1+\delta} \epsilon_{\nu+n}|,$$

and consequently

$$A_{\nu}^{(\sigma+\delta)} \leq \sum_{n=\nu}^{\infty} A_n^{(\sigma+\delta)} |\Delta^{\sigma+1+\delta} \epsilon_n| = o(1),$$

and this is equivalent to the result stated.

7. Proof of the Bohr-Hardy theorem.

If we suppose that $\sum_{\nu=0}^{\infty} u_{\nu}$ is summable (C, r) , where r is a positive integer (zero included), and that its sum is U , and further, if we assume that

$$(1) \quad \epsilon_{\nu} \rightarrow 0,$$

$$(2) \quad \sum_{\nu=0}^{\infty} A_{\nu}^{(r)} |\Delta^{r+1} \epsilon_{\nu}| \text{ is convergent,}$$

we ought to be able to prove that the series $\sum_{\nu=0}^{\infty} u_{\nu} \epsilon_{\nu}$ is also summable (C, r) , and that its sum is equal to the sum of the absolutely convergent series $\sum_{\nu=0}^{\infty} S_{\nu}^{(r)} \Delta^{r+1} \epsilon_{\nu}$, or, in other words, that

$$(29) \quad \frac{S_n^{(r)}(u\epsilon)}{A_n^{(r)}} = \frac{\sum_{\nu=0}^n u_{\nu} \epsilon_{\nu} A_{n-\nu}^{(r)}}{A_n^{(r)}} \rightarrow \sum_{\nu=0}^{\infty} S_{\nu}^{(r)}(u) \Delta^{r+1} \epsilon_{\nu},$$

as $n \rightarrow \infty$.

The lemmas which have been given in the preceding pages were constructed to deal with the difficulties which are presented by the general case. In the special case which we now consider, we shall use only the general Abelian transformation and the special case of Lemma 3 (and consequently of Lemma 2), which was discussed by Bromwich (see f.n.†, page 60).

The general Abelian transformation, given by (10) and (11), may be used to introduce the quantities $S_{\nu}^{(r)}(u)$, thus :

$$\begin{aligned} \sum_{\nu=0}^n u_{\nu} \epsilon_{\nu} A_{n-\nu}^{(r)} &= \sum_{\nu=0}^n S_{\nu}^{(r)}(u) \Delta_n^{r+1} (\epsilon_{\nu} A_{n-\nu}^{(r)}) \\ &= \sum_{\nu=0}^{n-r-1} S_{\nu}^{(r)} \Delta_n^{r+1} (\epsilon_{\nu} A_{n-\nu}^{(r)}) + \sum_{\nu=n-r}^n S_{\nu}^{(r)} \Delta_n^{r+1} (\epsilon_{\nu} A_{n-\nu}^{(r)}). \end{aligned}$$

Of the two sums in the last line, the first alone is of importance. The second effectively consists of a finite number of terms, each of which is of the form $O(S_n^{(r)} \cdot \epsilon_n) = o(n^r)$.

In order to express the essential sum in a convenient form, we expand the difference, Δ^{r+1} , of the product $\epsilon_\nu A_{n-\nu}^{(r)}$ in terms of differences of the two factors ϵ_ν and $A_{n-\nu}^{(r)}$ by the well known formula (24). Remembering that it is ν , and not n , which varies, we find that

$$\begin{aligned}\Delta^{r+1}(\epsilon_\nu A_{n-\nu}^{(r)}) &= \sum_{p=0}^{r+1} \binom{r+1}{p} (\Delta^p \epsilon_\nu) (\Delta^{r+1-p} A_{n-\nu-p}^{(r)}) \\ &= \sum_{p=0}^{r+1} \binom{r+1}{p} (\Delta^p \epsilon_\nu) A_{n-\nu-p}^{(p-1)},\end{aligned}$$

and consequently we have to consider only the $r+2$ sums

$$\binom{r+1}{p} \sum_{\nu=0}^{n-r-1} S_\nu^{(r)} (\Delta^p \epsilon_\nu) A_{n-\nu-p}^{(p-1)},$$

as $n \rightarrow \infty$, or, rather, what comes to the same thing, the sums

$$T_p^{(n)} = \binom{r+1}{p} \sum_{\nu=0}^n S_\nu^{(r)} (\Delta^p \epsilon_\nu) A_{n-\nu}^{(p-1)} \quad (0 \leq p \leq r+1),$$

which prove to be slightly more amenable to our analysis. It will be found that, of these sums, $T_{r+1}^{(n)}$ is the only one which is of importance. In fact,

$$\frac{T_{r+1}^{(n)}}{A_n^{(r)}} = \frac{\binom{r+1}{r+1} \sum_{\nu=0}^n (S_\nu^{(r)} \Delta^{r+1} \epsilon_\nu) A_{n-\nu}^{(r)}}{A_n^{(r)}} \rightarrow \sum_{\nu=0}^{\infty} S_\nu^{(r)} \Delta^{r+1} \epsilon_\nu$$

because a series which is convergent is also summable (C, r) ; and, for $0 \leq p \leq r$, we have

$$\begin{aligned}T_p^{(n)} &= \sum_{\nu=0}^n O(\nu^r) o(\nu^{-p}) A_{n-\nu}^{(p-1)} \\ &= \sum_{\nu=0}^n o(A_\nu^{(r-p)}) A_{n-\nu}^{(p-1)} \\ &= o\left(\sum_{\nu=0}^n A_\nu^{(r-p)} A_{n-\nu}^{(p-1)}\right) \\ &= o(n^r).\end{aligned}$$

The two operations, (i) the general Abelian transformation and (ii) the expansion of the difference $\Delta^{r+1}(\epsilon_\nu A_{n-\nu}^{(r)})$, are consequently sufficient to establish the relation (29). The first is used to introduce the sums $S_\nu^{(r)}$, the second brings into play the differences $\Delta^p \epsilon_\nu$, whose order of magnitude is a direct consequence of the regularity prescribed for the sequence (ϵ_ν) .

If the sequence (ϵ_ν) tends to a limit ϵ which is not zero, while the condition of regularity (2) is still satisfied, we have only to consider the numbers $\epsilon'_\nu = \epsilon_\nu - \epsilon$ instead of the numbers ϵ_ν . The sequence (ϵ'_ν) satisfies the two conditions (1) and (2), while $\Delta^\rho \epsilon'_\nu = \Delta^\rho \epsilon_\nu$ for every value of ρ greater than zero. The series

$$\sum_{\nu=0}^{\infty} u_\nu \epsilon_\nu = \sum_{\nu=0}^{\infty} u_\nu (\epsilon'_\nu + \epsilon)$$

will consequently be summable (C, r) with the sum

$$\sum_{\nu=0}^{\infty} S_\nu^{(r)} \Delta^{r+1} \epsilon_\nu + U\epsilon.$$

8. Proof of the general theorem.

1. By virtue of the last proposition of the preceding section, we may suppose, without loss of generality, that the limit of the sequence (ϵ_ν) under consideration is zero.

We examine first the case of convergence, that is to say we shall prove that the series $\sum_{\nu=0}^{\infty} S_\nu^{(r-\delta)} \Delta^{r+1-\delta} \epsilon_\nu$ is convergent when $0 < \delta \leq 1$, and that its sum is $S = \sum_{\nu=0}^{\infty} S_\nu^{(r)} \Delta^{r+1} \epsilon_\nu$.

If we again apply the general Abelian transformation, we have

$$\sum_{\nu=0}^n S_\nu^{(r-\delta)} \Delta^{r+1-\delta} \epsilon_\nu = \sum_{\nu=0}^n S_\nu^{(r)} \Delta_n^\delta (\Delta^{r+1-\delta} \epsilon_\nu);$$

in this expression we now express the broken difference $\Delta_n^\delta (\Delta^{r+1-\delta} \epsilon_\nu)$, which is equal to $\sum_{p=0}^{n-\nu} A_p^{(-\delta-1)} \Delta^{r+1-\delta} \epsilon_{\nu+p}$, in terms of the complete difference $\Delta^\delta (\Delta^{r+1-\delta} \epsilon_\nu) = \Delta^{r+1} \epsilon_\nu$, and we get

$$\sum_{\nu=0}^n S_\nu^{(r-\delta)} \Delta^{r+1-\delta} \epsilon_\nu = \sum_{\nu=0}^n S_\nu^{(r)} \Delta^{r+1} \epsilon_\nu - a_n,$$

where

$$a_n = \sum_{\nu=0}^n S_{\nu}^{(r)} \sum_{p=n-\nu+1}^{\infty} A_p^{(-\delta-1)} \Delta^{r+\delta-1} \epsilon_{\nu+p}.$$

We are therefore concerned with proving that $a_n \rightarrow 0$ as $n \rightarrow \infty$. Now, observing that $A_p^{(-\delta-1)} \leq 0$ when $p \geq 1$, we find that

$$\begin{aligned} |a_n| &\leq K \cdot A_n^{(r)} \sum_{\nu=0}^n (-A_{n-\nu+1}^{(-\delta-1)} |\Delta^{r+1-\delta} \epsilon_{n+1}| - A_{n-\nu+2}^{(-\delta-1)} |\Delta^{r+1-\delta} \epsilon_{n+2}| - \dots) \\ &= K \cdot A_n^{(r)} [|\Delta^{r+1-\delta} \epsilon_{n+1}| (A_0^{(-\delta)} - A_{n+1}^{(-\delta)}) + |\Delta^{r+1-\delta} \epsilon_{n+2}| (A_1^{(-\delta)} - A_{n+2}^{(-\delta)}) + \dots] \\ &\leq K \cdot A_n^{(r)} \sum_{p=0}^{\infty} A_p^{(-\delta)} |\Delta^{r+1-\delta} \epsilon_{n+1+p}| + K_1 \sum_{p=n+1}^{\infty} A_p^{(r-\delta)} |\Delta^{r+1-\delta} \epsilon_p| \\ &= o(1), \end{aligned}$$

by virtue of Lemmas 4 and 2; and this completes the investigation of the case of convergence.

2. The case of summability which we now proceed to investigate is evidently more troublesome. Assuming that $0 \leq \rho < r$, we have to prove that the series $\sum_{\nu=0}^{\infty} S_{\nu}^{(\rho-1)} \Delta^{\rho} \epsilon_{\nu}$ is summable $(C, r-\rho)$ with sum S , or, in other words, that

$$(30) \quad \frac{\sum_{\nu=0}^n (S_{\nu}^{(\rho-1)} \Delta^{\rho} \epsilon_{\nu}) A_{n-\nu}^{(r-\rho)}}{A_n^{(r-\rho)}} \rightarrow S$$

as $n \rightarrow \infty$.

It is not now convenient to make the transition to the sums $S_{\nu}^{(r)}$ in a single step, unless the difference $r-\rho$ is an integer. If we write

$$r-\rho = p+\delta$$

where ρ is a positive integer (zero included) and $0 \leq \delta < 1$, it is simpler to proceed in two stages, by first introducing the sums $S_{\nu}^{(\rho+p)} = S_{\nu}^{(r+\delta)}$ and subsequently making the final transition to the sums $S_{\nu}^{(r)}$. For this procedure enables us to effect the first transition by an Abelian transformation whose order is an *integer*, which, since it involves complete differences, lends itself much more readily to our investigations than the transformations of non-integral order which lead to broken differences

of a much less amenable character. Accordingly it is only in the last stage that the real difficulties arise. This last stage completely disappears when $r-\rho$ is an integer, in which case the proof is effected on the lines of the proof of the special case examined in the previous section.

We therefore make use of the transformation (11) of order p , whence we deduce that

$$\begin{aligned} \sum_{\nu=0}^n S_{\nu}^{(p-1)}(\Delta^p \epsilon_{\nu}) A_{n-\nu}^{(r-\rho)} \\ &= \sum_{\nu=0}^n S_{\nu}^{(p+p)} \Delta_n^{p+1} (\Delta^p \epsilon_{\nu} A_{n-\nu}^{(r-\rho)}) \\ &= \sum_{\nu=0}^{n-p-1} S_{\nu}^{(r-\delta)} \Delta_n^{p+1} (\Delta^p \epsilon_{\nu} A_{n-\nu}^{(r-\rho)}) + \sum_{\nu=n-p}^n S_{\nu}^{(r-\delta)} \Delta_n^{p+1} (\Delta^p \epsilon_{\nu} A_{n-\nu}^{(r-\rho)}), \end{aligned}$$

in which the last sum is negligible, since it consists of a finite number of terms, each of which is of order

$$O(S_n^{(r-\delta)} \Delta^p \epsilon_n) = O(n^r) \cdot o(n^{-\rho}) = o(n^{r-\rho}).$$

If we again apply formula (24), we get

$$\begin{aligned} \Delta^{p+1} (\Delta^p \epsilon_{\nu} A_{n-\nu}^{(r-\rho)}) &= \sum_{q=0}^{p+1} \binom{p+1}{q} \Delta^q (\Delta^p \epsilon_{\nu}) \cdot \Delta^{p+1-q} (A_{n-\nu-q}^{(r-\rho)}) \\ &= \sum_{q=0}^{p+1} \binom{p+1}{q} (\Delta^{p+q} \epsilon_{\nu}) A_{n-\nu-q}^{(q-1+\delta)}, \end{aligned}$$

and accordingly we have to consider merely the $p+2$ sums

$$\binom{p+1}{q} \sum_{\nu=0}^{n-p-1} S_{\nu}^{(r-\delta)} (\Delta^{p+q} \epsilon_{\nu}) A_{n-\nu-q}^{(q-1+\delta)} \quad (0 \leq q \leq p+1),$$

or, rather, since

$$S_n^{(r-\delta)} \Delta^{p+q} \epsilon_n = O(n^r) \cdot o(n^{-\rho}) = o(n^{r-\rho}),$$

we deal with the more convenient sums

$$\underline{T_q^{(n)} = \binom{p+1}{q} \sum_{\nu=0}^n S_{\nu}^{(r-\delta)} (\Delta^{p+q} \epsilon_{\nu}) A_{n-\nu}^{(q-1+\delta)} \quad (0 \leq q \leq p+1).}$$

Let us take first $T_{p+1}^{(n)}$ which, of all these, is the sum which is really important. On using the equation $r-\rho = p+\delta$, we find that

$$T_{p+1}^{(n)} = \binom{p+1}{p+1} \sum_{\nu=0}^n S_{\nu}^{(r-\delta)} (\Delta^{r-\delta+1} \epsilon_{\nu}) A_{n-\nu}^{(r-\rho)},$$

and, since the series $\sum_{\nu=0}^{\infty} S_{\nu}^{(r-\delta)} \Delta^{r-\delta+1} \epsilon_{\nu}$ is convergent, with sum S (as we have just proved), it is also summable $(C, r-\rho)$ with sum S , which means that, as $n \rightarrow \infty$,

$$\frac{T_{p+1}^{(n)}}{A_{n-\nu}^{(r-\rho)}} \rightarrow S = \sum_{\nu=0}^{\infty} S_{\nu}^{(r)} \Delta^{r+1} \epsilon_{\nu}.$$

3. It remains for us to prove that, as $n \rightarrow \infty$,

$$(31) \quad R_q^{(n)} = \sum_{\nu=0}^n S_{\nu}^{(r-\delta)} (\Delta^{\rho+q} \epsilon_{\nu}) A_{n-\nu}^{(q-1+\delta)} = o(n^{r-\rho}) \quad (0 \leq q \leq p).$$

As we have already remarked, it is an easy matter to prove this when $\delta = 0$, because then the sums $S_{\nu}^{(r)}$ already appear in the expressions under consideration. But, if $0 < \delta < 1$, it is a more tedious piece of work, while it is not sufficient merely to use the condition that

$$S_{\nu}^{(r-\delta)} = O(\nu^r);$$

in fact, this procedure shows that

$$\begin{aligned} \sum_{\nu=0}^n S_{\nu}^{(r-\delta)} (\Delta^{\rho+q} \epsilon_{\nu}) A_{n-\nu}^{(q-1+\delta)} &= \sum_{\nu=0}^n O(\nu^r) \cdot o(\nu^{-\rho-q}) A_{n-\nu}^{(q-1+\delta)} \\ &= o \left(\sum_{\nu=0}^n A_{\nu}^{(r-q-\rho)} A_{n-\nu}^{(q-1+\delta)} \right) \\ &= o(n^{r-\rho+\delta}). \end{aligned}$$

Consequently we have to discover transformations which are capable of introducing the sums $S_{\nu}^{(r)}$; it turns out that it is convenient to divide the set of sums (31) into two groups, of which one contains the sums corresponding to $1 \leq q \leq p$, and the other contains the single remaining sum ($q = 0$).

First group: $1 \leq q \leq p$.

In this case we can proceed almost exactly as in the case of convergence discussed at the beginning of this section.

In order to get rid of the factors $A_{n-\nu}^{(q-1+\delta)}$ we begin with an ordinary Abelian transformation :

$$\begin{aligned} (32) \quad R_q^{(n)} &= \sum_{\nu=0}^n S_{\nu}^{(r-\delta)} (\Delta^{\rho+q} \epsilon_{\nu}) A_{n-\nu}^{(q-1+\delta)} \\ &= \sum_{\nu=0}^n A_{n-\nu}^{(q-2+\delta)} \sum_{\mu=0}^{\nu} S_{\mu}^{(r-\delta)} \Delta^{\rho+q} \epsilon_{\mu}, \end{aligned}$$

which enables us to concentrate our attention on the expressions

$$\sum_{\mu=0}^{\nu} S_{\mu}^{(r-\delta)} \Delta^{\rho+q} \epsilon_{\mu}.$$

We find, in the usual manner, that

$$\begin{aligned} \sum_{\mu=0}^{\nu} S_{\mu}^{(r-\delta)} \Delta^{\rho+q} \epsilon_{\mu} &= \sum_{\mu=0}^{\nu} S_{\mu}^{(r)} \Delta_{\nu}^{\delta} (\Delta^{\rho+q} \epsilon_{\mu}) \\ &= \sum_{\mu=0}^{\nu} S_{\mu}^{(r)} \Delta^{\rho+q+\delta} \epsilon_{\mu} - a_{\nu}, \end{aligned}$$

where

$$a_{\nu} = \sum_{\mu=0}^{\nu} S_{\mu}^{(r)} \sum_{p=\nu-\mu+1}^{\infty} A_p^{(-\delta-1)} \Delta^{\rho+q} \epsilon_{\mu+p}.$$

Also, observing that the coefficients $A_p^{(-1-\delta)}$ under consideration, with $p \geq 1$, are all of them either negative or zero, we have

$$\begin{aligned} |a_{\nu}| &\leq K A_{\nu}^{(r)} \sum_{\mu=0}^{\nu} (-A_{\nu-\mu+1}^{(-\delta-1)} |\Delta^{\rho+q} \epsilon_{\nu+1}| - A_{\nu-\mu+2}^{(-\delta-1)} |\Delta^{\rho+q} \epsilon_{\nu+2}| - \dots) \\ &= K A_{\nu}^{(r)} [|\Delta^{\rho+q} \epsilon_{\nu+1}| (A_0^{(-\delta)} - A_{\nu+1}^{(-\delta)}) + |\Delta^{\rho+q} \epsilon_{\nu+2}| (A_1^{(-\delta)} - A_{\nu+2}^{(-\delta)}) + \dots] \\ &\leq K A_{\nu}^{(r)} \sum_{s=0}^{\infty} A_s^{(-\delta)} |\Delta^{\rho+q} \epsilon_{\nu+1+s}| + K A_{\nu}^{(r)} \sum_{s=\nu+1}^{\infty} A_s^{(-\delta)} |\Delta^{\rho+q} \epsilon_s| \\ &= O(\nu^r) O(\nu^{-\rho-q-\delta+1}) + O(\nu^{r-\rho-q-\delta+1}) \sum_{s=\nu+1}^{\infty} A_s^{(\rho+q-1)} |\Delta^{\rho+q} \epsilon_s| \\ &= O(\nu^{r-\rho-q-\delta+1}), \end{aligned}$$

by reason of Lemmas 4 and 2.

If we introduce into (32) the expression just obtained for

$$\sum_{\mu=0}^{\nu} S_{\mu}^{(r-\delta)} \Delta^{\rho+q} \epsilon_{\mu},$$

we have, with the help of the transformation which is the reverse of

the initial transformation,

$$R_q^{(n)} = \sum_{\nu=0}^n S_\nu^{(r)} (\Delta^{\rho+q+\delta} \epsilon_\nu) A_{n-\nu}^{(q-1+\delta)} + \sum_{\nu=0}^n A_{n-\nu}^{(q-2+\delta)} o(\nu^{r-\rho-q-\delta+1}),$$

and consequently, remembering that $q-2+\delta \geq -1+\delta$ (since $q \geq 1$), so that all the expressions $A_{n-\nu}^{(q-2+\delta)}$ are positive, (or zero), we have

$$\begin{aligned} |R_q^{(n)}| &= \sum_{\nu=0}^n o(\nu^{r-\rho-q-\delta}) A_{n-\nu}^{(q-1+\delta)} + o(n^{r-\rho}) \\ &= o(n^{r-\rho}). \end{aligned}$$

Second group: $q = 0$.

We have

$$\begin{aligned} R_0^{(n)} &= \sum_{\nu=0}^n S_\nu^{(r-\delta)} (\Delta^\rho \epsilon_\nu) A_{n-\nu}^{(-1+\delta)} = \sum_{\nu=0}^n S_\nu^{(r)} \Delta_n^\delta \{ (\Delta^\rho \epsilon_\nu) A_{n-\nu}^{(-1+\delta)} \} \\ &= \sum_{\nu=0}^{n-1} S_\nu^{(r)} \Delta_n^\delta \{ (\Delta^\rho \epsilon_\nu) A_{n-\nu}^{(-1+\delta)} \} + S_n^{(r)} \Delta^\rho \epsilon_n \\ &= \sum_{\nu=0}^{n-1} S_\nu^{(r)} \Delta_n^\delta \{ (\Delta^\rho \epsilon_\nu) A_{n-\nu}^{(-1+\delta)} \} + o(n^{r-\rho}); \end{aligned}$$

$$\begin{aligned} \text{now } \Delta_n^\delta \{ (\Delta^\rho \epsilon_\nu) A_{n-\nu}^{(-1+\delta)} \} &= \sum_{p=0}^{n-\nu} A_p^{(-1-\delta)} A_{n-\nu-p}^{(-1+\delta)} \Delta^\rho \epsilon_{\nu+p} \\ &= \sum_{p=0}^{n-\nu-1} \sigma_p \Delta^{\rho+1} \epsilon_{\nu+p}, \end{aligned}$$

$$\text{where } \sigma_p = A_0^{(-1-\delta)} A_{n-\nu}^{(-1+\delta)} + A_1^{(-1-\delta)} A_{n-\nu-1}^{(-1+\delta)} + \dots + A_p^{(-1-\delta)} A_{n-\nu-p}^{(-1+\delta)}$$

and $\sigma_{n-\nu} = A_{n-\nu}^{(-1)} = 0$, since $n-\nu \geq 1$.

At this stage it is essential to notice that all the coefficients σ_p ($p \leq n-\nu-1$) satisfy the inequalities

$$0 < \sigma_p \leq A_{n-\nu}^{(-1+\delta)} A_p^{(-\delta)}.$$

These inequalities follow from the facts that

$$\sigma_0 > \sigma_1 > \sigma_2 > \dots > \sigma_{n-\nu} = 0,$$

$$A_p^{(-1-\delta)} \leq 0 \quad (p \geq 1), \quad A_{n-\nu-s}^{(-1+\delta)} > A_{n-\nu}^{(-1+\delta)},$$

for every positive integral value of s (less, of course, than $n-\nu$).

We consequently have, on again using Lemma 4,

$$\begin{aligned}
 \left| \sum_{\nu=0}^{n-1} S_{\nu}^{(r)} \Delta_n^{\delta} \{ (\Delta^{\rho} \epsilon_{\nu}) A_{n-\nu}^{(-1+\delta)} \} \right| &\leq K \sum_{\nu=0}^{n-1} A_{\nu}^{(r)} \sum_{p=0}^{n-\nu-1} A_{n-\nu}^{(-1+\delta)} A_p^{(-\delta)} | \Delta^{\rho+1} \epsilon_{\nu+p} | \\
 &\leq K \sum_{\nu=0}^{n-1} A_{\nu}^{(r)} A_{n-\nu-1}^{(-1+\delta)} \sum_{p=0}^{\infty} A_p^{(-\delta)} | \Delta^{\rho+1} \epsilon_{\nu+p} | \\
 &= K \sum_{\nu=0}^{n-1} A_{\nu}^{(r)} A_{n-\nu-1}^{(-1+\delta)} o(\nu^{-\rho-\delta}) \\
 &= o \left(\sum_{\nu=0}^{n-1} A_{\nu}^{(r-\rho-\delta)} A_{n-\nu-1}^{(-1+\delta)} \right) \\
 &= o(n^{r-\rho}).
 \end{aligned}$$

The relation (31), and consequently the theorem under consideration, are therefore now completely established.

COMPLETE SYSTEMS OF DIFFERENTIAL INVARIANTS

By OLIVER E. GLENN.

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IN this paper I consider a new theory of complete systems of absolute differential parameters of an arbitrary binary differential form*.

The notation used is the symbolism of Maschke. For brevity, also, partial derivatives are represented by subscripts which are not enclosed in parentheses, according to the rule

$$F_i = \frac{\partial F}{\partial x_{(i)}}.$$

The expression (f, g) will represent the functional determinant of two functions of $x_{(1)}, x_{(2)}$, and $(f; g)^k$ indicates algebraic transvectants of two quantics f, g in the differentials $dx_{(1)}, dx_{(2)}$.

By an adaptation of Hermite's method† of typical representation to the case of a differential form‡

$$(1) \quad f = (da)^m = (a_1 dx_{(1)} + a_2 dx_{(2)})^m = a_{dx}^m,$$

it results that f may be represented typically by means of the linear transformations upon $dx_{(1)}, dx_{(2)}$ which are furnished by

$$(2) \quad dP = P_1 dx_{(1)} + P_2 dx_{(2)}, \quad dQ = Q_1 dx_{(1)} + Q_2 dx_{(2)},$$

P, Q being any functionally independent, non-covariant parameters. For absolute differential invariants it will then follow that the corresponding associated forms and their derivatives of the orders from 0 to τ form a complete system for the order τ .

* See my paper in the *Proc. Internat. Math. Congress, Toronto, 1924* (not yet published).

† Hermite, *Journal für Math.*, 52 (1856), 1.

‡ Maschke, *Trans. Amer. Math. Soc.*, 1 (1900), 197, and *ibid.*, 4 (1903), 445.

I. *Associated systems.* The initial transformation of f is by means of arbitrary functions

$$(3) \quad x_{(i)} = x_{(i)}(y_{(1)}, y_{(2)}), \quad dx_{(i)} = \frac{\partial x_{(i)}}{\partial y_{(1)}} dy_{(1)} + \frac{\partial x_{(i)}}{\partial y_{(2)}} dy_{(2)} \quad (i = 1, 2).$$

A linear differential covariant which is absolute may be multiplied by an integrating factor and equated to an exact differential. Let two covariants be

$$(4) \quad dP = Pdx, \quad dQ = Qdx.$$

The inverse of (4) substituted in (1) gives

$$(5) \quad f = [(a, Q) dP + (P, a) dQ]^m \div (P, Q)^m \\ = \sum_{i=0}^m \binom{m}{i} B_{(i)} (dP)^{m-i} (dQ)^i,$$

where

$$(6) \quad B_{(i)} = A_{(i)} \div (P, Q)^m, \quad A_{(i)} = (a, Q)^{m-i} (P, a)^i,$$

and (P, Q) is the symbol of the Jacobian. Then (5) is a typical representation of f ; $A_{(i)}$, ($i = 0, \dots, m$), (P, Q) being associated forms.

We have relations of invariancy as follows:

$$(7) \quad B'_{(i)} = B_{(i)}, \quad A'_{(i)} = D^m A_{(i)}, \quad (P', Q') = D(P, Q), \\ \frac{\partial B'_{(i)}}{\partial P'} = \frac{\partial B_{(i)}}{\partial P}, \quad \frac{\partial B'_{(i)}}{\partial Q'} = \frac{\partial B_{(i)}}{\partial Q}$$

where D is the determinant of (3), *i.e.*

$$D = \begin{vmatrix} \frac{\partial x_{(1)}}{\partial y_{(1)}} & \frac{\partial x_{(1)}}{\partial y_{(2)}} \\ \frac{\partial x_{(2)}}{\partial y_{(1)}} & \frac{\partial x_{(2)}}{\partial y_{(2)}} \end{vmatrix}.$$

From (4) it is clear that P and Q are integral invariants

$$(8) \quad P = \int (P_1 dx_{(1)} + P_2 dx_{(2)}), \quad Q = \int (Q_1 dx_{(1)} + Q_2 dx_{(2)}).$$

II. *Absolute invariants.* A relative differential invariant is made absolute by introducing as a factor a negative fractional power of some

invariant. For example, if $m = 2$, $f = \Sigma a_{(ik)} x_{(i)} x_{(k)}$, ($a_{(ik)} = a_{(ki)}$), then a relation of absolute invariancy is

$$|a'_{(ik)}|^{-\frac{1}{2}}(P', Q') = |a_{(ik)}|^{-\frac{1}{2}}(P, Q),$$

where P' , Q' are evidently functions of $y_{(1)}$, $y_{(2)}$. In some parameters the complementing factor is involved in a more complicated way, as in the case of Beltrami's parameter* ($m = 2$, $a = |a_{(ik)}|^{-\frac{1}{2}}$),

$$\Delta_{(2)}u = a[a, a(a, u)].$$

All of the current standard processes of generating differential invariants (absolute) require the use of a complementing factor, but there is usually a considerable latitude of choice of this invariant. In general, therefore, an absolute parameter is not integral, although it requires only a multiplier in the form of a power of the complementing factor to make it integral.

The invariantive relation for a parameter I , under (3), will be of the form

$$(9) \quad I(a_{(0)}, \dots, a_{(m)}, \frac{\partial^r a_{(i)}}{\partial x_{(1)}^{r-s} \partial x_{(2)}^s}, \frac{\partial^t u}{\partial x_{(1)}^{t-w} \partial x_{(2)}^w}, dx_{(1)}, dx_{(2)}) \\ = I(a'_{(0)}, \dots, a'_{(m)}, \frac{\partial^r a'_{(i)}}{\partial y_{(1)}^{r-s} \partial y_{(2)}^s}, \frac{\partial^t u'}{\partial y_{(1)}^{t-w} \partial y_{(2)}^w}, dy_{(1)}, dy_{(2)}),$$

assuming that

$$f = (a_{(0)}, \dots, a_{(m)}) \mathfrak{J} dx_{(1)}, dx_{(2)}^m.$$

The typical form (5) has the same force as f' [under (3)], since it is the transform of f by an arbitrary transformation†.

The parameter (P, Q) may be employed as the complementing factor of I .

We have

$$(10) \quad I(a_{(0)}, \dots) = I(B_{(0)}, \dots, B_{(m)}, \frac{\partial^r B_{(i)}}{\partial P^{r-s} \partial Q^s}, \frac{\partial^t u'}{\partial P^{t-w} \partial Q^w}, dP, dQ).$$

The right-hand side is a parameter, also complemented by a power of (P, Q) , but otherwise a polynomial in the set (11) below. This set forms

* Cf. J. E. Wright, *Invariants of quadratic differential forms* (1908), 6.

† Clebsch u. Lindemann, *Vorlesungen über Geometrie*, zweite Aufl. (1906-10), § 630.

a complete system and not merely an associated system, that is, we have the following

THEOREM. *A fundamental system of absolute differential parameters of f , of orders not exceeding τ , is composed of the following functions :*

$$(11) \quad B_{(i)}, \quad \frac{\partial^r B_{(i)}}{\partial P^{r-s} \partial Q^s}, \quad \frac{\partial^t u_{(i)}}{\partial P^{t-w} \partial Q^w}, \quad dP, \quad dQ$$

($i = 0, \dots, m$; $r = 1, \dots, \tau$; $j = 1, \dots, v$; $t = 1, \dots, \tau$; $s = 0, \dots, r$; $w = 0, \dots, t$).

The number of parameters in this system is

$$N = \frac{1}{2}(\tau+1)(\tau+2)(m+v+1)+2,$$

v being the number of arbitrary functions $u_{(1)}, u_{(2)}, \dots$. This is a theorem of extensive content with reference to particular orders m and different choices of P, Q .

III. *The explicit forms of parameters.* Among the various forms which a parameter may take, the ultimate form is its expression as a function of the coefficients of f and their derivatives. From (1), (9),

$$a_{(r)} = a_1^{m-r} a_2^r = b_1^{m-r} b_2^r = \dots,$$

but only certain combinations of derivatives of symbols a_i are capable of interpretation in terms of derivatives of actual coefficients $a_{(r)}$. Thus, if $m = 2$ and

$$f = \Sigma a_{(ik)} dx_{(i)} dx_{(k)}, \quad (a_{(ik)} = a_{(ki)}),$$

the following is a known interpretation (cf. Maschke : *loc. cit.*, 451),

$$a_i a_{kl} = \frac{1}{2} \left[\frac{\partial a_{(ik)}}{\partial x_{(l)}} + \frac{\partial a_{(il)}}{\partial x_{(k)}} - \frac{\partial a_{(kl)}}{\partial x_{(i)}} \right] = [kl]_i.$$

This is Christoffel's triple index symbol.

If the parameter (9) is expressed symbolically, therefore, only such combinations of derivatives of symbols as satisfy certain criteria will occur.

Assuming (9) in the symbolical form, if we operate on both sides by any operator Δ which satisfies a relation of invariancy, ($\Delta' = p\Delta$), for example the operator of total differentiation

$$\frac{\partial}{\partial y_{(1)}} dy_{(1)} + \frac{\partial}{\partial y_{(2)}} dy_{(2)} = \frac{\partial}{\partial x_{(2)}} dx_{(1)} + \frac{\partial}{\partial x_{(2)}} dx_{(2)},$$

the resulting parameter will necessarily contain the symbols and their derivatives in interpretable combinations only.

The invariantive derivatives with regard to P and Q in the formulæ (11) may also be evaluated in terms of the actual coefficients of f and the derivatives thereof. The explicit expressions may be found as follows. We have

$$B_{(i)} = A_{(i)} / (P, Q)^m,$$

of which the following for the quadratic is an example [Cf. IV (12)] :

$$B_0 = (aN^2 - 2bNM + cM^2) / (P_1N - P_2M)^2.$$

Also

$$(11_1) \quad \frac{\partial B_{(i)}}{\partial P} = \frac{\partial B_{(i)}}{\partial x_{(1)}} \frac{\partial x_{(1)}}{\partial P} + \frac{\partial B_{(i)}}{\partial x_{(2)}} \frac{\partial x_{(2)}}{\partial P}.$$

A method of expressing $B_{(i)}$ in terms of $A_{(r)}$ ($r = 0, \dots, m$) is given by (2). It is an invariant of definite order w . The derivatives $\partial x_{(i)} / \partial P$ are invariantive functions of P and Q . Hence (11₁) gives $B \partial_{(i)} / \partial P$ as an explicit invariantive function, of order $w+1$, of $a_{(r)}$, ($r = 0, \dots, m$), and their derivatives.

IV. *The quadratic form.* The quantic*

$$f = (d\xi)^2 = \xi_a^2 = (a, b, c \xi dx_{(1)}, dx_{(2)})^2,$$

has the parameter $P = a^2 (\xi, u)^2$, and the covariant

$$(12) \quad M dx_{(1)} + N dx_{(2)},$$

$$M = a \begin{bmatrix} 21 \\ 1 \end{bmatrix} - a \begin{bmatrix} 11 \\ 2 \end{bmatrix} + aa_2 - ba_1, \quad N = a \begin{bmatrix} 22 \\ 1 \end{bmatrix} - a \begin{bmatrix} 12 \\ 2 \end{bmatrix} + ba_2 - ca_1,$$

$\begin{bmatrix} ij \\ k \end{bmatrix}$ being a Christoffel symbol. If λ is an integrating factor and $Q_1 = \lambda M$, $Q_2 = \lambda N$, we have

$$(13) \quad dP = P_1 dx_{(1)} + P_2 dx_{(2)}, \quad dQ = Q_1 dx_{(1)} + Q_2 dx_{(2)},$$

where

$$P_1 = 2a(\xi, u)[a_1(\xi, u) + a(\xi_1, u) + a(\xi, u_1)],$$

$$P_2 = 2a(\xi, u)[a_2(\xi, u) + a(\xi_2, u) + a(\xi, u_2)].$$

* Forsyth *Phil. Trans.*, A. 201 (1903), 329-402, and *ibid.*, 202 (1903), 277-333.

The corresponding fundamental system of order τ is then given by (11), with

$$(14) \quad \begin{aligned} P &= \alpha^2(au_2^2 - 2bu_2u_1 + cu_1^2), \quad Q = \int \lambda(Mdx_{(1)} + Ndx_{(2)}), \\ (P, Q) &= \lambda(P_1N - P_2M), \\ A_{(0)} &= \lambda^2(aN^2 - 2bMN + cM^2), \\ A_{(1)} &= \lambda[-aP_2N + b(P_1N + P_2M) - cP_1M], \\ A_{(2)} &= aP_2^2 - 2bP_2P_1 + cP_1^2. \end{aligned}$$

Thus only one extraneous arbitrary function is required in order to construct the system of a quadratic.

V. *The cubic form.* The operator,

$$\Delta = \beta u_2 \frac{\partial}{\partial(dx_{(1)})} - \beta u_1 \frac{\partial}{\partial(dx_{(2)})},$$

β being $R^{-1/k}$ and R any parameter of index k , applied to any absolute covariant, gives an absolute parameter, on account of the cogrediency* of βu_2 , $-\beta u_1$ with $dx_{(1)}$, $dx_{(2)}$. If

$$f = \xi_{dx}^3 = \eta_{dx}^3 = (a, b, c, d\xi dx_{(1)}, dx_{(2)})^3,$$

then the algebraic Hessian of f is

$$H = (\xi, \eta)^2 \xi_{dx} \eta_{dx},$$

while

$$\frac{1}{6}\Delta^2 f = \beta^2(\xi, u)^3 \xi_{dx}, \quad \frac{1}{2}\beta^2\Delta H = \beta^3(\xi, \eta)^2(\xi, u) \eta_{dx}.$$

The latter two parameters are absolute, and H is of index 2.

If λ, μ are integrating factors and

$$J = \left| \begin{array}{cc} (\xi, u)^2 \xi_1(\eta, \theta)^2(\eta, u)\theta_1 \\ (\xi, u)^2 \xi_2(\eta, \theta)^2(\eta, u)\theta_2 \end{array} \right| = (\xi, \theta)(\eta, \theta)^2(\xi, u)^2(\xi, u),$$

then

$$dP = J^{-\frac{1}{2}}\lambda(\xi, u)^3 \xi_{dx}, \quad dQ = J^{-\frac{1}{2}}\mu(\xi, \eta)^2(\xi, u) \eta_{dx},$$

and also

$$P = \int J^{-\frac{1}{2}}\lambda(\xi, u)^3 \xi_{dx}, \quad Q = \int J^{-\frac{1}{2}}\mu(\xi, \eta)^2(\xi, u) \eta_{dx},$$

are absolute invariants.

* *Encyk. der math. Wiss.*, 1 (I B2), 385.

For brevity we may write

$$dP = A\xi_{dx}, \quad dQ = B\eta_{du},$$

whence

$$(P, Q) = AB(\xi, \eta),$$

$$f = (B_{(0)}, B_{(1)}, B_{(2)}, B_{(3)} \mathfrak{X} dx_{(1)}, dx_{(2)})^3,$$

and we find that

$$A_0 = a(B\eta_2)^3 - 3b(B\eta_2)^2(B\eta_1) + 3c(B\eta_2)(B\eta_1)^2 - d(B\eta_1)^3.$$

Let

$$\Omega = A\xi_1 \frac{\partial}{\partial(-B\eta_1)} - A\xi_2 \frac{\partial}{\partial(B\eta_2)},$$

then

$$(15) \quad A_{(1)} = \frac{1}{3}\Omega A_{(0)}, \quad A_{(2)} = \frac{1}{6}\Omega^2 A_{(0)}, \quad A_{(3)} = \frac{1}{6}\Omega^3 A_{(0)}.$$

The symbolical form of $A_{(0)}$, for instance, is as follows :

$$(16) \quad A_{(0)} = J^{-3} \mu^3(\xi, \eta)^2(\xi, u)(\theta, l)^2(\theta, u)(K, \lambda')^2(K, u) \\ \times [a\eta_2 l_2 \lambda'_2 - 3b\eta_2 l_2 \lambda'_1 + 3c\eta_2 l_1 \lambda'_1 - d\eta_1 l_1 \lambda'_1],$$

and, on the basis of (15), we may readily calculate

$$(17) \quad A_{(1)} = J^{-3} \lambda \mu^2(\xi, u)^2(\eta, \theta)^2(\eta, u)(l, K)^2(l, u) \\ \times [-a\xi_2 \theta_2 K_2 + 2b\xi_2 \theta_2 K_1 + b\xi_1 \theta_2 K_2 - c\xi_2 \theta_1 K_1 \\ - 2c\xi_1 \theta_2 K_1 + d\xi_1 \theta_1 K_1], \\ A_{(2)} = J^{-7} \lambda^2 \mu(\xi, u)^2(\eta, u)^2(\theta, l)^2(\theta, u) \\ \times [a\xi_2 \eta_2 l_2 - 2b\xi_2 \eta_1 l_2 - b\xi_2 \eta_2 l_1 + c\xi_1 \eta_1 l_2 + 2c\xi_2 \eta_1 l_1 - d\xi_1 \eta_1 l_1], \\ A_{(3)} = J^{-1} \lambda^3(\xi, u)^2(\eta, u)^2(\theta, u)^2 \\ \times [-a\xi_2 \eta_2 \theta_2 + 3b\xi_2 \eta_2 \theta_1 + 3c\xi_2 \eta_1 \theta_1 + d\xi_1 \eta_1 \theta_1].$$

The complete system of absolute parameters of orders not exceeding τ is furnished by $A_{(i)}$, ($i=0, \dots, 3$), [and $B_{(i)}$, ($i=0, \dots, 3$)], together with the formulae (11).

It is worthy of note that this system also contains but one arbitrary function u .

VI. *Systems for the forms of orders $m > 3$.* The fundamental system of algebraic concomitants of the differential form f of order

$$m = 2n + 1 > 3,$$

contains two irreducible linear covariants which are expressible as transvectants. These were identified in the system of the quintic by Gordan and in the system of the quantic of order 7 by Von Gall*.

Assume that

$$(18) \quad f = a_{dx}^m = b_{dx}^m, \quad i = (f; f)^{2n} = (a, b)^{2n} a_{dx} b_{dx},$$

then the two linear covariants are

$$\alpha_{dx} = (i^n; f)^{2n}, \quad \beta_{dx} = (i^{n+1}; f)^{2n+1}.$$

The resultant of α_{dx} , β_{dx} is an invariant R of degree $4n+4$ and index $2(n+1)(2n+1) = k$. The following covariants are absolute :

$$(19) \quad dP = \lambda R^{-s/k} \alpha_{dx}, \quad dQ = \mu R^{-t/k} \beta_{dx},$$

$$\text{where} \quad s = 2n(n+1), \quad t = 2n^2 + 4n + 1,$$

and λ , μ are integrating factors. Therefore :

THEOREM. *A complete system for the order τ may be formed for a quantic of any odd order greater than 3, containing no extraneous arbitrary functions. It is given by (11) with†*

$$P = \int \lambda R^{-s/k} d_{dx}, \quad Q = \int \mu R^{-t/k} \beta_{dx}.$$

Considerably different is the case when $m = 2n > 2$, since there exist no linear covariants of the pure algebraic type. One extraneous arbitrary function is sufficient if we choose parameters as follows (Cf. § V) :

$$(20) \quad \alpha_{dx} = \frac{1}{(2n)!} \Delta^{2n-1} f, \quad \beta_{dx} = \frac{1}{2^4} \Delta^3 (f; f)^{2(n-1)}.$$

The complementing factor R is the algebraic invariant of degree 2, $R = (f; f)^{2n}$. The index is $k = 2n$. The following parameters are absolute :

$$dP = \lambda \alpha_{dx}, \quad dQ = \mu R^{-s/k} \beta_{dx}.$$

* *Math. Annalen*, 31 (1888), 318-336.

† Cartan, *Leçons sur les invariants intégraux* (1922), Introduction.

λ and μ being the appropriate integrating factors, and

$$s = 2n - 2.$$

THEOREM. *A fundamental system for the form f of even order $2n > 2$ is determined by (11) with $P = \int \lambda d_{dx}$, $Q = \int \mu R^{-s/k} \beta_{dx}$.*

VII. *Remarks concerning relative invariants.*

The simplicity and the generality of the foregoing theory are largely results of the assumption that differential invariants are absolute. If this restriction is not made we may deal with quantics which are both rational and integral, and the method of associated forms then no longer suffices, alone, to give complete systems. The present writer originated a theory of relative parameters by employing the method of invariant elements*. The method is, in general, that of the theory of orthogonal invariants. The transition to the parameters (11) of the more regular type is made by the use of Hilbert's lemma† on infinite systems of polynomials.

* *Amer. Journal of Math.*, 46 (1924), 473.

† *Math. Annalen*, 36 (1890), 473.

SOME APPLICATIONS OF A TRANSFORMATION OF SERIES

By J. R. WILTON.

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Contents.

1. Theorems A and B, involving a transformation of series, with lemmas and corollaries.
2. Applications of Theorem A to the analytic continuation of functions defined by certain series.
3. The summation of certain series in finite terms, with special reference to a class of null-series.
4. Applications of the theorems of § 1 to an extension of the Euler-Maclaurin sum-formula, to the Gibbs Phenomenon, and to certain other asymptotic developments.

1. The present communication is a considerably revised form of a paper received by the Society on 6 November, 1924, under the title "Some applications of a transformation of series due to Schlömilch"* Almost the whole of § 1 has been added in order to meet suggestions made by Prof. Hardy and the Referee. I have also added two new paragraphs to § 4, containing applications of the formulae to the Gibbs phenomenon†, and to the determination of the behaviour of the series $\sum_0^{\infty} r^{\lambda(n)} \frac{\cos nx}{\sin nx}$ as $r \rightarrow 1$, $\lambda(n)$ being a function which tends steadily to infinity with n . On the other hand, §§ 2 and 3 have been considerably curtailed, and, to a less extent, omissions have been made from the earlier part of § 4.

* Since it has not been possible for me to refer to the work of Schlömilch, it seems better to omit his name from the title. I am indebted to Prof. Hardy for the suggestion that I should make the analytic continuation of certain functions the chief concern of the paper.

† A brief historical account of the Gibbs Phenomenon in Fourier series is given by H. S. Carslaw, *Bull. Amer. Math. Soc.*, 31 (1925), 420-424. See also C. N. Moore, *ibid.*, 417-419.

The paper discusses the analytic continuation of a Laurent series of general form, and of a restricted class of Dirichlet series, by a method which, at least in particular applications, is not new*, although no proof of the formula on which it depends appears to have been published, and no systematic examination of its consequences has been undertaken.

In order to avoid constant repetition it is to be understood throughout that N is always a positive integer or zero, and n an integer, positive, negative, or zero; that the accent attached to the sign of summation in $\sum_a^b f(n)$ means that, if a is an integer, the first term is $\frac{1}{2}f(a)$, and if b is an integer, the last term of the sum is $\frac{1}{2}f(b)$; that the symbol $\sum_{-\infty}^{\infty}$ is an abbreviation for $\lim_{N \rightarrow \infty} \sum_{-N}^N$; and that $\sum_{-\infty}^{(0)} f(n)$ indicates that the term corresponding to $n = 0$ is to be omitted from the summation. As usual $\epsilon > 0$, and is otherwise arbitrary.

The symbol $V_a^b f(t)$ will be used for the total variation of $f(t)$ over the interval $a \leq t \leq b$. By the statement that $f(t)$ is of bounded variation over the infinite interval $t \geq a$, I mean that $\lim_{b \rightarrow \infty} V_a^b f(t)$ is finite, and I denote the limit by $V_a^\infty f(t)$. If $V_a^b f(t) \rightarrow \infty$ with b , the variation is unbounded.

1.1. The main purpose of this section is the proof of the following theorem:

THEOREM A. (i) Let $f(t)$ be of bounded variation over every finite interval $a \leq t \leq T$; (ii) let $g'(t)$, over the infinite interval $t \geq T$, be either (iia) of bounded variation, or (iib) absolutely integrable ("summable" in the sense of Lebesgue); (iii) for values of $t \geq T$, let

$$f(t) = g(t) \sum_x \{a(x) \cos xt + b(x) \sin xt\},$$

where the sum consists of a finite number of terms, among which $a(0)$ may occur, and it is assumed (with no loss of generality) that

* See, for instance, two examples in Carslaw's *Fourier's series and integrals*, 1st ed. (1906), 124; also Bromwich's *Theory of infinite series* (1908), 447 (Ex. 4) and 472 (Ex. 26), where a proof of Poisson's relation and other formulae by means of a particular case of Theorem A is referred to Schlömilch. Reference should also be made to H. M. Macdonald, *Proc. Edinburgh Math. Soc.*, 44 (1926), 2-7, where the equation 1.12 (below) is stated without proof, and is employed to deduce some familiar results.

$-\pi \leq x \leq \pi$; (iv) let $f(t)$ be continuous for integer values of t ; and (v) let the series $\sum_a^{\infty} f(n)$ be convergent.

With these conditions the series of integrals on the right of (1.11) is convergent*, and

$$(1.11) \quad \sum_a^{\infty} f(n) = \int_a^{\infty} f(t) dt + 2 \sum_1^{\infty} \int_a^{\infty} f(t) \cos 2n\pi t dt.$$

Conversely, let conditions (i)-(iv) be satisfied, and (vi) let $\int_a^{\infty} f(t) dt$ be convergent. Then it follows that $\sum_a^{\infty} f(n)$ is convergent, and the relation (1.11) is again true.

If $f(t)$ is of bounded variation in the interval $-T' \leq t \leq T$, and a further set of conditions, corresponding to (ii)-(vi), is satisfied when $t \leq -T'$, then the convergence of either $\sum f(n)$ or $\int f(t) dt$ in (1.12) implies that of the other, and

$$(1.12) \quad \sum_{-\infty}^{\infty} f(n) = \int_{-\infty}^{\infty} f(t) dt + 2 \sum_1^{\infty} \int_{-\infty}^{\infty} f(t) \cos 2n\pi t dt.$$

For many of the applications it is convenient to rewrite (1.11) and (1.12) in a form which may be derived by consideration of the function $f(t)e^{t \log z}$ in place of $f(t)$ and by addition of two real series, when necessary; namely, within the circle of convergence $|z| = r$,

$$(1.13) \quad \sum_a^{\infty} z^n f(n) = \sum_{-\infty}^{\infty} \int_a^{\infty} f(t) e^{(2n\pi i + \log z)t} dt;$$

and, within the annulus included between the two circles of convergence $|z| = r'$, $|z| = r$ ($r' < r$),

$$(1.14) \quad \sum_{-\infty}^{\infty} z^n f(n) = \sum_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{(2n\pi i + \log z)t} dt.$$

In the statement of conditions for the truth of Theorem A, (iv) is entirely unimportant; it merely avoids an undesirable complication of expression; if it be not made, $\sum f(n)$ must be replaced by

$$\frac{1}{2} \sum \{f(n+0) + f(n-0)\}.$$

* The particular case of (1.11) in which $f(t) = t^{-s}$ ($Rs > 0$) has been given by Hardy and Littlewood, *Math. Zeitschrift*, 10 (1921), 283-317 (284). The method which I have finally adopted for the proof of Theorem A resembles theirs in some respects, particularly in the proof of the second part of my Lemma 3.

Since it is the Laurent series, rather than the series of integrals, which is under consideration, the restrictions placed upon $f(t)$ involve no very great loss of generality; for example, if the Laurent series is absolutely convergent, all the conditions, including both (iia) and (iib), are satisfied by the function whose graph consists of the portions of straight lines joining consecutive points of the set for which a corresponding abscissa and ordinate are, respectively, n and $a_n = f(n)$. The conditions stated for the truth of the theorem are, of course, a sufficient, not a necessary, set; that neither (iia) nor (iib) is necessary is evident from the fact that neither includes the other.

1.2. I begin with a series of lemmas, the first of which amounts to a particular case of Theorem A.

LEMMA 1. *If $f(t)$ is of bounded variation over the finite interval $a \leq t \leq b$, and if $f(t)$ is continuous for integer values* of t , the series of integrals in (1.21) is convergent, and*

$$(1.21) \quad \sum_a^b f(n) = \int_a^b f(t) dt + 2 \sum_1^\infty \int_a^b f(t) \cos 2n\pi t dt.$$

The proof consists in adding together a finite number of convergent Fourier series†.

On replacing the cosine by its exponential form, and on addition of two real series, if necessary, we find that, for all finite values of z (except that $z \neq 0$ if $a \leq -1$),

$$(1.22) \quad \sum_a^b z^n f(n) = \sum_{-\infty}^\infty \int_a^b f(t) e^{(2n\pi i + \log z)t} dt.$$

LEMMA 2. *If $f'(t)$ is either (i) of bounded variation, or (ii) absolutely integrable, over the infinite interval $t \geq T$, and if either $\sum f(n)$ or $\int_a^\infty f(t) dt$ is convergent, then $f(t) \rightarrow 0$ as $t \rightarrow \infty$.*

For, if $f'(t)$ is integrable over (T, ∞) , $f(t)$ tends to a limit which must be zero if either series or integral is convergent.

* This condition is, as before, wholly unimportant.

† This remark I owe to Prof. Carslaw and to the Referee (whose communication reached me by the same post as Prof. Carslaw's). I am also indebted to the Referee for a considerable simplification of the proof of § 4.3 (i), and for other suggestions in matters of detail.

Lemma 1 is, essentially, due to Dirichlet, *Werke*, 1 (1889), 474-5, Equations (1) and (2).

LEMMA 3. If $f(t)$ satisfies conditions (ii) and (iii) of Theorem A, a number $T_1 \gg T$ may be chosen so that, when $\tau' > \tau \gg T_1$,

$$(1.23) \quad \left| 2 \sum_1^{\infty} \int_{\tau}^{\tau'} f(t) \cos 2n\pi t \, dt \right| < \epsilon, \quad \left| 2 \sum_1^{\infty} \int_{\tau}^{\infty} f(t) \cos 2n\pi t \, dt \right| < \epsilon,$$

provided that $g(t) \rightarrow 0$ as $t \rightarrow \infty$.

It will be sufficient to assume that $f(t) = g(t) \cos xt$; the appropriate modifications due to a term in $\sin xt$ are obvious. With this assumption

$$\begin{aligned} 2 \sum_1^{\infty} \int_{\tau}^{\tau'} f(t) \cos 2n\pi t \, dt \\ = \sum_{-\infty}^{(0)} \left\{ \left[g(t) \frac{\sin(2n\pi+x)t}{2n\pi+x} \right]_{\tau}^{\tau'} - \int_{\tau}^{\tau'} g'(t) \frac{\sin(2n\pi+x)t}{2n\pi+x} \, dt \right\}. \end{aligned}$$

If $[t]$ denotes*, as usual, the integer part of t ,

$$\sum_{-\infty}^{(0)} \frac{\sin(2n\pi+x)t}{2n\pi+x} = \frac{\sin x(\frac{1}{2} + [t])}{2 \sin \frac{1}{2}x} - \frac{\sin xt}{x} \quad \text{or} \quad \left(\frac{1}{2} \cot \frac{1}{2}x - \frac{1}{x} \right) \sin xt,$$

according as t is not, or is, an integer. When $-\pi \leq x \leq \pi$ it is easy to see that, for all values of t , the sum of the series lies between $\pm(\frac{1}{2} + \pi^{-1})$. Since $g(t) \rightarrow 0$ we may choose T_1 so that the modulus of the series of integrated parts is $< \frac{1}{2}\epsilon$, and the same is true as $\tau' \rightarrow \infty$.

(i) If $g'(t)$ is of bounded variation,

$$\begin{aligned} \left| \sum_{-\infty}^{(0)} \int_{\tau}^{\tau'} g'(t) \frac{\sin(2n\pi+x)t}{2n\pi+x} \, dt \right| &\leq 2V_{\tau}^{\tau'} g'(t) \sum_{-\infty}^{(0)} \frac{1}{(2n\pi+x)^2} \\ &\leq 2V_{\tau}^{\infty} g'(t) \left(\frac{1}{4} \operatorname{cosec}^2 \frac{1}{2}x - \frac{1}{x^2} \right) \leq \left(\frac{1}{2} - \frac{2}{\pi^2} \right) V_{\tau}^{\infty} g'(t) < \frac{1}{2}\epsilon, \end{aligned}$$

if T_1 be properly chosen; whence the inequalities (1.23) follow in this case.

(ii) The sequence
$$\sum_{-N}^{(0)} \frac{\sin(2n\pi+x)t}{2n\pi+x},$$

as $N \rightarrow \infty$, is boundedly convergent for all values of t , and its limit differs from

$$\frac{1}{2} \operatorname{cosec} \frac{1}{2}x \sin x(\frac{1}{2} + [t]) - \sin xt/x$$

at points of a set of content zero; hence, if $g'(t)$ is absolutely integrable over the infinite interval $t \gg T$, it is, by a known theorem†, permissible

* Bromwich, *Infinite series*, 257 (Ex. 17).

† Stated by W. H. Young, *Proc. London Math. Soc.* (2), 9 (1911), 463-485 (468).

to change the order of integration and summation; in this case, when $\tau' > \tau \gg T_1$,

$$\begin{aligned} \left| \sum_{-\infty}^{\infty} \int_{\tau}^{\tau'} g'(t) \frac{\sin(2n\pi + x)t}{2n\pi + x} dt \right| &\ll \left(\frac{1}{2} + \frac{1}{\pi} \right) \int_{\tau}^{\tau'} |g'(t)| dt \\ &\ll \left(\frac{1}{2} + \frac{1}{\pi} \right) \int_{\tau}^{\infty} |g'(t)| dt < \frac{1}{2}\epsilon, \end{aligned}$$

if T_1 be properly chosen; and the inequalities (1.23) follow in this case also.

LEMMA 4. *If $f'(t)$ is either (i) of bounded variation, or (ii) absolutely integrable, over the infinite interval $t \gg T$, the convergence of $\int^{\infty} f(t) dt$ is a necessary and sufficient condition for the convergence of the series $\sum f(n)$.*

When any one of the four possibilities holds $f(t) \rightarrow 0$ as $t \rightarrow \infty$ by Lemma 2; hence, by Lemma 3, $T_1 \gg T$ may be chosen so that, when $\tau' > \tau \gg T_1$,

$$\left| 2 \sum_1^{\infty} \int_{\tau}^{\tau'} f(t) \cos 2n\pi t dt \right| < \frac{1}{2}\epsilon;$$

whence, by employing Lemma 1,

$$\left| \sum_{\tau}^{\tau'} f(n) - \int_{\tau}^{\tau'} f(t) dt \right| < \frac{1}{2}\epsilon,$$

which proves the lemma.

For example, let $f(t) = \cos x t^{\frac{1}{2}} / (t^{\frac{1}{2}} \log t)$, which is of unbounded variation, but integrable over $t \gg 2$; here $f'(t)$ is of bounded variation but is not absolutely integrable over $t \gg 2$. The series $\sum_2^{\infty} \frac{\cos x \sqrt{n}}{n^{\frac{1}{2}} \log n}$ is convergent ($x \neq 0$).

LEMMA 5. *If (i) $g(t) \rightarrow 0$ as $t \rightarrow \infty$; and if, over the infinite interval $t \gg T$, (ii) $|g(t)\lambda'(t)|$ is integrable, (iii) $-(2N+1)\pi \ll \lambda'(t) \ll (2N+1)\pi$, (iv) $f(t) = g(t)e^{i\lambda(t)}$, and either (va) $|g'(t)|$ is integrable, or (vb) $g'(t)$ and $\lambda'(t)$ are both of bounded variation, then a number $T_1 \gg T$ may be chosen so that, when $\tau' > \tau \gg T_1$,*

$$(1.24) \quad \left| 2 \sum_{N+1}^{\infty} \int_{\tau}^{\tau'} f(t) \cos 2n\pi t dt \right| < \epsilon, \quad \left| 2 \sum_{N+1}^{\infty} \int_{\tau}^{\infty} f(t) \cos 2n\pi t dt \right| < \epsilon.$$

This lemma is not required in the proof of Theorem A, but is used in the analogous Theorem B below. The proof follows the lines of

Lemma 3. When $N = 0$, Lemmas 2 and 4 may be immediately adapted to suit this case. I use the lemma only in this weaker form.

Let

$$\sum_{-\infty}^{\infty (N)} \equiv \sum_{-\infty}^{\infty} - \sum_{-N}^N$$

when $N > 0$; then

$$\begin{aligned} 2 \sum_{N+1}^{\infty} \int_{\tau}^{\tau'} f(t) \cos 2n\pi t \, dt &= -i \sum_{-\infty}^{\infty (N)} \int_{\tau}^{\tau'} \frac{g(t)}{\lambda'(t) + 2n\pi} \frac{d}{dt} e^{i[\lambda(t) + 2n\pi t]} \, dt \\ &= -i \sum_{-\infty}^{\infty (N)} \left\{ \left[\frac{g(t) e^{i(\lambda(t) + 2n\pi t)}}{\lambda'(t) + 2n\pi} \right]_{\tau}^{\tau'} - \int_{\tau}^{\tau'} \left(\frac{g'(t)}{\lambda' + 2n\pi} - \frac{g\lambda''}{(\lambda' + 2n\pi)^2} \right) e^{i(\lambda + 2n\pi t)} \, dt \right\}. \end{aligned}$$

As in Lemma 3, the series of integrated parts is boundedly convergent for all values of t , and as $g(t) \rightarrow 0$, T_1 may be chosen so that the modulus of the integrated portion is less than $\epsilon/3$. The modulus of the second part of the integral does not exceed.

$$\sum_{-\infty}^{\infty (N)} \int_{\tau}^{\tau'} \frac{|g\lambda''| \, dt}{(\lambda' + 2n\pi)^2} \leq \frac{1}{2} \int_{\tau}^{\infty} |g\lambda''| \, dt < \frac{\epsilon}{3},$$

on proper (renewed) choice of T_1 . As regards the first part of the integral :

(i) If $g'(t)$ is absolutely integrable, as in Lemma 3, part (ii), the order of summation and integration may be changed, and the modulus of the resulting integral does not exceed

$$\int_{\tau}^{\tau'} g' \left| \sum_{-\infty}^{\infty (N)} \frac{e^{2n\pi i t}}{\lambda' + 2n\pi} \right| \, dt \leq K \int_{\tau}^{\tau'} |g'(t)| \, dt \leq K \int_{\tau}^{\infty} |g'(t)| \, dt < \frac{\epsilon}{3},$$

on proper choice of T_1 , where K is a number which depends only on N . Thus (1.24) follows in this case.

(ii) If $g'(t)$ and $\lambda'(t)$ are of bounded variation, the modulus of the first part of the integral is

$$\left| \int_{\tau}^{\tau'} \frac{g' \, dt}{(\lambda' + 2n\pi)^2} \frac{d}{dt} e^{i(\lambda + 2n\pi t)} \right| \leq 4 V_{\tau}^{\infty} \frac{g'}{(\lambda' + 2n\pi)^2},$$

and, if G' be the upper bound of $|g'(t)|$ when $t \geq \tau$, the sum does not exceed

$$\begin{aligned} 8 \sum_1^{\infty} \frac{1}{(2n-1)^2 \pi^2} V_{\tau}^{\infty} g'(t) + 4G' \sum_{-\infty}^{\infty (N)} V_{\tau}^{\infty} \frac{1}{(\lambda' + 2n\pi)^2} \\ \leq V_{\tau}^{\infty} g'(t) + 8G' V_{\tau}^{\infty} \lambda'(t) \sum_1^{\infty} \{(2n-1)\pi\}^{-3} < \epsilon/3, \end{aligned}$$

is T_1 is sufficiently large. Hence, in this case also, (1.24) is satisfied.

1.3. *Proof of Theorem A.* Under conditions (i)–(v) of the theorem

$$\begin{aligned} & \left| \sum_a^{\infty} f(n) - \int_a^{\infty} f(t) dt - 2 \sum_1^{\infty} \int_a^{\infty} f(t) \cos 2n\pi t dt \right| \\ &= \left| \sum_{\tau}^{\infty} f(n) - \int_{\tau}^{\infty} f(t) dt - 2 \sum_1^{\infty} \int_{\tau}^{\infty} f(t) \cos 2n\pi t dt \right| \quad (\text{Lemma 1}) \\ &< (\epsilon/3) + (\epsilon/3) + (\epsilon/3) = \epsilon \quad (\text{Lemmas 3 and 4}), \end{aligned}$$

provided that $\tau \gg T_1$, T_1 being a properly chosen number exceeding T .

Precisely the same conclusion may be drawn if $f(t)$ satisfies conditions (i)–(iv) and (vi) of the theorem. In both cases the equation (1.11) follows, and Theorem A is proved.

It is evident that Lemma 4 is required only in case $x = 0$.

COROLLARY 1. If $f(t) = g(t)$ and satisfies (i), (iia), and (iv) of Theorem A, and if $\sum f(n)e^{inx}$ is convergent ($-\pi \leq x \leq \pi$), then

$$\begin{aligned} (1.31) \quad & \left| \sum_{n>a} f(n)e^{inx} - \int_a^{\infty} f(t)e^{ixt} dt - if(a) \left\{ \frac{e^{ix(\frac{1}{2} + [a])}}{2 \sin \frac{1}{2}x} - \frac{e^{ixa}}{x} \right\} \right| \\ & \leq \frac{1}{2} V_a^{\infty} f(t) (\operatorname{cosec}^2 \frac{1}{2}x - 4x^{-2}) \leq \frac{1}{2} \{1 - (4/\pi^2)\} V_a^{\infty} f(t); \end{aligned}$$

$$(1.32) \quad \lim_{x \rightarrow 0, N \rightarrow \infty} \left\{ \sum_N^{\infty} f(n)e^{inx} - \int_N^{\infty} f(t)e^{ixt} dt \right\} = 0.$$

The proof is obvious from Lemma 3, part (i).

COROLLARY 2. If $f(t) = g(t)$ and satisfies (i), (iib), and (iv) of Theorem A, and if $\sum f(n)e^{inx}$ is convergent ($-\pi \leq x \leq \pi$), then

$$(1.33) \quad \left| \sum_{n>a} f(n)e^{inx} - \frac{if(a)}{2 \sin \frac{1}{2}x} e^{ix(\frac{1}{2} + [a])} \right| \leq \frac{1}{2} |\operatorname{cosec} \frac{1}{2}x| V_a^{\infty} f(t),$$

$$\begin{aligned} (1.331) \quad & \left| \sum_{n>a} f(n)e^{inx} - \frac{1}{2}x \cot \frac{1}{2}x \int_a^{\infty} f(t)e^{ixt} dt \right. \\ & \left. + \frac{if(a)e^{ixa}}{2 \sin \frac{1}{2}x} \{ \cos \frac{1}{2}x - e^{ix(\frac{1}{2} - a + [a])} \} \right| \leq \frac{1}{2} V_a^{\infty} f(t), \end{aligned}$$

and the double limit formula (1.32) holds unchanged.

As in Lemma 3, part (ii), the expression whose modulus occurs on the left of (1.31) is equal to

$$(1.34) \quad i \int_a^\infty f'(t) e^{ixt} \left(\frac{e^{ix(\frac{1}{2}-t+[t])}}{2 \sin \frac{1}{2}x} - \frac{1}{x} \right) dt$$

$$(1.341) \quad = \frac{if(a)}{x} e^{ixa} - \int_a^\infty f(t) e^{ixt} dt + \frac{\frac{1}{2}i e^{\frac{1}{2}ix}}{\sin \frac{1}{2}x} \int_a^\infty f'(t) e^{ix[t]} dt$$

$$(1.342) \quad = \left(\frac{1}{x} - \frac{1}{2} \cot \frac{1}{2}x \right) \left\{ if(a) e^{ixa} - x \int_a^\infty f(t) e^{ixt} dt \right\} \\ + \frac{\frac{1}{2}i}{\sin \frac{1}{2}x} \int_a^\infty f'(t) e^{ixt} (e^{ix(\frac{1}{2}-t+[t])} - \cos \frac{1}{2}x) dt.$$

The inequality (1.33) now follows from (1.341), while (1.331) follows from (1.342), since

$$|e^{ix(\frac{1}{2}-t+[t])} - \cos \frac{1}{2}x|^2 = 1 + \cos^2 \frac{1}{2}x - 2 \cos \frac{1}{2}x \cos x(\frac{1}{2}-t+[t]) \leq \sin^2 \frac{1}{2}x.$$

To prove (1.32) it is merely necessary to observe that

$$\lim_{x \rightarrow 0} x^2 \int_N^\infty f(t) e^{ixt} dt = -i \lim x \int_N^\infty f'(t) e^{ixt} dt = 0.$$

It may be noted that (1.341) is, formally, a quite obvious transformation of the series, and a direct proof would present no difficulty.

The series on the right of (1.13) and (1.14) are both of the form

$$(1.35) \quad \sum_{-\infty}^{\infty} F(2n\pi i + \log z),$$

and, if F is definite for every n , they are convergent for those values of $\log z$ for which the series $\sum z^n f(n)$ on the left are convergent. It does not seem possible to adduce any general argument as to the way in which a change in the value of z will affect the convergence of (1.35), but it is clear that in many cases (and, in fact, in all those examined) z will render (1.35) divergent only by passing through, or tending to, a singularity of one or more of the functions F . It thus frequently—in a sense, usually—happens that (1.13) and (1.14) exhibit the singularities of the functions represented by the series $\sum z^n f(n)$, and give their analytic continuations over the whole plane. Occasionally the right-hand side of (1.13) or (1.14) will serve to express in finite terms the sum of the series on the left*.

* Examples are given in §3.

1.4. THEOREM B. (i) Let $f(t)e^{-s\lambda(t)}$, where $s \equiv \sigma + i\tau$, be of bounded variation over every finite interval $a \leq t \leq T$; when $t \gg T$ (ii) let $-\pi \leq \tau\lambda'(t) \leq \pi$, and (iii) let $g(t) = f(t)e^{-\sigma\lambda(t)}$ satisfy conditions (ii) and (v) of Lemma 5; (iv) let $\sum f(n)e^{-s\lambda(n)}$ be convergent. Then corresponding to (1.11),

$$(1.41) \quad \sum_a' f(n)e^{-s\lambda(n)} = \int_a^\infty f(t)e^{-s\lambda(t)} dt + 2 \sum_1 \int_a^\infty f(t)e^{-s\lambda(t)} \cos 2n\pi t dt.$$

The proof of Theorem B; and a converse corresponding to that of Theorem A, will be evident from Lemma 5.

COROLLARY. If the conditions satisfied by $g(t)$ in Theorem B are (ii) and (va) of Lemma 5,

$$(1.42) \quad \lim_{\tau \rightarrow 0, N \rightarrow \infty} \left\{ \sum_N' f(n)e^{-s\lambda(n)} - \int_N^\infty f(t)e^{-s\lambda(t)} dt \right\} = 0.$$

The proof follows the lines of Lemma 5 and Corollary 2 to Theorem A.

2.1. In practice the method of application of the formulae (1.13) and (1.14) is tentative; it depends upon knowledge of the value of the integral $\int^\infty f(t)e^{xt} dt$ for complex or purely imaginary x , the imaginary part being unrestricted in value. There is, of course, a great variety of such integrals known; but many of them lead to familiar results, or to results which follow readily, and more directly, from Mellin's integral*.

A number of integrals involving Bessel functions may be employed to provide illustrations of the method; that chosen is Hankel's generalization of Weber's integral† ($Ra > 0$, $Rv > -1$),

$$(2.11) \quad \int_{-c}^\infty e^{-at} \frac{J_\nu \{2x(t+c)^{\frac{1}{2}}\}}{\{x(t+c)^{\frac{1}{2}}\}^\nu} (t+c)^\nu dt = \frac{e^{ac}}{a^{\nu+1}} e^{-(x^2/a)}.$$

The result of applying (1.13) is, when $|z| < 1$, $0 < c < 1$,

$$(2.12) \quad \sum_0^\infty (n+c)^\nu z^n \frac{J_\nu \{2x(n+c)^{\frac{1}{2}}\}}{\{x(n+c)^{\frac{1}{2}}\}^\nu} \\ = z^{-c} \sum_{-\infty}^\infty e^{-2n\pi ic} \left(\log \frac{1}{z} - 2n\pi i \right)^{-\nu-1} \exp \frac{x^2}{\log z + 2n\pi i}.$$

* I owe this remark to Prof. Hardy.

† G. N. Watson, *Theory of Bessel functions*, 394 (4).

The function represented by the series on the left has a singularity at $z = 1$, and it has no other singularities; the series on the right is its analytic continuation over the whole plane. If $R\nu > 0$, (2.12) holds for $c = 0$, and on making $z \rightarrow 1$,

$$(2.13) \quad \lim_{z \rightarrow 1} \left\{ \sum_1^{\infty} n^{\nu} z^n \frac{J_{\nu}(2x\sqrt{n})}{(x\sqrt{n})^{\nu}} - \left(\log \frac{1}{z} \right)^{-\nu-1} \exp \frac{x^2}{\log z} \right\} \\ = \frac{2}{(2\pi)^{\nu+1}} \sum_1^{\infty} \frac{1}{n^{\nu+1}} \left\{ \sin \left(\frac{x^2}{2n\pi} - \frac{1}{2}\nu\pi \right) + \sin \frac{1}{2}\nu\pi \right\} - \frac{2 \sin \frac{1}{2}\nu\pi}{(2\pi)^{\nu+1}} \zeta(\nu+1);$$

when written in this form, the result is true when $R\nu > -1$. It is, in fact, permissible to make* $\nu \rightarrow -1$; on writing $-J_1$ for J_{-1} ,

$$(2.14) \quad \lim_{z \rightarrow 1} \left\{ x^2 \sum_1^{\infty} z^n \frac{J_1(2x\sqrt{n})}{x\sqrt{n}} + \exp \frac{x^2}{\log z} \right\} = 1 + 2 \sum_1^{\infty} \left(1 - \cos \frac{x^2}{2\pi n} \right) \\ = 1 + \frac{B_1}{2!} \frac{x^4}{2!} - \frac{B_2}{4!} \frac{x^8}{4!} + \dots$$

2.2. The series (2.12) is a power series representing a function with only one singularity in the finite part of the plane. The example which follows may be regarded as typical of the application of the method to Laurent series; an essential singularity at the origin is analysed into an enumerable sequence of simple poles tending to $z = 0$. The series, which converges everywhere outside the unit circle, may be derived from the Gamma function integral, taken in the form ($Rx > 0$, $Ra > 0$, $c > 0$)

$$(2.21) \quad \int_{-\infty}^{\infty} \exp(at - xe^{t/c}) dt = cx^{-ac} \Gamma(ac).$$

On applying (1.14), with $|z| > 1$,

$$(2.22) \quad \sum_{-\infty}^{\infty} z^n \exp(-xe^{n/c}) - cz^{-c \log x} \Gamma(c \log z) \\ = cz^{-c \log x} \sum_1^{\infty} \{ e^{-2n\pi i c \log x} \Gamma(c \log z + 2n\pi i c) \\ + e^{2n\pi i c \log x} \Gamma(c \log z - 2n\pi i c) \},$$

and the series on the right is convergent if $|z| \gg \delta > 0$. The function represented by the series on the left of (2.22) has therefore a simple

* It is easy to verify, by actually proceeding to the limits in the reverse order, that $\lim_{z \rightarrow 1} \lim_{\nu \rightarrow -1}$ of the expression on the right of (2.12) is equal to $\lim_{\nu \rightarrow -1} \lim_{z \rightarrow 1}$ [as in (2.14)].

pole on the unit circle at $z = 1$, with residue unity, and an infinite sequence of simple poles within the circle at the points $z = e^{-m/c}$, with residues $(-)^m x^m e^{-m/c} / m!$, for positive integer values of m .

The poles of the function in (2.22) arise, of course, solely from the terms for which n is negative. Write $1/\log a$ for c , and then, if $a > 1$,

$$(2.23) \quad \sum_0^\infty z^n e^{-x a^n} = \frac{z^{-\log z / \log a}}{\log a} \sum_{-\infty}^\infty x^{2n\pi i / \log a} \Gamma\left(\frac{\log z - 2n\pi i}{\log a}\right) \\ - \frac{1}{z-1} + \sum_1^\infty z^{-n} (1 - e^{-x a^{-n}}).$$

This (with $z = e^{i\pi}$) is a slightly modified form of an equation due to Hardy*. In it make $z \rightarrow 1$, and write z instead of e^{-x} , $|z| < 1$,

$$(2.24) \quad \sum_0^\infty z^{a^n} = \sum_1^\infty (1 - z^{a^{-n}}) - \frac{\gamma + \log \log 1/z}{\log a} \\ + \frac{1}{\log a} \sum_{-\infty}^{(0)} \Gamma\left(\frac{2n\pi i}{\log a}\right) \exp\left(-\frac{2n\pi i \log \log 1/z}{\log a}\right).$$

3.1. When the integral on the right of (1.13) or (1.14) is discontinuous it not infrequently happens that the series on the left, which will then be a Fourier series, can be summed in finite terms. The number of discontinuous integrals which contain Bessel functions leads to a correspondingly large number of Schlömilch series—actually, slight extensions of Schlömilch series—among those for which the method succeeds. Incidentally it leads to certain null-series of very general form†. The two examples which follow are illustrations of this fact. The first arises from a remarkable integral due to Ramanujan‡, which may

* G. H. Hardy, "On certain oscillating series", *Quart. J. of Math.*, 38 (1907), 269-288 (281). Hardy's equation (3) is obtained from the consideration of a contour integral resembling Mellin's integral.

† [Footnote added June, 1927]. It should perhaps be stated that these results were contained in my paper as originally submitted to the Society, in November, 1924. Since the revised MS. was submitted there have appeared two papers in which Equations (3.22) and (3.23) (in a somewhat less general form) and a special case of (3.12) have been obtained by a closely allied method: C. Fox, "A class of null series", *Proc. London Math. Soc.* (2), 24 (1926), 479-493 (484), and "Some further contributions to the theory of null series, etc.", *ibid.* (2), 26 (1927), 35-80 (71-2 and Theorem 1).

‡ *Quart. J. of Math.*, 48 (1920), 294-310 (304); see also Watson, *loc. cit.*, 449. Ramanujan does not explicitly state, but from the context appears to imply, the condition $|\xi| = |\eta|$; this is unnecessary, and no such restriction is stated by Watson.

be written

$$(3.11) \quad \int_{-\infty}^{\infty} \frac{J_{\mu+yt}(\xi)}{\xi^{\mu+yt}} \frac{J_{\nu-yt}(\eta)}{\eta^{\nu-yt}} e^{2\pi i x t} dt = \frac{e^{\pi i (\nu-\mu)/y}}{y} \left\{ \frac{2 \cos (\pi x/y)}{\xi^2 e^{-\pi i x/y} + \eta^2 e^{\pi i x/y}} \right\}^{\frac{1}{2}(\mu+\nu)} \\ \times J_{\mu+\nu} \left\{ 2 \cos \frac{\pi x}{y} (\xi^2 e^{-\pi i x/y} + \eta^2 e^{\pi i x/y}) \right\}^{\frac{1}{2}},$$

where $-\xi$ and η are arbitrary, x is real, $y > 0$, $R(\mu+\nu) > 1$, if $-\frac{1}{2}y < x < \frac{1}{2}y$; for other real values of x the integral is zero.

Application of (1.14) leads to the sum of the series

$$(3.12) \quad \sum_{-\infty}^{\infty} \frac{J_{\mu+ny}(\xi)}{\xi^{\mu+ny}} \frac{J_{\nu-ny}(\eta)}{\eta^{\nu-ny}} e^{2n\pi i x} \\ = \frac{e^{\pi i (\nu-\mu)/y}}{y} \sum_{-\frac{1}{2}y-x}^{\frac{1}{2}y-x} e^{n\pi i (\nu-\mu)/y} \left\{ \frac{2 \cos \pi (n+x)/y}{\xi^2 e^{-\pi i (n+x)/y} + \eta^2 e^{\pi i (n+x)/y}} \right\}^{\frac{1}{2}(\mu+\nu)} \\ \times J_{\mu+\nu} \left\{ 2 \cos [\pi (n+x)/y] (\xi^2 e^{-\pi i (n+x)/y} + \eta^2 e^{\pi i (n+x)/y}) \right\}^{\frac{1}{2}},$$

a result which is true if $R(\mu+\nu) > 0^*$. When $x = \frac{1}{2}$ we have the remarkably general null-series,

$$(3.13) \quad \sum_{-\infty}^{\infty} (-)^n (\eta/\xi)^{ny} J_{\mu+ny}(\xi) J_{\nu-ny}(\eta) \quad (0 < y \leq 1).$$

Clearly the sign of y may be changed without altering the result. Further, when $y = 0$, the series is summable (C1) to the sum zero; if this be interpreted to be its value when $y = 0$, the series is null in the closed interval $-1 \leq y \leq 1$.

The equation (3.12) is, in a somewhat disguised form, a generalization of Graf's extension[†] of Neumann's addition theorem. This may be seen by writing $2\pi i x = i\theta + y \log \xi - y \log \eta + \pi i y$, so that $I\theta = y \log |\xi/\eta|$. When $y = 1$,

$$(3.14) \quad \sum_{-\infty}^{\infty} (-)^n e^{in\theta} J_{\mu+n}(\xi) J_{\nu-n}(\eta) \\ = e^{i\nu\theta} e^{(2N+1)\nu\pi i} \left(\frac{\xi - \eta e^{-i\theta}}{\xi - \eta e^{i\theta}} \right)^{\frac{1}{2}(\mu+\nu)} J_{\mu+\nu}(\xi^2 + \eta^2 - 2\xi\eta \cos \theta)^{\frac{1}{2}},$$

* The series (3.12) is then absolutely convergent; it is semi-convergent when

$$-1 < R(\mu+\nu) \leq 0,$$

provided that neither $x + \frac{1}{2}y$ nor $x - \frac{1}{2}y$ is an integer or zero, and the series (3.13) is still null in the open interval $-1 < y < 1$.

† Watson, *loc. cit.*, 359.

where* $N \equiv [\frac{1}{2} - x] = [i \log(\xi/\eta) - \theta]/2\pi$. Graf's formula, when the conditions $I\theta = \log|\xi/\eta|$, $R\mu > 0$ are satisfied, follows on putting $\nu = 0$; and the general formula follows by the principle of analytic continuation.

It is, perhaps, worth while to record the simple form taken by (3.12) on putting

$$y = 1, \quad \pi x = \theta + \phi, \quad \xi = re^{i\phi}, \quad \eta = re^{-i\phi},$$

in the special case when r , θ , and ϕ are real, $R(\mu + \nu) > -1$, namely

$$(3.15) \quad \sum_{-\infty}^{\infty} e^{2ni\theta} J_{\mu+n}(re^{i\phi}) J_{\nu-n}(re^{-i\phi}) \\ = e^{i(\nu-\mu)(\theta+N\pi)} \left\{ \frac{\cos(\theta+\phi)}{\cos(\theta-\phi)} \right\}^{\frac{1}{2}(\mu+\nu)} J_{\mu+\nu} \{ 2r \cos^{\frac{1}{2}}(\theta+\phi) \cos^{\frac{1}{2}}(\theta-\phi) \},$$

provided that, when $-1 < R(\mu + \nu) \leq 0$, $\theta + \phi$ is not an odd multiple of $\frac{1}{2}\pi$. $N \equiv [\frac{1}{2} - (\theta + \phi)\pi^{-1}]$, so that N is zero if $\theta + \phi$ is acute.

3.2. The second example chosen is a generalized form of Schlömilch-Fourier series arising from the particular case of Sonine's integral† when one of the Bessel functions is of order $-\frac{1}{2}$, in which case the integral may be written, when $R\mu > \frac{1}{2}$, b is real, and $y > 0$,

$$(3.21) \quad \int_{-\infty}^{\infty} \frac{J_{\mu}[y\sqrt{\{(t+c)^2+z^2\}}]}{\{(t+c)^2+z^2\}^{\frac{1}{2}\mu}} e^{ibt} dt \\ = \begin{cases} 0 & (|b| \gg y), \\ \frac{e^{-ibc}(2\pi)^{\frac{1}{2}}}{y^{\mu} z^{\mu-\frac{1}{2}}} (y^2 - b^2)^{\frac{1}{2}\mu-\frac{1}{4}} J_{\mu-\frac{1}{2}}\{z(y^2 - b^2)^{\frac{1}{2}}\} & (|b| \leq y), \end{cases}$$

for all values of c and z . It follows, by (1.14), that

$$(3.22) \quad \sum_{-\infty}^{\infty} \frac{J_{\mu}[2\pi y\sqrt{\{(n+c)^2+z^2\}}]}{\{(n+c)^2+z^2\}^{\frac{1}{2}\mu}} e^{2n\pi ix} \\ = \frac{e^{-2\pi icx}}{y^{\mu} z^{\mu-\frac{1}{2}}} \sum_{-y-x}^{y-x} e^{-2n\pi ic} \{y^2 - (x+n)^2\}^{\frac{1}{2}\mu-\frac{1}{4}} J_{\mu-\frac{1}{2}}[2\pi z\sqrt{\{y^2 - (x+n)^2\}}],$$

a formula which, by the principle of analytic continuation, is true if $R\mu > -\frac{1}{2}$, provided that, when $-\frac{1}{2} < R\mu \leq \frac{1}{2}$, neither $x+y$ nor $x-y$

* Except in §3.2, and in a few cases where misunderstanding is impossible, square brackets are consistently used to denote the integral part of the expression contained within them.

† Watson, *loc. cit.*, 415 (1).

is an integer, positive, negative, or zero. As in many other series of this type, if $0 < x \leq \frac{1}{2}$, the sum is zero if $0 < y \leq x$ ($y \neq x$, if $R\mu \leq \frac{1}{2}$); and, in particular, if $x = \frac{1}{2}$, we have the very general null-series,

$$(3.23) \quad \sum_{-\infty}^{\infty} (-)^n \frac{J_{\mu}[2\pi y \sqrt{\{(n+c)^2 + z^2\}}]}{\{(n+c)^2 + z^2\}^{\frac{1}{2}\mu}} = 0 \quad \begin{cases} 0 < y \leq \frac{1}{2} \ (R\mu > \frac{1}{2}), \\ 0 < y < \frac{1}{2} \ (-\frac{1}{2} < R\mu \leq \frac{1}{2}), \end{cases}$$

of which the special case $c = z = 0$ is Nielsen's well known null-series*,

$$(3.231) \quad \sum_0^{\infty} (-)^n \frac{J_{\mu}(2\pi n y)}{n^{\mu}}.$$

4. It is convenient to set down here a series which has already been employed in Lemmas 3 and 5. If x is real, then, for all except integer values of c , it is true that

$$(4.0) \quad \sum_{-\infty}^{\infty} \frac{e^{2\pi i x}}{n+c} = \begin{cases} \pi \operatorname{cosec} \pi c e^{2\pi i \xi c} & \text{if } x \text{ is not an integer,} \\ \pi \cot \pi c & \text{if } x \text{ is an integer,} \end{cases}$$

where $\xi \equiv \frac{1}{2} - x + [x]$.

The result of differentiating the left-hand side of (4.0) with regard to c is a series which is uniformly convergent for all except integer values of c , and the result of differentiating the right-hand side with regard to c is a continuous function of x ; therefore

$$(4.01) \quad \sum_{-\infty}^{\infty} \frac{e^{2\pi i x}}{(n+c)^2} = \pi^2 \operatorname{cosec} \pi c e^{2\pi i \xi c} (\cot \pi c - 2i\xi);$$

and, more generally ($m > 0$),

$$(4.02) \quad \sum_{-\infty}^{\infty} \frac{e^{2\pi i x}}{(n+c)^{m+1}} = (-)^m \frac{\pi}{m!} \frac{d^m}{dc^m} (e^{2\pi i \xi c} \operatorname{cosec} \pi c).$$

If $2\pi ic = \log z$, the series may be rewritten as

$$(4.03) \quad \sum_{-\infty}^{\infty} \frac{e^{(\log z + 2\pi i)x}}{(\log z + 2\pi i)^{m+1}} = (-)^m \frac{z^x}{m!} \left(z \frac{d}{dz} \right)^m \left(\frac{z^{1-x+[x]}}{z-1} \right).$$

When $m = 0$ and x is an integer, the right-hand side must be replaced by $\frac{1}{2} z^x (z+1)/(z-1)$. Otherwise the formula is correct for $m \geq 0$.

* Nielsen, *Handbuch der Theorie der Cylinderfunktionen*, 347-351 (§ 134); Watson, *loc. cit.*, 635 (2). Watson treats μ as real; but Nielsen considers the more general case.

4.1. Throughout this section questions concerned with the asymptotic behaviour of series are discussed. The tentative character of the analysis disappears, and it becomes possible to give some results of a general nature, and to attack directly particular problems of the type considered.

Let $z^t f(t)$ satisfy conditions (i)–(v) of Theorem A, and, in addition, let $f^{(r)}(t)$ exist, while

$$\lim_{b \rightarrow \infty} z^b f^{(m)}(b) = 0 \quad (m = 1, 2, \dots, r);$$

then from (1.13), on integrating by parts,

$$(4.11) \quad \sum'_a z^n f(n) = \phi(a, z) - \sum_{-\infty}^{\infty} \int_a^{\infty} f^{(r)}(t) \frac{e^{(\log z + 2n\pi i)t}}{\log z + 2n\pi i} dt,$$

where

$$\phi(a, z) = \begin{cases} z^{1+[a]} f(a)/(1-z) & \text{if } a \text{ is not an integer,} \\ \frac{1}{2} z^a (1+z) f(a)/(1-z) & \text{if } a \text{ is an integer,} \end{cases}$$

and, in either case,

$$(4.12) \quad \sum_{n>a} z^n f(n) = z^a \sum_{m=0}^{r-1} \frac{f^{(m)}(a)}{m!} \left(z \frac{d}{dz} \right)^m \left(\frac{z^{1-a+[a]}}{1-z} \right) + R_r,$$

$$(4.120) \quad |R_r| \leq Z_r \int_a^{\infty} e^{t \log |z|} |f^{(r)}(t)| dt,$$

$$(4.121) \quad Z_r \equiv \sum_{-\infty}^{\infty} \{ \log^2 |z| + (2n\pi + \arg z)^2 \}^{-\frac{1}{2}r}.$$

On writing b instead of a , and subtracting from (4.12), we obtain the corresponding expansion for the case of a finite number of terms.

The particular case of (4.12), when a is a large positive integer, furnishes an asymptotic expansion of the remainder after N terms of a convergent power series*, namely, if $|z| \neq 1$,

$$(4.13) \quad \sum_N^{\infty} z^n f(n) = z^N \sum_{m=0}^{r-1} \frac{f^{(m)}(N)}{m!} \left(z \frac{d}{dz} \right)^m \left(\frac{1}{1-z} \right) + R_r,$$

where R_r is controlled by the inequality (4.120) (with N instead of a as the lower limit of the integral).

* The formula fails to give any information in the case of an integral function, for the order of magnitude of $f^{(m)}(N)$ then increases with m .

4.2. When $|z|=1$ it is better to employ (4.02) and (4.0) in place of (4.03). In this case, if $-\pi \ll x \ll \pi$, and

$$\lim_{b \rightarrow \infty} f^{(m)}(b) = 0 \quad (m = 1, 2, \dots, r),$$

$$(4.21) \quad \sum_{n > a} e^{inx} f(n) = -e^{iNx} \sum_{m=0}^{2r-1} \frac{f^{(m)}(a)}{i^{m+1} m!} \frac{d^m}{dx^m} \left(\frac{e^{iax}}{2 \sin \frac{1}{2}x} \right) + R_{2r},$$

$$(4.210) \quad \alpha \equiv \frac{1}{2} - a + [a], \quad |R_{2r}| \ll \frac{-1}{2r-1!} \frac{d^{2r-1}}{dx^{2r-1}} \left(\frac{1}{2} \cot \frac{1}{2}x \right) \int_a^\infty |f^{(2r)}(t)| dt,$$

if the integral converges. As in (4.12), when a is a large positive integer (4.21) gives an asymptotic formula for the remainder after N terms of a trigonometric series:

$$(4.22) \quad \sum_N^\infty e^{inx} f(n) = -e^{iNx} \sum_{m=0}^{2r-1} \frac{f^{(m)}(N)}{i^{m+1} m!} \frac{d^m}{dx^m} \left(\frac{1}{e^{ix} - 1} \right) + R_{2r},$$

in which $|R_{2r}|$ satisfies the inequality (4.210) with N instead of a as the lower limit of the integral.

It is not necessary that the series should be a Fourier series; with $f(n) = 1/\log n$, for instance, the sine series is the classical example* of a convergent trigonometric series which is not a Fourier series. In this case

$$\sum_N^\infty \frac{e^{inx}}{\log n} = -e^{iNx} \left\{ \frac{1}{(e^{ix} - 1) \log N} - \frac{1}{4 \sin^2 \frac{1}{2}x N \log^2 N} \right\} + R_2,$$

$$|R_2| \ll (4N \log^2 N \sin^2 \frac{1}{2}x)^{-1},$$

which establishes, incidentally, the convergence (when $x \neq 0$) of the repeated series

$$\sum_{N=2}^\infty \sum_{n=N}^\infty \frac{e^{inx}}{\log n}.$$

The formulae (4.12) and (4.21) may, finally, be still further specialized by putting $x=0$; under the same conditions as are required for the truth of (4.21), on integrating (1.11) by parts,

$$(4.23) \quad \sum_{n > a} f(n) = \int_a^\infty f(t) dt + \sum_{r=1}^{2m} \frac{(-)^{r-1} f^{(r-1)}(a)}{r+1!} \phi'_{r+1}(a - [a]) + R_m,$$

* Due to Fatou; see Lebesgue, *Les séries trigonométriques*, 124. Further references are given by Hardy and Littlewood, *Proc. London Math. Soc.* (2), 24 (1925), 211-246 (211, footnote).

where $\phi_r(x)$ is the r -th Bernoullian polynomial, and

$$(4.230) \quad (-)^m R_m = 2 \int_a^\infty f^{(2m)}(t) \sum_1^\infty \frac{\cos 2n\pi t}{(2n\pi)^{2m}} dt,$$

$$(4.231) \quad |R_m| \leq \frac{B_m}{2m!} \int_a^\infty |f^{(2m)}(t)| dt.$$

When a is an integer, (4.23) is the Euler-Maclaurin sum-formula for the case of an infinite series.

4.3. In the remainder of this section I apply the formulae of § 1, particularly the corollaries, to the elucidation of certain asymptotic questions somewhat different from those already treated. I begin with an application of Corollary 1.

(i) From (1.31), if $f(t) = 1/\log t$,

$$(4.31) \quad \sum_N^\infty \frac{\sin nx}{\log n} = \int_N^\infty \frac{\sin xt}{\log t} dt - \frac{\cos Nx}{\log N} \left(\frac{1}{x} - \frac{1}{2} \cot \frac{1}{2}x \right) + R,$$

$$|R| \leq (2N \log^2 N)^{-1}.$$

The behaviour of the integral may be seen as follows; on integrating by parts, and applying the second theorem of the mean,

$$(4.311) \quad \int_N^\infty \frac{\sin xt}{\log t} dt = \frac{\cos Nx}{x \log N} - \frac{1}{x \log^2 N} \int_{Nx}^{\mu x} \frac{\cos t}{t} dt \quad (\mu > N)$$

$$= \frac{\cos \theta}{x \log(\theta/x)} - \frac{1}{x \log^2(\theta/x)} \int_\theta^{\theta'} \frac{\cos t}{t} dt \quad (\theta' > \theta),$$

where $Nx = \theta$, a definite number greater than 0. In this case

$$(4.312) \quad \int^x dx \int_N^\infty \frac{\sin xt}{\log t} dt = -\cos \theta \log \log \frac{\theta}{x} + O\left(\frac{1}{\log 1/x}\right)$$

$$\sim -\cos \theta \log \log 1/x.$$

These equations show that, as $N \rightarrow \infty$ and $x \rightarrow +0$, the integral (4.311) and the repeated integral (4.312) both oscillate infinitely, thus verifying that $\Sigma(\log n)^{-1} \sin nx$ is not a Fourier series.

(ii) Some interesting cases of the Gibbs Phenomenon as $x \rightarrow 0$ in certain Dirichlet series are furnished by the corollary to Theorem B. If $|f'(t)|$ and $|f(t)\lambda''(t)|$ are integrable over $t \geq N$, if $\lambda(t)$ tends steadily to infinity, and $0 < \lambda'(t) \leq \Delta$ ($t \geq a$), $0 \leq x \leq \pi/\Delta$, then, from (1.42),

$$(4.32) \quad \lim_{x \rightarrow 0, N \rightarrow \infty} \left\{ \sum_N^\infty f(n) \sin x\lambda(n) - \int_N^\infty f(t) \sin x\lambda(t) dt \right\} = 0.$$

Let $F(x) \equiv \sum_n^{\infty} f(n) \sin x\lambda(n)$, $F(0) = 0$, $F(+0) = \lim_{x \rightarrow +0} F(x)$.

It is then evident that (4.32) may be written

$$(4.321) \quad \lim_{x \rightarrow +0, N \rightarrow \infty} \left\{ \sum_n^N f(n) \sin x\lambda(n) + \int_N^{\infty} f(t) \sin x\lambda(t) dt \right\} = F(+0).$$

The two repeated limits exist in the case of the series, and are $F(0)$ and $F(+0)$ respectively; they must therefore exist in the case of the integral, and

$$(4.322) \quad \lim_{N \rightarrow \infty} \lim_{x \rightarrow +0} \int_N^{\infty} f(t) \sin x\lambda(t) dt = F(+0) - F(0) = F(+0).$$

Since the sign of $f(t)$ is at our disposal, we may assume that $F(+0) > 0$, and therefore

$$\varliminf_{x, N} \int_N^{\infty} f \sin x\lambda dt \leq 0 < F(+0) \leq \varlimsup_{x, N} \int_N^{\infty} f \sin x\lambda dt.$$

The condition for the existence of the Gibbs Phenomenon is

$$\varlimsup_{x, N} \sum_n^N f(n) \sin x\lambda(n) > F(+0),$$

or, on account of (4.321),

$$(4.323) \quad \varliminf_{x, N} \int_N^{\infty} f(t) \sin x\lambda(t) dt < 0.$$

A particularly simple case arises if

$$f(t) \equiv \lambda'(t)/\lambda(t) \quad \text{and} \quad \lambda''(t) \leq 0^* \quad \text{when } t \geq N;$$

it is then evident that f' and $f\lambda''$ are both absolutely integrable over $t \geq N$. And the lower limit as $x \rightarrow +0$ and $N \rightarrow \infty$ of the integral (4.323), *i.e.* of

$$\int_{x\lambda(N)}^{\infty} \frac{\sin \lambda}{\lambda} d\lambda \quad \text{is} \quad \int_{\pi}^{\infty} \frac{\sin \lambda}{\lambda} d\lambda = -0.281.$$

Hence, if $\lambda''(t) \leq 0$ and $0 < \lambda'(t) \leq \Delta$ for sufficiently large values of t , the series

$$(4.324) \quad F(x) \equiv \sum_n^{\infty} \frac{\lambda'(n)}{\lambda(n)} \sin x\lambda(n), \quad F(+0) = \frac{1}{2}\pi,$$

exhibits the Gibbs Phenomenon as $x \rightarrow 0$.

* This condition is, of course, unnecessarily restrictive. All that is required for the truth of the theorem is that $|\lambda''| |\lambda^{-1}|$ should be integrable over $t \geq N$.

Simple examples are

$$(4.325) \quad F(x) \equiv \sum_1^{\infty} \frac{\sin xn^m}{n}, \quad F(+0) = \frac{\pi}{2m} \quad (0 < m \leq 1);$$

$$(4.326) \quad F(x) \equiv \sum_2^{\infty} \frac{\sin x \log n}{n \log n}, \quad F(+0) = \frac{1}{2}\pi;$$

$$(4.327) \quad F(x) \equiv \sum_2^{\infty} \frac{1}{n} \sin \frac{xn}{(\log n)^{1+\kappa}} \quad (\kappa > 0), \quad F(+0) = \frac{1}{2}\pi.$$

The last series is obtained by adding to a particular case of (4.324) a series whose sum is continuous at $x=0$. That the sum of the series (4.326) tends to $\frac{1}{2}\pi$ as $x \rightarrow +0$ is clear from the properties of the ζ -function, but that the series exhibits the Gibbs Phenomenon does not seem to be immediately obvious.

(iii) Another case of the Gibbs Phenomenon as $x \rightarrow 0$ in a series of fairly general form may be derived from Corollary 2 to Theorem A. If $f(0) = 0$, $x > 0$, and $\int_0^{\infty} \left| \frac{d}{dt} \frac{f(t)}{t} \right| dt$ is convergent, (1.34) leads to

$$(4.33) \quad \sum_1^{\infty} \frac{f(nx)}{n} - \int_x^{\infty} \frac{f(t)}{t} dt = f(x) + \int_1^{\infty} (t - [t]) \left\{ \frac{xf'(xt)}{t} - \frac{f(xt)}{t^2} \right\} dt.$$

The modulus of the integral on the right is less than

$$x \int_x^{\infty} \left| \frac{d}{dt} \frac{f(t)}{t} \right| dt = x \int_x^{\delta} \left| \frac{d}{dt} \frac{f(t)}{t} \right| dt + O(x),$$

where $\delta > 0$, and it is assumed that $0 < x < \delta$. Let there be definite positive numbers H , K , and κ such that, when $0 < t \leq \delta$,

$$|f(t)| \leq Ht^{\kappa}, \quad |f'(t)| \leq Kt^{\kappa-1}.$$

Then it is evident that the integral on the right of (4.33) is $O(x^{\kappa}) + O(x)$ and if $F(x) \equiv \sum_1^{\infty} \frac{f(nx)}{n}$, then

$$(4.331) \quad F(+0) = \lim_{x \rightarrow +0} \sum_1^{\infty} \frac{f(nx)}{n} = \int_0^{\infty} \frac{f(t)}{t} dt, \quad F(0) = 0.$$

Corresponding to (1.32) is the double limit theorem

$$(4.332) \quad \lim_{x \rightarrow +0, N \rightarrow \infty} \left\{ \sum_N^{\infty} \frac{f(nx)}{n} - \int_{Nx}^{\infty} \frac{f(t)}{t} dt \right\} = 0;$$

and, assuming that $F(+0) > 0$, the series for $F(x)$ will exhibit the Gibbs Phenomenon as $x \rightarrow 0$ if there is a positive value of ξ for which

$$\int_{\xi}^{\infty} \frac{f(t)}{t} dt < 0.$$

The Bessel function $J_\nu(t)$ satisfies all the conditions imposed upon $f(t)$, if $\nu > 0$.

4. 4. When the complete analytical expression for the continuation of a function defined by a series of one of the types considered in § 1 cannot be obtained—as, for example, when the integrals in (1.13) and (1.14) cannot be evaluated—it is still possible to employ the corollaries to Theorem A or B, or one of the formulae of the present section to obtain information as to the behaviour of the function when the variable tends to a singularity. It will be sufficient to take a single example of some degree of generality. The series chosen is really a Dirichlet series, with real variable, since it is not necessary that $\lambda(n)$ should be an integer. Let $\lambda(t) \rightarrow \infty$ with t , and let

$$(4.41) \quad S(r, x) \equiv \sum_0^\infty r^{\lambda(n)} e^{inx} \quad (0 \leq r < 1).$$

If $x \neq 0$, it follows from (1.33) that, as $r \rightarrow 1$, the sum $S(r, x)$ oscillates finitely, provided that $\int |\lambda' \log r| e^{\lambda \log r} dt$ is convergent,—a condition which includes such a series as

$$\sum_0^\infty \left\{ r^{2n} e^{2nix} + r^{(2n+1)^m} e^{(2n+1)ix} \right\}.$$

If $\lambda(t)$ is monotonic, so that $\lambda'(t) \geq 0$ for $t \geq 0$,

$$f'(t) = -\lambda'(t) r^{\lambda(t)} \log 1/r \leq 0,$$

and if $\lambda(0) \geq 0$, then $f(0) \leq 1$, so that, from (1.34), $|S(r, x)| \leq \frac{1}{2} |\cot \frac{1}{4} x|$, and, as $r \rightarrow 1$,

$$|\overline{\lim} S(r, x)| \leq \frac{1}{2} |\cot \frac{1}{4} x| \quad (x \neq 0).$$

In particular, if $x = \pi$, the sum of the series $\sum (-)^n r^{\lambda(n)}$, as $r \rightarrow 1$, oscillates at most between 0 and 1, as is otherwise obvious.

It is possible to obtain more definite information when $\lambda(t)$ is further restricted. For example, for $t \geq N$, let

$$(4.42) \quad |\lambda''| \leq \lambda' \phi(\lambda), \quad \lambda' \leq \lambda \psi(\lambda),$$

where ϕ and ψ are positive monotonic functions which tend to zero as $t \rightarrow \infty$. From (4.22), if $x \neq 0$,

$$\sum_N^\infty r^{\lambda(n)} e^{inx} - \frac{1}{2} i r^{\lambda(N)} e^{iNx} \cot \frac{1}{2} x = \frac{1}{4} \operatorname{cosec}^2 \frac{1}{2} x \left\{ -\lambda'(N) r^{\lambda(N)} e^{iNx} \log r + R \right\},$$

$$|R| \leq \log \frac{1}{r} \int_N^\infty \left(|\lambda''| + \lambda'^2 \log \frac{1}{r} \right) e^{\lambda \log r} dt$$

$$\leq \log \frac{1}{r} \int_N^\infty \left\{ \phi(\lambda) + \lambda \psi(\lambda) \log \frac{1}{r} \right\} \lambda' e^{\lambda \log r} dt.$$

Divide the path of integration into (N, ρ) and (ρ, ∞) , and choose ρ , dependent upon r , so that $\rho \rightarrow \infty$ as $r \rightarrow 1$, while $\lambda(\rho) \log 1/r \rightarrow 0$, i.e. $(1-r)\lambda(\rho) \rightarrow 0$; this is always possible. The part of the integral from N to ρ does not exceed

$$\{\phi(N) + \psi(N)\lambda(\rho) \log 1/r\} \{e^{\lambda(N) \log r} - e^{\lambda(\rho) \log r}\} = o(1),$$

and the second part does not exceed

$$\phi(\rho) + \psi(\rho) = o(1).$$

Thus $R \rightarrow 0$ as $r \rightarrow 1$, and

$$\lim_{r \rightarrow 1} \sum_N r^{\lambda(n)} \cos nx = \frac{1}{2} \cos Nx - \frac{1}{2} \cot \frac{1}{2}x \sin Nx,$$

$$\lim_{r \rightarrow 1} \sum_N r^{\lambda(n)} \sin nx = \frac{1}{2} \cot \frac{1}{2}x \cos Nx.$$

When $\lambda = t^k$ ($k > 0$), this result is a particular case of a theorem due to Fejér*.

It is not likely that the conditions (4.42) can be greatly improved; for it is easy to establish the oscillation as $r \rightarrow 1$ of a well-known series for which, if $\delta \geq 0$, $\lambda' < \delta\lambda$. It is only necessary to take $1 < c < e^\delta$ in the series $\sum r^{c^n} e^{i n x}$. The oscillation of this series as $r \rightarrow 1$ may be shown by means of (1.331).

In the general case a more precise result may be obtained by a closer examination of the remainder term in (1.33). From (1.34) and (1.341), if $x \neq 0$,

$$(4.43) \quad \sum_0^\infty r^{\lambda(n)} e^{i n x} - \frac{1}{2} i r^{\lambda(0)} \cot \frac{1}{2}x = \frac{\frac{1}{2} i e^{\frac{1}{2} i x}}{\sin \frac{1}{2}x} \int_0^\infty e^{i x [t]} \lambda' \log r e^{\lambda \log r} dt.$$

Divide the interval $(0, \infty)$ into the three parts $(0, \rho)$, (ρ, ρ_1) , (ρ_1, ∞) , where ρ and ρ_1 are so chosen that

$$\lambda(\rho) \log 1/r = \delta, \quad \lambda(\rho_1) \log 1/r = 1/\delta \quad (0 < \delta < 1).$$

(i) The modulus of the integral over $(0, \rho)$ does not exceed

$$(4.431) \quad e^{\lambda(0) \log r} - e^{-\delta} \leq 1 - e^{-\delta} \leq \delta;$$

(ii) that of the integral over (ρ_1, ∞) does not exceed

$$(4.432) \quad e^{\lambda(\rho_1) \log r} = e^{-1/\delta} < \delta.$$

* See Hardy's paper referred to in footnote * on page 92 *supra*, where the oscillation of the series (with $x = \pi$) is proved when $\lambda = c^t$, for a sufficiently large c . Some numerical results (when $c = 2$ and $c = 6$) are given by Bromwich, *Infinite series*, 490 (Ex. 30).

(iii) Let $\lambda(t) = e^{\phi(t)}$; then

$$\phi(\rho) = \log \frac{1}{\log 1/r} - \log \frac{1}{\delta}, \quad \phi(\rho_1) = \log \frac{1}{\log 1/r} + \log \frac{1}{\delta},$$

$$(4.433) \quad \phi(\rho_1) - \phi(\rho) = 2 \log 1/\delta.$$

In the case of the series $\sum r^{c^n} e^{inx}$, $(\rho_1 - \rho) \log c = 2 \log 1/\delta$, and the resulting analysis (when $x = \pi$) is precisely that of Hardy's paper*, page 277. If $c \gg 5$, it is sufficient to take $\delta = c^{-1}$, so that $\rho_1 - \rho = 1$, to obtain a rough numerical estimate of the amount of the oscillation when $x = \pi$.

For smaller values ($c > 1$) it is always possible to determine a value of θ not exceeding π such that oscillation certainly occurs if $-\theta \leq x \leq \theta$.

If $\phi'(t)$ tends steadily to infinity with t , it is easy to show that the series (4.41) oscillates over the whole possible range. For if, in (4.433), $\rho_1 = \rho + \kappa$ ($\kappa > 0$),

$$\kappa \phi'(\rho + \theta \kappa) = 2 \log 1/\delta \quad (0 < \theta < 1).$$

Choose r so that

$$(4.434) \quad 1 > r > e^{-\delta^{1+\Delta}} \quad (\Delta > 0),$$

$$\phi(\rho) > \Delta \log 1/\delta.$$

It is then possible to choose Δ so that $\phi'(\rho) > 2\epsilon^{-1} \log 1/\delta$ ($\epsilon < 1$), and therefore $\kappa < \epsilon$. Avoiding values of r for which an integer N lies in the interval $\rho < N < \rho + \epsilon$,

$$(4.435) \quad \int_{\rho}^{\rho_1} e^{ix[t]} \lambda' \log \frac{1}{r} e^{\lambda \log r} dt = e^{ix[\rho]} (e^{-\delta} - e^{-1/\delta}).$$

Collecting results, it follows from (4.431)–(4.435) that, for almost all values of r which satisfy (4.434) (Δ being chosen so that $\kappa < \epsilon$),

$$(4.436) \quad \left| \int_0^{\infty} e^{ix[t]} \lambda' \log r e^{\lambda \log r} dt + e^{ix[\rho]} \right| < 4\delta;$$

and since, within $(0, 1)$, δ is arbitrary, both the real and imaginary parts of the integral oscillate between ± 1 as $r \rightarrow 1$.

Let $\mu(t)$ be the function inverse to $\lambda(t)$, so that, since

$$\log 1/r \sim 1-r, \quad \text{we have} \quad \rho \sim \mu \{1/(1-r)\};$$

and from (4.43), if $x \neq 0$,

$$(4.44) \quad \sum_0^{\infty} r^{\lambda(n)} e^{inx} \sim \frac{\sin \frac{1}{2}x(1+[\mu])}{\sin \frac{1}{2}x} e^{\frac{1}{2}ix[\mu]},$$

* See the foot-note immediately preceding this.

and, in particular,

$$(4.441) \quad \sum_0^{\infty} (-)^n r^{\lambda(n)} \sim \frac{1}{2} (1 + e^{i\pi[\mu]}),$$

where $[\mu]$ denotes the greatest integer in $\mu \{1/(1-r)\}$.

There is, however, a set of values of r in its progress towards 1 corresponding to which an integer N occurs as an interior point of the interval (ρ, ρ_1) , and, for such values of r , the above equations are no longer true. It is easy to see, for example, that (4.436) must be replaced by

$$\left| \int_0^{\infty} e^{ix[t]} \lambda' \log r e^{\lambda \log r} dt + e^{ix[\rho]} \{1 - \theta(1 - e^{ix})\} \right| < 4\delta,$$

$$e^{-1/\delta} < \theta = e^{\lambda(N) \log r} < e^{-\delta};$$

and, as r increases, θ steadily increases, so that the change, in (4.436), from $e^{ix[\rho]}$ to $e^{ix([\rho]+1)}$ is continuous, but the rate at which the change occurs is unbounded as $r \rightarrow 1$. The same remark applies, of course, to (4.44) and (4.441).

The corresponding result when $x = 0$ is that

$$(4.442) \quad \sum_0^{\infty} r^{\lambda(n)} - \int_0^{\infty} e^{\lambda(t) \log r} dt$$

oscillates between 0 and 1; from this result it is a simple matter to deduce that, as $r \rightarrow 1$,

$$(4.443) \quad \sum_0^{\infty} r^{\lambda(n)} \sim \mu \{1/(1-r)\},$$

and this shows that (4.44) remains true as $x \rightarrow 0$.

The formulae (4.44)–(4.443) are proved on the hypothesis that $\phi'(t) \rightarrow \infty$; they are, for example, true when $\phi(t) = \log \Gamma(1+t)$, *i.e.* $\lambda(t) = \Gamma(1+t)$, or when $\lambda(t) = e^{t \log \log t}$, etc.; and (4.443) happens to be true when $\lambda(t) = c^t$. These results are in agreement with those obtained in the last paragraph of Hardy's paper*.

The methods of this section are not sufficiently delicate to determine the behaviour of the series (4.443) as $r \rightarrow e^{i\theta}$, if $\theta \neq 0$, unless $\lambda'(t)$ is bounded. In the particular case when $\lambda(t) = c^t$ the information is provided by the equation (2.24), and such formulae might be obtained in other particular cases.

* See foot-note * on p. 92.

ON PLANE CURVES

By S. POLLARD.

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Introduction.

1. The elementary theory of plane curves, to which is devoted a chapter of nearly every text-book on the Differential Calculus, contains a number of problems which at first sight appear to be merely trivial, but which eventually require for their solution the employment of quite modern technique and analytical ideas which are not particularly simple. For instance, there is the familiar problem of the arc and its chord. Nearly all the elementary propositions which occur in the chapter on plane curves, to which reference has been made, are based on the assumption that the arc and its chord are asymptotically equivalent, *i.e.*

$$(1.1) \quad \frac{\text{arc}}{\text{chord}} \rightarrow 1$$

or; what is substantially the same thing,

$$(1.2) \quad ds^2 = dx^2 + dy^2;$$

it is convenient to write this in the form

$$(1.3) \quad s'(u) = \phi(u)$$

where, here and throughout the paper, $\phi(u)$ is defined by the equation

$$(1.4) \quad \phi(u) = \sqrt{\{f'(u)^2 + g'(u)^2\}},$$

the curve being given by the equations

$$(1.5) \quad x = f(u), \quad y = g(u).$$

Now (1.3) is far from being universally true. The most that can be asserted, it being assumed that $s'(u)$ and $\phi(u)$ both exist, is

$$s'(u) \geq \phi(u)$$

and it is an easy matter to construct examples of the definite inequality

$$s'(u) > \phi(u).$$

This raises the problem of finding reasonably general conditions under which the fundamental identity (1.3) holds. The usual condition is the continuity of $f'(u)$ and $g'(u)$ throughout an interval in which the point u considered lies. This, of course, assumes a good deal, and the proof is easy.

A much less restricted condition is considered by Jordan, who requires only the continuity of $f'(u)$ and $g'(u)$ at the single point u considered. The proof is naturally somewhat harder.

The desirable condition is plainly the continuity at the point considered of $\phi(u)$. This, however, introduces difficulties; for it can be seen that the proof must eventually depend on an application of the mean-value theorem to the expression

$$\sqrt{[f(v)-f(u)]^2 + [g(v)-g(u)]^2}$$

for the length of the chord joining the points u , v , and, unfortunately, an application in the usual way yields

$$(v-u)\sqrt{\{f'(w_1)\}^2 + \{g'(w_2)\}^2}$$

in which w_1 and w_2 are not necessarily the same; this disagreement seems to make the introduction of the function $\phi(u)$, which would require the two w 's to be the same, impossible.

In § 2 the problem is solved by splitting up (1.3) into its component inequalities, which are then proved by appropriate arguments, one of which expresses the fact that the arc is at least as long as its chord, while the other depends on a somewhat roundabout application of the mean-value theorem.

This proof, in its turn, raises a new problem of considerable interest, that of the relations between $s(u)$ and $\phi(u)$. It appears that $\phi(u)$ is a simple example of a class of functions which may be called pseudo-derivatives. They are not actual derivatives, but they behave like derivatives in that the functions with which they are associated obey the mean-value theorem with the pseudo-derivative in place of the actual

derivative. The existence of these pseudo-derivatives is no accident, but can be traced to the presence of non-additive functions of intervals associated with the primitive function. In the case considered $\phi(u)$ is a pseudo-derivative associated with $s(u)$, and its existence is due to the presence of the length of the chord (u, v) , which is a non-additive function of the interval (u, v) .

It may be observed, in passing, that the proof (§ 26) of this property of $\phi(u)$ is an interesting elementary example of the new technique introduced by Denjoy in his memoir on derivatives.

Another problem is introduced by the fact that a number of the propositions given by the text-books are really propositions in Analysis Situs, and require to be established in accordance with the canons of this subject instead of being merely "verified" by an appeal to a diagram. In particular, there is the question of the meaning of the sign of the radius of curvature, about which opinions differ, and there is a certain amount of confusion.

Once the right method is found this problem presents no serious difficulties, and all that remains is the straightforward task of writing out the theory. Accordingly, only the briefest possible developments are given in the text.

A similar problem arises when a curve is considered, in accordance with modern ideas, as an ordered set of points. It is plain that the same curve can be defined by means of an infinity of equations of the form (1.5), and some account is required of the invariance of the various formulae when one parameter is replaced by another. The solution of this particular problem would appear to be the abandonment of the theory which forbids dy/dx to be anything but another way of writing the x -derivative of y and the employment of the proper differential notation in which dy and dx have separate existences, and dy/dx means dy divided by dx . When this reform is carried out the formulae become a little more complicated, but they have the advantage of being invariant. Thus

$$\rho = p + \frac{d^2p}{d\psi^2},$$

which is not invariant and should preferably be written

$$\rho = p + D_\psi^2 p,$$

is replaced by

$$\rho = p + \frac{d^2p}{d\psi^2} - t \frac{d^2\psi}{d\psi^2},$$

which is invariant.

It should be observed that this new method of writing the formulae has the incontestable advantage of causing the various formulae to appear, as it were, without disguise, so that any selected formula can be verified as it stands by the simple process of calculating the differentials, in the usual systematic way, which it contains.

Quite a different class of problem is raised by the existence of various, apparently equivalent, definitions of the radius of curvature. In many accounts of plane curves the employment of these various definitions is a necessity, due to the fact that the formulae employed are not invariant. In plain language, the appeal to them is simply a device by means of which an author can avoid the troublesome changes of the independent variable which are the logical outcome of his employment of non-invariant formulae. Thus suppose the radius of curvature is originally defined, as is usual, as the ψ -derivative of s . This, of course, supposes that the equations of the curve can be put in the form (1. 5) with ψ in place of u . If it is required to deduce a second expression for ρ which involves derivatives other than ψ -derivatives, then the independent variable must be changed. A convenient but fallacious method of carrying out the change is the adoption of an "equivalent" definition of ρ .

The fact is that the various definitions of ρ are far from being equivalent, and the problem arises of determining the exact significance of each. A complete solution would, as may be supposed, involve rather lengthy analysis, and all that is attempted is the illustration, by means of the simplest possible examples, of the nature of the solution.

The notion of contact calls for some discussion. What difficulties it involves may be seen from the case in which one of the curves is a space-filling curve. In the text these are partly overcome by the introduction of a definition according to which contact is asymmetrical. A distinction is made between the two curves, one of which is selected as curve of reference, and the order of the contact is made to depend on distance measured parallel to the normal to the curve of reference.

The ideal definition of contact would depend on shortest distance, but this is, unfortunately, not easy to calculate. The definition actually adopted is deliberately framed in order to facilitate the discussion of the osculating circle. The natural conception of this is, of course, that of a circle having contact with the curve. It is found, however, that a more satisfactory conception is that of a circle with which the curve has contact. When the circle, and not the curve, is taken as curve of reference, the analysis is considerably simplified.

To make matters quite definite, it should be understood that the curves considered are defined by equations of the form (1. 5). It is assumed that

u varies in an interval (α, β) , in which the functions $f(u)$ and $g(u)$ are single-valued and continuous and, for the most part, differentiable and of bounded variation.

A special case is that of Lipschitz curves, for which $f(u)$ and $g(u)$ have bounded derivatives. These are discussed in § 23.

The fundamental theorem.

2. THEOREM. *If $\phi(u)$ is continuous at γ , then $s'(\gamma)$ exists and has the value*

$$\phi(\gamma) = \sqrt{[f'(\gamma)^2 + g'(\gamma)^2]}.$$

It simplifies the discussion to consider separately the right- and left-hand differential coefficients of $s(u)$, the conclusion desired being equivalent to the two identities

$$s'(\gamma+0) = \phi(\gamma), \quad s'(\gamma-0) = \phi(\gamma).$$

Of these only the first need be discussed in detail, since the analysis required by the second is at once obtained by making obvious modifications.

This first identity can be still further divided into the inequalities

$$(2.1) \quad s_+(\gamma) \geq \phi(\gamma), \quad s^+(\gamma) \leq \phi(\gamma),$$

the first of which offers little difficulty, while the second is somewhat harder. By $s^+(\gamma)$ is meant, in accordance with the customary notation, the upper derivative on the right, *i.e.*

$$s^+(\gamma) = \lim_{h \rightarrow 0} \frac{s(\gamma+h) - s(\gamma)}{h},$$

where $h \rightarrow 0$ by positive values. The corresponding lower limit is $s_+(\gamma)$.

Let u be greater than γ , and take between γ and u a series of points

$$\gamma = u_0 < u_1 < \dots < u_r < \dots < u_n = u.$$

If

$$(2.2) \quad \psi_r(t) = \sqrt{[f(t) - f(u_{r-1})]^2 + [g(t) - g(u_{r-1})]^2},$$

then, as can be shown directly from the definition, $s(u) - s(\gamma)$ is the upper bound of the sums

$$\sigma(u) = \sum_{r=1}^n \psi_r(u_r).$$

An immediate consequence of this is

$$s(u) - s(\gamma) \geq \sqrt{[f(u) - f(\gamma)]^2 + [g(u) - g(\gamma)]^2},$$

which, after division by $u - \gamma$ and passage to the limit, yields

$$(2.3) \quad s_+(\gamma) \geq \sqrt{[f'(\gamma)]^2 + [g'(\gamma)]^2},$$

which is the first part of (2.1).

For the second part there follows, from (2.2),

$$\begin{aligned} \psi_r(t) \psi'_r(t) &= \{f(t) - f(u_{r-1})\} f'(t) + \{g(t) - g(u_{r-1})\} g'(t) \\ &\leq \psi_r(t) \phi(t), \end{aligned}$$

by the inequality

$$(pp' + qq')^2 \leq (p^2 + q^2)(p'^2 + q'^2).$$

The derivative $\psi'_r(t)$ exists and is not greater than $\phi(t)$.

Now, since ϕ is continuous at γ , it must be bounded in some interval $(\gamma - l, \gamma + l)$ surrounding γ . Restrict u to lie in this interval and let $\Phi(u)$ be the upper bound of ϕ in (γ, u) .

Then, for values of t between γ and u ,

$$\psi'_r(t) \leq \Phi(u)$$

so that

$$\psi_r(u_r) = \psi_r(u_r) - \psi_r(u_{r-1}) \leq (u_r - u_{r-1}) \Phi(u)$$

and

$$\sigma(u) \leq \sum_{r=1}^n (u_r - u_{r-1}) \Phi(u) = (u - \gamma) \Phi(u),$$

which gives, on passage to the upper bound,

$$(2.4) \quad s(u) - s(\gamma) \leq (u - \gamma) \Phi(u),$$

whence

$$s^+(\gamma) = \overline{\lim} \frac{s(u) - s(\gamma)}{u - \gamma} \leq \overline{\lim} \Phi(u).$$

But, since ϕ is continuous at γ , $\Phi(u) \rightarrow \phi(\gamma)$. This at once yields the second part of (2.1).

The perpendicular distance from a point to a line.

3. Let L be a line whose equations are

$$x = a + lv, \quad y = \beta + mv,$$

and let (a, b) be a point which does not lie on it. There is no loss of generality in supposing that

$$l^2 + m^2 = 1.$$

Expressions for p , the perpendicular distance from (a, b) to L , are well known, but they are usually obtained with the aid of a diagram. The following analysis yields an expression free from this defect.

The line through (a, b) perpendicular to L is given by the equations

$$x = a + mr, \quad y = b - lr.$$

It meets L at the point at which

$$r = \begin{vmatrix} l & a-a \\ m & b-\beta \end{vmatrix} \\ = \Delta(a-a, \quad b-\beta),$$

say, so that

$$p = \pm \Delta(a-a, \quad b-\beta).$$

To decide between these expressions, consider the angle θ which the segment $(a, b) \rightarrow (x, y)$ makes with the x -axis. Plainly

$$\tan \theta = \frac{y-b}{x-a} = \frac{mv + \beta - b}{lv + a - a} \\ \sec^2 \theta \frac{d\theta}{dv} = \frac{\Delta(a-a, \quad b-\beta)}{(lv + a - a)^2}$$

and the sign of $d\theta/dv$ is that of $\Delta(a-a, \quad b-\beta)$.

Now it is usual to fix the signs of l, m so that the direction of the line, *i.e.* the order of its points, is that of increasing v . This being so, it is plain that

$$p = +\Delta(a-a, \quad b-\beta)$$

is a reasonable choice, for, with p determined in this way, θ increases if p is positive and decreases if p is negative, as (x, y) moves along the line in the prescribed direction.

The foot of the perpendicular is given by

$$x_0 = l^2 a + lm(b-\beta) + m^2 a, \quad y_0 = m^2 b + lm(a-a) + l^2 \beta.$$

Arithmetical definitions of $x_0, y_0, r, t, p, \theta, \psi, \chi$.

4. It is assumed that the particular point under consideration on the curve corresponds to a value of u at which $\phi(u)$ is continuous and does not vanish, so that $ds \neq 0$ and

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1,$$

and $dx/ds, dy/ds$ are a pair of direction cosines.

The following definitions, which do not depend upon a diagram, may now be made :

$$\lambda = \frac{dx}{ds}, \quad \mu = \frac{dy}{ds};$$

$$\cos \psi = \lambda, \quad \sin \psi = \mu;$$

$$p = \begin{vmatrix} \lambda & 0-x \\ \mu & 0-y \end{vmatrix} = x\mu - y\lambda;$$

$$x_0 = \mu(x\mu - y\lambda), \quad y_0 = \lambda(y\lambda - x\mu);$$

$$t = \frac{x-x_0}{\lambda} = \frac{y-y_0}{\mu};$$

$$r = \sqrt{x^2 + y^2};$$

$$\cos \theta = \frac{x}{r}, \quad \sin \theta = \frac{y}{r};$$

$$\chi = \psi - \theta.$$

In accordance with § 3, θ increases if p is positive, and decreases if p is negative, as (x, y) moves along the curve in the direction of increasing u .

First order formulae.

5. From the above definitions the usual formulae follow :

$$dx = ds \cos \psi, \quad dy = ds \sin \psi;$$

$$dx \sin \psi - dy \cos \psi = 0, \quad dx \cos \psi + dy \sin \psi = ds;$$

$$rdr = xdx + ydy;$$

$$r^2 d\theta = xdy - ydx = pds;$$

$$x_0 = p \frac{dy}{ds}, \quad y_0 = -p \frac{dx}{ds};$$

$$t = \frac{xdx + ydy}{ds} = x \cos \psi + y \sin \psi = r \frac{dr}{ds};$$

$$\cos \chi = \frac{dr}{ds}, \quad \sin \chi = r \frac{d\theta}{ds}.$$

Equivalent parameters.

6. If U is a continuous strictly increasing function of u , then there is a one-one continuous correspondence between u and U which preserves the order, and the curve under consideration is equally well defined by the equations

$$x = F(U), \quad y = G(U),$$

where

$$F(U) = f\{u(U)\}, \quad G(U) = g\{u(U)\}.$$

The new parameter U may be described as *equivalent* to the old. As an example, s is always equivalent in this way to u .

If, further, at the point considered U is a differentiable function of u with a differential coefficient not equal to zero, and therefore positive, then any function of u is differentiable if, and only if, the corresponding function of U is differentiable. Moreover, the same is true when differentiability of order n higher than the first is considered provided that, in addition, U is a function of u which is differentiable n times.

In these last mentioned circumstances, the parameters u and U will be said to be *equivalent for n -th order purposes*.

When the conditions of § 4 hold u and s are evidently equivalent for first order purposes.

With expressions such as dy/dx interpreted as quotients of differentials, neither the definitions of § 4 nor the formulae of § 5 are altered in any way when u is replaced by an equivalent parameter. In particular, they remain true when s is regarded as the independent variable.

The curvature.

7. The curvature is defined as the s -derivative of ψ :

$$(7.1) \quad \kappa = D_s \psi.$$

It is necessary and sufficient for the existence of a definite and finite curvature that ψ be a differentiable function of s .

The exact significance of this definition is a matter of some interest, and a brief investigation may be not out of place. To begin with, the definition requires, besides the existence of a finite length, the existence of a definite tangent at all points in the neighbourhood of the point considered. For this it is sufficient that $f'(u)$ and $g'(u)$ exist and do not vanish together, *i.e.* $\phi(u) \neq 0$.

Two cases may now be distinguished. In the first $f'(\gamma) \neq 0$, where γ is the parameter of the point under consideration, and in the second $g'(\gamma) \neq 0$.

Suppose that $f'(\gamma) \neq 0$. Then $\tan \psi(\gamma)$ is finite, and ψ is a differentiable function of s if, and only if, the same is true of $\tan \psi$, and it is readily seen that the necessary and sufficient condition for a curvature κ is

$$\frac{\frac{g'(u) - g'(\gamma)}{f'(u) - f'(\gamma)}}{s(u) - s(\gamma)} \rightarrow \frac{\phi(\gamma)^2}{f'(\gamma)^2} \kappa$$

or

$$(7.2) \quad \frac{f'(\gamma) g'(u) - g'(\gamma) f'(u)}{s(u) - s(\gamma)} \cdot \frac{f'(\gamma)}{f'(u)} \rightarrow \phi(\gamma)^2 \kappa.$$

Incidentally, $f'(u) \neq 0$ for all values of u sufficiently near γ and

$$(7.21) \quad \frac{g'(u)}{f'(u)} \rightarrow \frac{g'(\gamma)}{f'(\gamma)}.$$

When $g'(\gamma) \neq 0$, (7.2) is replaced by

$$(7.3) \quad \frac{f'(\gamma) g'(u) - g'(\gamma) f'(u)}{s(u) - s(\gamma)} \cdot \frac{g'(\gamma)}{g'(u)} \rightarrow \phi(\gamma)^2 \kappa.$$

It is easily verified that, when $f'(\gamma) \neq 0$ and $g'(\gamma) \neq 0$ are both true,

$$\frac{f'(u)}{f'(\gamma)} \sim \frac{g'(u)}{g'(\gamma)}.$$

From (7.21) and the result which replaces it when $g'(\gamma) \neq 0$ it follows that ψ is a continuous function of u , i.e. the curve has a continuously turning tangent at γ .

Next suppose that $\phi(u)$ is continuous at γ . Then it follows from (7.21) and its analogue that $f'(u)$ and $g'(u)$ are separately continuous at γ . Since, further, $s'(\gamma) = \phi(\gamma)$, (7.2) and (7.3) reduce to the single condition

$$(7.4) \quad \frac{f'(\gamma) g'(u) - g'(\gamma) f'(u)}{u - \gamma} \rightarrow \phi(\gamma)^3 \kappa,$$

$$\begin{aligned} \text{and} \quad \kappa &= \frac{1}{\phi(\gamma)^3} \lim_{u \rightarrow \gamma} \frac{f'(\gamma) g'(u) - g'(\gamma) f'(u)}{u - \gamma} \\ &= \frac{1}{\phi(\gamma)^3} \lim_{u \rightarrow \gamma} \left\{ f'(\gamma) \frac{g'(u) - g'(\gamma)}{u - \gamma} - g'(\gamma) \frac{f'(u) - f'(\gamma)}{u - \gamma} \right\}. \end{aligned}$$

Condition (7.4) is plainly equivalent to the existence of the second order derivative of $f'(\gamma)g(u) - g'(\gamma)f(u)$. If this function be denoted by $F(u)$, then

$$\kappa = \frac{F''(\gamma)}{\phi(\gamma)^3}.$$

Finally, suppose that $\phi(u)$ is continuous not merely at γ , but also throughout some neighbourhood of γ . Then, throughout this neighbourhood,

$$\cos \psi = D_s x, \quad \sin \psi = D_s y.$$

From these it follows that κ exists if, and only if, x and y are both twice differentiable functions of s .

Incidentally, in the usual way,

$$-\sin \psi \cdot \kappa = D_s^2 x, \quad \cos \psi \cdot \kappa = D_s^2 y$$

and

$$\kappa^2 = (D_s^2 x)^2 + (D_s^2 y)^2.$$

If $\phi'(\gamma)$ exists, then s and u are equivalent for second order purposes, and x and y must also be twice differentiable functions of u , *i.e.* the second order conditions of the next section must hold.

Second order conditions.

8. In addition to what has already been stated in § 4, it is now assumed that x and y are twice-differentiable functions of u .

An immediate consequence is the equivalence of u and s for second order purposes. Also κ is now always definite and finite.

The osculating circle and the radius and centre of curvature.

9. The osculating circle is defined precisely as the circle with which the curve has second order contact at the point considered, and its centre and radius are respectively the centre and radius of curvature.

With this definition the curvature and radius of curvature are quite distinct ideas, and it is not immediately apparent that there is any connexion between the two. A critical investigation of the matter is given in § 15, where it is shown that the existence of a definite curvature, different from zero, implies considerably more than the existence of an osculating circle. Indeed, a curve may possess an osculating circle without being even rectifiable.

When the second order conditions hold, the investigation of the osculating circle is a simple matter, conducted along classical lines.

If the circle is $(X-\xi)^2+(Y-\eta)^2=R^2$

(X and Y being current coordinates), necessary and sufficient conditions for second order contact are

$$dP = 0, \quad d^2P = 0,$$

where $P = (x-\xi)^2+(y-\eta)^2-R^2$.

Provided that $dx d^2y - dy d^2x \neq 0$, these yield

$$(9.1) \quad \xi - x = -\mu\rho, \quad \eta - y = \lambda\rho,$$

$$R = \pm\rho,$$

where

$$(9.2) \quad \rho = \frac{ds^3}{dx d^2y - dy d^2x}.$$

It is convenient to fix the sign of R by putting

$$R = \rho.$$

The radius and centre of curvature are now as given by (9.1) and (9.2).

It is easily verified that $dx d^2y - dy d^2x \neq 0$ is equivalent to $d\psi \neq 0$ and therefore to $\kappa \neq 0$, and that

$$\rho = \frac{1}{\kappa}.$$

If $d\psi = 0$ there is no circle with second order contact, its place being taken by the tangent, which now has second order contact.

Position of the centre of curvature.

10. With the notation of § 3,

$$\begin{aligned} \Delta(\xi-x, \eta-y) &= \begin{vmatrix} \lambda & \xi-x \\ \mu & \eta-y \end{vmatrix} \\ &= \begin{vmatrix} \lambda & -\mu\rho \\ \mu & \lambda\rho \end{vmatrix} \\ &= \rho. \end{aligned}$$

Thus the centre of curvature is on the left of the tangent if ρ is positive and it is on the right if ρ is negative.

The positive direction of the tangent may be described as that which a particle constrained to move along the curve in the sense of increasing u would take if it were allowed to leave the curve at (x, y) .

Second order formulae.

11. There are a great many formulae involving second order differentials, of which the following are specimens. They can all be obtained from the first order formulae by differentiation, account being taken of the relations between the various first order differentials. It is assumed throughout that the second order conditions are satisfied. Specimens are :

$$\bar{d}^2 s = \lambda \bar{d}^2 x + \mu \bar{d}^2 y,$$

$$\bar{d}^2 x = \lambda \bar{d}^2 s - dy d\psi = \lambda \bar{d}^2 s - \mu ds d\psi,$$

$$\bar{d}^2 y = \mu \bar{d}^2 s + dx d\psi = \mu \bar{d}^2 s + \lambda ds d\psi,$$

$$d\psi = \frac{dx \bar{d}^2 y - dy \bar{d}^2 x}{ds^2},$$

$$dp = t d\psi,$$

$$dt = ds - p d\psi,$$

$$\rho = \frac{ds}{d\psi} = p + \frac{dt}{d\psi} = \frac{ds^3}{dx \bar{d}^2 y - dy \bar{d}^2 x},$$

this last provided that $d\psi \neq 0$, i.e. the point on the curve is not one where the tangent is stationary.

All these formulae are invariant in the sense that they do not change when u is replaced by an equivalent parameter U .

The formula

$$\rho = \frac{\{1 + (dy/dx)^2\}^{3/2}}{d^2 y / dx^2},$$

which is so often employed, has no place in the present scheme because it is not invariant, being true, in general, only when x is the independent variable.

It should be written in the unambiguous form

$$\rho = \frac{\{1 + (D_x y)^2\}^{3/2}}{D_x^2 y},$$

or, preferably, replaced by

$$\rho = \frac{(dx^2 + dy^2)^{3/2}}{dx \, d^2y - dy \, d^2x},$$

which, besides being invariant, is also compact.

Third order formulae.

12. If x and y are thrice differentiable functions of u , third order formulae may be obtained, *e.g.*

$$d^2p = dt \, d\psi + t \, d^2\psi,$$

$$d^2t = d^2s - p \, d^2\psi - t \, d\psi^2,$$

$$\rho = p + \frac{d^2p}{d\psi^2} - t \frac{d^2\psi}{d\psi^2},$$

$$p = \frac{d^2(s-t)}{d^2\psi} - t \frac{d\psi^2}{d^2\psi},$$

provided that none of the differentials which occur as denominators vanish.

Reversion-invariance.

13. The reverse of the original curve is

$$x = f(a + \beta - v), \quad y = g(a + \beta - v),$$

where v increases from a to β .

The point $u = \gamma$ on the original curve is said to correspond to $v = a + \beta - \gamma$ on the reverse. The coordinates of these two points are, of course, identical.

The length of arc $\sigma(v)$ is, for the reverse curve, measured from $v = a$, which corresponds to $u = \beta$, so that

$$\sigma(v) - \sigma(a) = s(\beta) - s(u).$$

If dx, dy, ds denote differentials formed on the assumption that x and y are given by the original equations, and $\delta x, \delta y, \delta \sigma$ those formed on the assumption that x and y are given by the equations of the reverse, and if $du = dv$, then $\delta x = -dx, \delta y = -dy, \delta \sigma = ds$.

It follows that, if $r_1, t_1, p_1, \theta_1, \psi_1, \chi_1$ refer to the reverse curve, then

$$r_1 = r, \quad \delta r = -dr, \quad t_1 = -t, \quad p_1 = -p, \quad \theta_1 = \theta, \quad \delta\theta_1 = -d\theta,$$

$$\psi_1 = -\psi, \quad \chi_1 = -(\psi + \theta).$$

The foot of the perpendicular is the same for both curves.

Further differentiations yield

$$\delta^n x = (-)^n d^n x, \quad \delta^n y = (-)^n d^n y, \quad \delta^n \sigma = (-)^{n-1} d^n s.$$

Thus

$$\rho_1 = -\rho, \quad \xi_1 = \xi, \quad \eta_1 = \eta.$$

Transformation-invariance.

14. Let the original curve be transformed into a curve with equations

$$x = l_1 f(u) + m_1 g(u), \quad y = l_2 f(u) + m_2 g(u),$$

where

$$l_1^2 + m_1^2 = 1, \quad l_2^2 + m_2^2 = 1, \quad l_1 l_2 + m_1 m_2 = 0,$$

$$D = \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \pm 1.$$

The point $u = \gamma$ on the original curve is said to correspond to $u = \gamma$ on the new curve. For each curve, u increases from α to β .

If $r_1, t_1, \dots$ refer to the new curve, then it is easily verified that

$$r_1 = r, \quad t_1 = t, \quad p_1 = D \cdot p, \quad d\theta_1 = D \cdot d\theta, \quad d\psi_1 = D \cdot d\psi, \quad \chi_1 = D \cdot \chi,$$

$$\rho_1 = D \cdot \rho, \quad \xi_1 = l_1 \xi + m_1 \eta, \quad \eta_1 = l_2 \xi + m_2 \eta.$$

Conditions for the existence of an osculating circle.

15. The osculating circle has been defined as a circle with which the curve has second order contact*.

Let P be the point whose parameter is γ and Q a point whose parameter u tends to γ . Further, let C be any point different from P , and let the circle with centre C and radius CP cut CQ at Q' . Then, as readily

* See § 22.

follows from the definition, the necessary and sufficient condition for second order contact at P , the circle being the curve of reference, is

$$(15.01) \quad \frac{QQ'}{PQ^2} \rightarrow 0$$

as $u \rightarrow \gamma$.

If C is (ξ, η) and the radius CP , measured without regard to sign, is R , this condition becomes

$$\frac{\sqrt{[\{f(u) - \xi\}^2 + \{g(u) - \eta\}^2]} - R}{\{f(u) - f(\gamma)\}^2 + \{g(u) - g(\gamma)\}^2} \rightarrow 0.$$

Now, C being supposed different from P , $R \neq 0$. Thus, since

$$R + \sqrt{[\{f(u) - \xi\}^2 + \{g(u) - \eta\}^2]} \rightarrow 2R,$$

the above condition is equivalent to

$$(15.1) \quad \frac{\{f(u) - \xi\}^2 + \{g(u) - \eta\}^2 - R^2}{\{f(u) - f(\gamma)\}^2 + \{g(u) - g(\gamma)\}^2} \rightarrow 0.$$

This fixes (ξ, η) . For, if possible, let there be a second centre D satisfying the condition. If DQ meets the second circle at Q'' , then

$$\frac{QQ''}{PQ^2} \rightarrow 0$$

and eventually

$$\frac{Q'Q''}{PQ'^2} \rightarrow 0;$$

this shows that the two circles have second order contact at P , which is readily proved to be impossible.

When R is given its value (15.1) becomes

$$(15.2) \quad \frac{\{f(u) - f(\gamma)\} \{f(u) + f(\gamma) - 2\xi\} + \{g(u) - g(\gamma)\} \{g(u) + g(\gamma) - 2\eta\}}{\{f(u) - f(\gamma)\}^2 + \{g(u) - g(\gamma)\}^2} \rightarrow 0.$$

Either of the conditions

$$(15.3) \quad \frac{\{f(u) - f(\gamma)\} \{f(u) - \xi\} + \{g(u) - g(\gamma)\} \{g(u) - \eta\}}{\{f(u) - f(\gamma)\}^2 + \{g(u) - g(\gamma)\}^2} \rightarrow \frac{1}{2},$$

$$(15.4) \quad \frac{\{f(u) - f(\gamma)\} \{f(\gamma) - \xi\} + \{g(u) - g(\gamma)\} \{g(\gamma) - \eta\}}{\{f(u) - f(\gamma)\}^2 + \{g(u) - g(\gamma)\}^2} \rightarrow -\frac{1}{2},$$

is equivalent to (15.2).

These conditions do not require the curve to be either differentiable at γ or rectifiable in the neighbourhood of γ .

It is plain, however, that (15.01) requires the angle QPQ' to have zero as its limit, so that the direction PQ tends to the direction of the tangent at P to the circle. It follows that there are two numbers l, m having the ratio $\eta - g(\gamma) : f(\gamma) - \xi$ and not both vanishing, such that

$$(15.5) \quad \frac{f(u) - f(\gamma)}{\sqrt{\{f(u) - f(\gamma)\}^2 + \{g(u) - g(\gamma)\}^2}} \rightarrow l,$$

$$\frac{g(u) - g(\gamma)}{\sqrt{\{f(u) - f(\gamma)\}^2 + \{g(u) - g(\gamma)\}^2}} \rightarrow m.$$

If ρ is such that

$$\xi - f(\gamma) = -\rho m, \quad \eta - g(\gamma) = \rho l,$$

then (15.4) becomes

$$(15.6) \quad \frac{l\{g(u) - g(\gamma)\} - m\{f(u) - f(\gamma)\}}{\{f(u) - f(\gamma)\}^2 + \{g(u) - g(\gamma)\}^2} \rightarrow \frac{1}{2}\rho^{-1}.$$

The necessary and sufficient condition for an osculating circle thus takes the form that l and m must exist such that (15.5) and (15.6) hold, ρ being a finite limit different from zero.

Incidentally,

$$R^2 = \rho^2.$$

The investigation is greatly simplified if $f'(\gamma)$ and $g'(\gamma)$ exist and do not both vanish. For then the existence of l and m is assured, these being given by

$$l = \frac{f'(\gamma)}{\phi(\gamma)}, \quad m = \frac{g'(\gamma)}{\phi(\gamma)}.$$

Further

$$\{f(u) - f(\gamma)\}^2 + \{g(u) - g(\gamma)\}^2 \sim (u - \gamma)^2 \phi(\gamma)^2,$$

and the necessary and sufficient condition takes the simple form

$$(15.7) \quad \frac{f'(\gamma)\{g(u) - g(\gamma)\} - g'(\gamma)\{f(u) - f(\gamma)\}}{(u - \gamma)^2} \rightarrow \text{finite limit} \neq 0.$$

Also

$$\rho = \lim \frac{\frac{1}{2}(u - \gamma)^2 \phi(\gamma)^3}{f'(\gamma)\{g(u) - g(\gamma)\} - g'(\gamma)\{f(u) - f(\gamma)\}}.$$

With the notation of § 7, (15.7) may be rewritten in the form

$$(15.8) \quad \frac{F(u) - F(\gamma)}{(u - \gamma)^2} \rightarrow \text{finite limit} \neq 0.$$

This yields $F'(\gamma) = 0$

and also

$$\frac{F(\gamma + h) - F(\gamma)}{h^2}, \quad \frac{F(\gamma - h) - F(\gamma)}{h^2} \rightarrow \text{finite limit} \neq 0,$$

so that
$$\frac{F(\gamma + h) - 2F(\gamma) + F(\gamma - h)}{h^2} \rightarrow \text{finite limit} \neq 0,$$

and a necessary condition is the existence of the symmetrical second derivative at γ of $F(x)$.

A sufficient, but not necessary, condition is the existence of an ordinary second derivative $F''(\gamma)$ different from zero. With this condition

$$\rho = \frac{\phi(\gamma)^3}{F''(\gamma)}.$$

This last shows that, when $\phi(u)$ is continuous at γ , the existence of a definite curvature different from zero is sufficient to ensure the existence of an osculating circle. Moreover,

$$\rho = \frac{1}{\kappa}.$$

With the second order conditions, $F''(\gamma)$ plainly exists, and ρ , as here defined, coincides with ρ as defined in § 9.

Example of a non-rectifiable curve with an osculating circle.

16. Divide the interval $(0, 1)$ by the points

$$0, \dots, n^{-\frac{1}{2}}, \dots, 3^{-\frac{1}{2}}, 2^{-\frac{1}{2}}, 1.$$

Define
$$g(n^{-\frac{1}{2}}) = \frac{1}{2}n^{-\frac{1}{2}} + n^{-1} \quad (n \text{ odd}),$$

$$= \frac{1}{2}n^{-\frac{1}{2}} \quad (n \text{ even}),$$

and make $g(u)$ linear in the intervals between the points. Next, define

$g(u)$ for negative values of u by writing $g(-u) = g(u)$, and finally put $g(0) = 0$.

The curve $x = u, \quad y = g(u)$

lies between the curves

$$x = u, \quad y = \frac{1}{2}u^2;$$

$$x = u, \quad y = \frac{1}{2}u^2 + u^4;$$

and, therefore, as is easily shown by applying (15.01), has at $u = 0$ the osculating circle with centre $(0, 1)$ and radius 1.

The arc $0 \leq u \leq (2m+1)^{-\frac{1}{2}}$, however, has a length greater than

$$\frac{1}{2m+1} + \frac{1}{2m+3} + \dots = \infty,$$

and the curve is not rectifiable in any interval of u containing the origin.

It should be observed that, at $u = 0$, x and y are both differentiable functions of u .

Example of a curve at a point of which $ds^2 > dx^2 + dy^2$.

17. Write
$$u_n = \frac{1}{n^2} + \frac{1}{(n+1)^2} + \dots,$$

and define $g(u)$ by taking

$$\begin{aligned} g(u) &= (u_n - u_{n+1}) \sin \frac{2\pi(u - u_{n+1})}{u_n - u_{n+1}} \\ &= \frac{1}{n^2} \sin 2n^2\pi(u - u_{n+1}), \end{aligned}$$

in $u_{n+1} \leq u \leq u_n$, together with

$$g(-u) = g(u), \quad g(0) = 0.$$

The function $g(u)$, so defined, consists of a number of continuous differentiable pieces joined together at the points u_n , together with the limit point 0 at $u = 0$.

As $u \rightarrow u_n$ from the right

$$g(u) \rightarrow \frac{1}{(n-1)^2} \sin 2(n-1)^2\pi(u_n - u_n) = \frac{1}{(n-1)^2} \sin 0 = 0,$$

and as $u \rightarrow u_n$ from the left

$$g(u) \rightarrow \frac{1}{n^2} \sin 2n^2 \pi (u_n - u_{n+1}) = \frac{1}{n^2} \sin 2\pi = 0.$$

Since $g(u_n) = 0$, $g(u)$ is continuous at u_n .

Again

$$g'(u_n + 0) = \frac{1}{(n-1)^2} 2(n-1)^2 \pi \cos 2(n-1)^2 \pi (u_n - u_n) = 2\pi \cos 0 = 2\pi,$$

$$g'(u_n - 0) = \frac{1}{n^2} 2n^2 \pi \cos 2n^2 \pi (u_n - u_{n+1}) = 2\pi \cos 2\pi = 2\pi,$$

so that $g(u)$ is differentiable at u_n .

It immediately follows that $g(u)$ is continuous and differentiable at $-u_n$ also, and the pieces are continuously and smoothly joined together, so that $g(u)$ is continuous and differentiable, except possibly at the origin.

Corresponding to any value of u other than zero, let ν be the integer such that $u_{\nu+1} < |u| \leq u_\nu$. Then, as $u \rightarrow 0$, $\nu \rightarrow \infty$, and

$$|g(u)| \leq \frac{1}{\nu^2} \rightarrow 0,$$

$$\begin{aligned} \left| \frac{g(u)}{u} \right| &\leq \frac{1}{\nu^2} / u_{\nu+1} \\ &= \frac{1}{\nu^2} / \left\{ \frac{1}{(\nu+1)^2} + \frac{1}{(\nu+2)^2} + \dots \right\} \\ &< \frac{1}{\nu^2} / \left\{ \frac{1}{(\nu+1)^2} + \frac{1}{(\nu+2)^2} + \dots + \frac{1}{(2\nu)^2} \right\} \\ &\leq \frac{1}{\nu^2} / \frac{\nu}{(2\nu)^2} = \frac{4}{\nu} \rightarrow 0. \end{aligned}$$

It follows that $g(u)$ is continuous and differentiable at the origin, with $g'(0) = 0$.

In the interval (u_{n+1}, u_n) the total variation of $g(u)$ is plainly $2/n^2$, and so the total variation in $(0, u_n)$ is $2u_n$.

Consider now the curve

$$x = u, \quad y = g(u).$$

It is continuous and rectifiable, and it has everywhere a definite tangent. It does not cross itself.

In the interval (u_{n+1}, u_n) x_u and y_u are continuous and so

$$\begin{aligned} s(u_n)-s(u_{n+1}) &= \int_{u_{n+1}}^{u_n} \sqrt{\{1+g'(u)^2\}}\, du \\ &= \int_{u_{n+1}}^{u_n} \sqrt{\{1+4\pi^2 \cos^2 2n^2 \pi(u-u_{n+1})\}}\, du \\ &= \frac{1}{2\pi n^2} \int_0^{2\pi} \sqrt{\{1+4\pi^2 \cos^2 \theta\}}\, d\theta = \frac{\sigma}{n^2}, \end{aligned}$$

where σ is a positive constant.

By addition,
$$s(u_n)-s(0) = \sigma u_n.$$

Thus, if u is positive and v is determined as before, $s(u)-s(0)$ lies between σu_{v+1} and σu_v , so that

$$\frac{\sigma u_{v+1}}{u} \leq \frac{s(u)-s(0)}{u-0} \leq \frac{\sigma u_v}{u},$$

which gives

$$\frac{\sigma u_{v+1}}{u_v} \leq \frac{s(u)-s(0)}{u-0} \leq \frac{\sigma u_v}{u_{v+1}}.$$

Now

$$u_v > \int_v^\infty \frac{dt}{t^2} = \frac{1}{v},$$

so that

$$1-\frac{u_{v+1}}{u_v} = \frac{1}{v^2 u_v} < \frac{1}{v} \rightarrow 0,$$

and

$$\frac{u_{v+1}}{u_v} \rightarrow 1,$$

which shows that, as $u \rightarrow 0$ by positive values,

$$\frac{s(u)-s(0)}{u-0} \rightarrow \sigma,$$

and so eventually

$$s'(0) = \sigma.$$

Thus, at $u = 0$,
$$ds = \sigma du, \quad dx = du, \quad dy = 0,$$

$$ds^2 = \sigma^2 du^2, \quad dx^2 + dy^2 = du^2.$$

Plainly, however,

$$\sigma = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{\{1+4\pi^2 \cos^2 \theta\}}\, d\theta > 1,$$

and so

$$ds^2 > dx^2 + dy^2,$$

the inequality to be established.

It may be noted that

$$\begin{aligned}\sigma &= \frac{\sqrt{(1+4\pi^2)}}{2\pi} \int_0^{2\pi} \sqrt{\left\{1 - \frac{4\pi^2}{1+4\pi^2} \sin^2 \theta\right\}} d\theta \\ &\geq \int_0^{2\pi} \sqrt{\{1 - \sin^2 \theta\}} d\theta = 4.\end{aligned}$$

Example of a curve at a point of which dx and dy exist, but ds does not exist.

18. The procedure just adopted requires to be varied a little. The points u_n are taken as before, and the same definition given of $g(u)$, except that when n is of the form $4p^2+1$, where p is an integer,

$$g(u) = \frac{1}{n^2} \sin 2n^2 \pi (u - u_{n+1})$$

when either $u_{n+1} \leq u \leq \frac{3}{4}u_{n+1} + \frac{1}{4}u_n$ or $\frac{1}{4}u_{n+1} + \frac{3}{4}u_n \leq u \leq u_n$, and

$$g(u) = \frac{1}{n^2} \cos 2n^2 \pi (u - \frac{1}{4}u_n - \frac{3}{4}u_{n+1})$$

when $\frac{3}{4}u_{n+1} + \frac{1}{4}u_n \leq u_n \leq \frac{1}{4}u_{n+1} + \frac{3}{4}u_n$.

By the argument employed in § 17 it can be shown that this modification has no effect on the continuity and differentiability of $g(u)$. It is plain, however, that it makes the curve

$$x = u, \quad y = g(u)$$

have considerably more length than before in the interval (u_{n+1}, u_n) . The new length in this interval is in fact, as is easily verified,

$$s(u_n) - s(u_{n+1}) = \frac{1}{n^2} \frac{\sigma + \tau_n}{2},$$

where

$$\tau_n = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{\{1 + 4n^2 \pi^2 \sin^2 \theta\}} d\theta.$$

Now

$$\tau_n \leq \frac{1}{2\pi} \int_0^{2\pi} \sqrt{\{n^2 + 4n^2 \pi^2 \sin^2 \theta\}} d\theta = n\tau,$$

so that the total increase of length in $(0, u_1)$ is not more than

$$\sum_{p=0}^{\infty} \frac{\sigma}{2} \left\{ \frac{1}{(4p^2+1)^2} + \frac{1}{(4p^2+1)} \right\}$$

which is finite, and the new curve is rectifiable.

If possible, let ds exist for $u = 0$. Then

$$\begin{aligned} \frac{s(u_n) - s(u_{n+1})}{u_n} &= \frac{s(u_n) - s(0)}{u_n - 0} - \frac{s(u_{n+1}) - s(0)}{u_{n+1} - 0} \frac{u_{n+1}}{u_n} \\ &\rightarrow s'(0) - s'(0) = 0. \end{aligned}$$

But when n is of the form $4p^2+1$,

$$\frac{s(u_n) - s(u_{n+1})}{u_n} = \frac{\sigma + \tau_n}{2n^2 u_n} \sim \frac{\tau_n}{2n} > \frac{1}{2} \int_0^{2\pi} |\sin \theta| d\theta = 2,$$

and cannot tend to zero. Thus ds does not exist for $u = 0$.

Example of a rectifiable curve which has an osculating circle but no definite curvature.

19. The methods of § 17 readily yield rectifiable curves for which the existence of an osculating circle does not imply a definite curvature.

For let $g(u)$ be the function defined in § 17, and consider the curve

$$x = u, \quad y = \frac{1}{2}u^2 + ug(u).$$

In the interval $u_{n+1} \leq u \leq u_n$,

$$u \geq u_{n+1} > \frac{1}{n+1} \geq \frac{1}{3n},$$

so that

$$\frac{1}{n^2} < 9u^2$$

and

$$(19.1) \quad |g(u)| \leq 9u^2,$$

which ensures that the curve in question lies between

$$x = u, \quad y = \frac{1}{2}u^2 \pm 9u^3,$$

and so, at $u = 0$, has the osculating circle, with centre $(0, 1)$ and radius 1, which is common to these curves.

The length of the arc (u_{n+1}, u_n) is given by

$$\begin{aligned}
 s(u_n) - s(u_{n+1}) &= \int_{u_{n+1}}^{u_n} \sqrt{1+y'^2} \, du \\
 &\leq \int_{u_{n+1}}^{u_n} (1+|y'|) \, du \\
 &\leq \int_{u_{n+1}}^{u_n} \{1+u+|g(u)|+u|g'(u)|\} \, du \\
 &\leq \int_{u_{n+1}}^{u_n} \{1+u+9u^2+2\pi u\} \, du \\
 &= \left[u + \left(\frac{1}{2} + \pi\right)u^2 + 3u^3 \right]_{u_{n+1}}^{u_n}
 \end{aligned}$$

so that

$$s(u_n) - s(0) \leq u_n + \left(\frac{1}{2} + \pi\right)u_n^2 + 3u_n^3,$$

which shows that the curve is rectifiable.

The curvature at the origin, however, is not definite. For

$$\frac{\psi(u) - \psi(0)}{s(u) - s(0)} \rightarrow \kappa$$

is only possible, since $\psi(0) = 0$, if

$$(19.2) \quad \frac{\tan \psi(u)}{s(u) - s(0)} \rightarrow \kappa.$$

But, if ν is the integer such that $u_{\nu+1} \leq u < u_\nu$ and u is positive,

$$s(u) - s(0) \leq u_\nu + \left(\frac{1}{2} + \pi\right)u_\nu^2 + 3u_\nu^3 \sim u.$$

Also, $s(u) - s(0) \geq x(u) - x(0) = u$, so that $s(u) - s(0) \sim u$ and (19.2) implies that

$$\frac{\tan \psi(u)}{u} \rightarrow \kappa,$$

i.e.

$$1 + \frac{1}{u} g(u) + g'(u) \rightarrow \kappa,$$

or, in virtue of (19.1),

$$g'(u) \rightarrow \kappa - 1,$$

which is not the case since $g'(u) = 2\pi \cos 2\nu^2\pi (u - u_{\nu+1})$ and oscillates between -2π and $+2\pi$.

On the intersection of two normals.

20. Let the curve be supposed differentiable at all its points, but not necessarily rectifiable. Further, suppose that $f'(u)$ and $g'(u)$ never vanish together, so that at each point there is a definite tangent and a definite normal.

The equations of the normal at $u = \gamma$ are

$$x = f(\gamma) - \omega g'(\gamma), \quad y = g(\gamma) + \omega f'(\gamma),$$

where ω is a variable parameter.

If
$$f'(\gamma) g'(u) - g'(\gamma) f'(u) \neq 0$$

this normal intersects the normal at u , and the point of intersection corresponds to

$$\omega = \frac{f'(u) \{f(u) - f(\gamma)\} + g'(u) \{g(u) - g(\gamma)\}}{f'(\gamma) g'(u) - g'(\gamma) f'(u)}.$$

The condition for a "point of intersection of consecutive normals" is, with the notation of § 7,

$$\frac{f'(u) \{f(u) - f(\gamma)\} + g'(u) \{g(u) - g(\gamma)\}}{F'(u)} \rightarrow \text{finite limit.}$$

Suppose now that $f'(u)$ and $g'(u)$ are continuous at γ , so that the curve is *smooth*. Then

$$f'(u) \{f(u) - f(\gamma)\} + g'(u) \{g(u) - g(\gamma)\} \sim (u - \gamma) \phi(\gamma)^2$$

and the condition becomes

$$\frac{u - \gamma}{F'(u)} \rightarrow \text{finite limit.}$$

In order that there may be a limit different from the point on the curve $\lim \omega$ must be different from zero, the condition for which is

$$\frac{F'(u)}{u - \gamma} \rightarrow \text{finite limit,}$$

i.e. $F''(\gamma)$ exists.

Thus there is a point of intersection of consecutive normals different from the point on the curve if, and only if, there is a definite curvature different from zero. It is easily verified that the point of intersection is, in this case, the centre of curvature.

It is not difficult to show that the curve of § 19 is smooth, so that, as it has no definite curvature, there is no point of intersection of consecutive normals, although the osculating circle exists.

Examples of a smooth curve which has no tangent and of a curve which has a continuously turning tangent but is not smooth.

21. There are two properties of a curve which should be carefully distinguished. One is the property of having a continuously turning tangent: the other is the continuity of the derivatives $f'(u)$ and $g'(u)$. These are not equivalent.

If $f'(\gamma)$ and $g'(\gamma)$ are both zero, then there may not be a tangent at $u = \gamma$, even though $f'(u)$ and $g'(u)$ are continuous.

For take

$$f(u) = u^3, \quad g(u) = u^3 \sin \frac{1}{u} \quad (u \neq 0),$$

$$f(0) = 0, \quad g(0) = 0,$$

so that

$$f'(u) = 3u^2, \quad g'(u) = 3u^2 \sin \frac{1}{u} - u \cos \frac{1}{u} \quad (u \neq 0),$$

$$f'(0) = 0, \quad g'(0) = 0,$$

and both the derivatives are continuous at $u = 0$.

The curve, however, has no tangent at $u = 0$, for the gradient of the chord is

$$\frac{g(u) - g(0)}{f(u) - f(0)} = \sin \frac{1}{u},$$

which does not tend to a limit.

On the other hand, a curve may have a continuously turning tangent without being smooth.

For take

$$f(u) = 2u + u^2 \sin \frac{1}{u}, \quad g(u) = u^3 \sin \frac{1}{u} \quad (u \neq 0),$$

$$f(0) = 0, \quad g(0) = 0.$$

Then

$$f'(u) = 2 + 2u \sin \frac{1}{u} - \cos \frac{1}{u}, \quad g'(u) = 3u^2 \sin \frac{1}{u} - u \cos \frac{1}{u} \quad (u \neq 0),$$

$$f'(0) = 2, \quad g'(0) = 0$$

and, as is easily seen, as $u \rightarrow 0$,

$$\frac{g'(u)}{f'(u)} \rightarrow 0 = \frac{g'(0)}{f'(0)},$$

so that the curve has a tangent which turns continuously at γ . But $f'(u)$ does not tend to $f'(0)$, and the curve is not smooth at γ .

On the notion of contact.

22. Let C and Γ be two curves which pass through a common point P . Further, let Γ have a tangent at P , and be such that an arc of it containing P not as an end point can be found which is not cut in more than one point by a parallel to the normal at P . Denote this arc by γ .

If Q is a point on C sufficiently near to P , the line through Q parallel to the normal at P will cut γ , and cut it at exactly one point Q' .

The contact, referred to Γ , between C and Γ is said to be of order r if, as $Q \rightarrow P$ along C ,

$$\frac{QQ'}{PQ^r} \rightarrow 0.$$

In this definition Γ is described as the curve of reference.

The definition is asymmetrical in that C may be such that lines parallel to the normal cut it in several points, so that complications occur when C is taken as curve of reference which are not present when Γ is taken as curve of reference.

Inequalities for the length of a differentiable Lipschitz curve.

23. The curve

$$x = f(u), \quad y = g(u)$$

is said to be a Lipschitz curve if there exist constants K, L such that

$$|f(v) - f(u)| \leq K |v - u|, \quad |g(v) - g(u)| \leq L |v - u|.$$

These involve

$$|f'(u)| \leq K, \quad |g'(u)| \leq L,$$

so that

$$\phi(u) \leq \sqrt{(K^2 + L^2)},$$

and $\phi(u)$ possesses upper and lower Riemann integrals. With the notation of § 2,

$$\Phi(u) \leq \sqrt{(K^2 + L^2)}$$

and (2.4) yields

$$|s(v) - s(u)| \leq \sqrt{K^2 + L^2} |v - u|,$$

so that $s(u)$ has bounded derivatives.

If $D(u)$ is any one of $s^+(u)$, $s^-(u)$, $s_+(u)$, $s_-(u)$, it follows from the continuity of $s(u)$, by a known theorem, that

$$\int_a^\beta D(u) du \leq s(\beta) - s(a) \leq \bar{\int}_a^\beta D(u) du.$$

Since, by (2.3), $s_+(u) \geq \phi(u)$, the first half of this gives

$$(23.1) \quad \int_a^\beta \phi(u) du \leq s(\beta) - s(a).$$

Again, divide the interval (a, β) by means of points

$$a = u_0 < u_1 < \dots < u_r < \dots < u_n = \beta,$$

and let M_r be the upper bound of $\phi(u)$ in $u_{r-1} \leq u \leq u_r$. Then, by (2.4),

$$s(u_r) - s(u_{r-1}) \leq M_r(u_r - u_{r-1}),$$

so that

$$s(\beta) - s(a) \leq \sum_{r=1}^n M_r(u_r - u_{r-1})$$

and hence, on passage to the lower bound,

$$(23.2) \quad s(\beta) - s(a) \leq \int_a^\beta \phi(u) du.$$

Thus, in view of (23.1),

$$\int_a^\beta \phi(u) du \leq s(\beta) - s(a) \leq \bar{\int}_a^\beta \phi(u) du.$$

If $\phi(u)$ is Riemann integrable, then, as is already known,

$$s(\beta) - s(a) = \int_a^\beta \phi(u) du.$$

Tonelli's theorem.

24. For any rectifiable curve, not necessarily everywhere differentiable, Tonelli has shown*, with the aid of the Lebesgue integral, that

$$s(\beta) - s(a) \geq \int_a^\beta \phi(u) du,$$

* L. Tonelli, *Atti di Torino*, 43 (1908), 783-799.

equality being obtained if, and only if, $f(u)$ and $g(u)$ are completely continuous functions.

The integrand is considered only where it exists. In virtue of a theorem given by Lebesgue the points of exception form a set of measure zero and do not affect the value of the (Lebesgue) integral.

A particular case of Tonelli's theorem is that equality is obtained for a differentiable curve, because, as $f(u)$ and $g(u)$ are of bounded variation, they can only be differentiable everywhere if they are completely continuous.

Construction of the case of inequality in Tonelli's theorem.

25. Let Π be the perfect set obtained by subtracting from $(0, 1)$ the successive complementary intervals

$$\left(\frac{1}{3}, \frac{2}{3}\right); \left(\frac{1}{9}, \frac{2}{9}\right), \left(\frac{7}{9}, \frac{8}{9}\right); \left(\frac{1}{27}, \frac{2}{27}\right), \left(\frac{7}{27}, \frac{8}{27}\right), \left(\frac{19}{27}, \frac{20}{27}\right), \left(\frac{25}{27}, \frac{26}{27}\right); \dots$$

It is well known that the measure of Π is zero.

Define $g(u)$ by attributing to it, in the above complementary intervals, the respective values

$$\frac{1}{2}; \quad \frac{1}{4}, \frac{3}{4}; \quad \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}; \quad \dots$$

At the remaining points of Π define $g(u)$ by making it continuous.

This procedure, as is well known, is consistent with itself, and yields a function which is continuous and increases from 0 to 1 as u goes from 0 to 1, and has zero for its derivative at all points interior to a complementary interval.

Consider now the curve

$$x = u, \quad y = g(u) \quad (0 \leq u \leq 1).$$

It is rectifiable, because u and $g(u)$ are both increasing functions. Its length is at least as great as that of the segment determined by the points $u = 0, 1$, so that

$$s(1) - s(0) \geq \sqrt{2}.$$

Within each complementary interval

$$g'(u) = 0, \quad \phi(u) = 1,$$

and, therefore, since the remaining points form a set of measure zero, the Lebesgue integral being used,

$$\int_0^1 \phi(u) du = 1.$$

Plainly
$$s(1) - s(0) > \int_0^1 \phi(u) du,$$

and the case of inequality in Tonelli's theorem is realized.

A mean-value theorem.

26. If $D(u)$ is any one of the four derivatives of $s(u)$, then, by a known theorem, there are points ξ_1, ξ_2 in (α, β) such that

$$D(\xi_1) \leq \frac{s(\beta) - s(\alpha)}{\beta - \alpha} \leq D(\xi_2).$$

It will now be shown that, if $\phi(u)$ exists, then this is still true if $D(u)$ is replaced by $\phi(u)$, so that $\phi(u)$, although not actually in all cases a derivative, behaves very like a derivative.

Since, as has been proved, $s_+(u) \geq \phi(u)$, the first half of the mean-value theorem is obtained without difficulty.

If $\phi(u)$ is unbounded there must be values of $\phi(u)$ greater than

$$\frac{s(\beta) - s(\alpha)}{\beta - \alpha}$$

and the second half follows.

When $\phi(u)$ is bounded, let M be its upper bound. Unless

$$(26.1) \quad M = \frac{s(\beta) - s(\alpha)}{\beta - \alpha},$$

there is no difficulty, as there are values of $\phi(u)$ greater than any number less than M .

Suppose, then, that (26.1) holds. Since

$$\frac{1}{\beta - \alpha} \int_a^{\bar{\beta}} \phi(u) du \leq M,$$

it follows from (23.2) that

$$\int_a^{\bar{\beta}} \phi(u) du = M,$$

so that, no matter what may be the positive number ϵ , the points where $\phi(u) > M - \epsilon$ must be everywhere dense in (α, β) .

If γ is such a point, there can be found other points p, q, r satisfying $p < \gamma < q < r$ such that, if π is any point in (p, q) , then

$$F(r, \pi) = \frac{\sqrt{[f(r) - f(\pi)]^2 + [g(r) - g(\pi)]^2}}{r - \pi} \geq M - \epsilon.$$

For, as $r \rightarrow \gamma$,

$$F(r, \gamma) \rightarrow \sqrt{[f'(\gamma)^2 + g'(\gamma)^2]} = \phi(\gamma) > M - \epsilon,$$

and so, if r is sufficiently near to γ ,

$$(26.2) \quad F(r, \gamma) > M - \epsilon.$$

Take r to be on the right of γ and as close to it as may be required.

If, now r is regarded as fixed, $F(r, \pi)$ is a continuous function of π which is greater than $M - \epsilon$ at γ . By one of the fundamental properties of continuous functions, $F(r, \pi)$ must be greater than $M - \epsilon$ throughout some interval (p, q) surrounding γ , and this yields what is required, q , if necessary, being taken less than r .

It is convenient to describe an interval such as (p, r) as a *special interval* and (p, q) as its *shaded portion*. In general there will be many such special intervals in (α, β) . Of these a definite one may be selected by the following rule.

First replace ϵ by $\frac{1}{2}\epsilon$ and obtain the special intervals with their shaded portions which correspond. In this way an inequality which gives rise to a closed set, from which a choice is easily made, can be temporarily substituted for (26.2).

Of the numbers

$$\frac{1}{2}\alpha + \frac{1}{2}\beta; \quad \frac{3}{4}\alpha + \frac{1}{4}\beta, \quad \frac{1}{4}\alpha + \frac{3}{4}\beta; \quad \dots$$

there is a first which lies in a shaded portion of some special interval. Call this γ_1 .

Choose r to be the extreme point on the right, within (α, β) , such that $F(r, \gamma_1) \geq M - \frac{1}{2}\epsilon$. Such a point necessarily exists because $F(r, \gamma_1)$ is a continuous function of r . The chosen point is evidently such that $F(r, \gamma_1) > M - \epsilon$.

Now revert to ϵ and choose p and q to be the extreme points satisfying the stated condition relative to r . Continuity, again, shows that these extremes exist.

The points p, q, r in the interval (α, β) are now definitely fixed, being dependent on ϵ alone.

Take $\epsilon = 1$ and determine p_1, q_1, r_1 in the manner just described.

Next, denote by (α_1, β_1) the interval $(\frac{1}{2}p_1 + \frac{1}{2}q_1, q_1)$, and, with $\epsilon = \frac{1}{2}$, determine p_2, q_2, r_2 in the interval (α_1, β_1) . Since the points γ are everywhere dense this determination is possible.

Then denote by (α_2, β_2) the interval $(\frac{1}{2}p_2 + \frac{1}{2}q_2, q_2)$, and repeat the process with $\epsilon = \frac{1}{3}$. Call the points now obtained p_3, q_3, r_3 .

Continued indefinitely, this process yields a decreasing sequence of shaded closed intervals (p_n, q_n) and a sequence of points r_n such that $r_n - q_n \rightarrow 0$.

If ξ_2 is the point common to all the intervals (p_n, q_n) ,

$$F(r_n, \xi_2) \geq M - \frac{1}{n},$$

which shows that

$$\phi(\xi_2) \geq M,$$

which is what is required to complete the proof.

It is not possible to improve this mean-value theorem by finding a single point ξ which does duty for both ξ_1 and ξ_2 , so that

$$(26.3) \quad \frac{s(\beta) - s(\alpha)}{\beta - \alpha} = \phi(\xi).$$

For consider the case

$$\begin{aligned} f(u) &= 2u^2 \sin \frac{1}{u}, & g(u) &= 2u^2 \cos \frac{1}{u} & (u > 0), \\ f(u) &= \frac{1}{2\sqrt{2}} u^2, & g(u) &= \frac{1}{2\sqrt{2}} u^2 & (u \leq 0). \end{aligned}$$

It can be verified that

$$\begin{aligned} \phi(u) &= 2\sqrt{1+4u^2} & (u > 0), \\ &= |u| & (u \leq 0), \end{aligned}$$

in virtue of which $\phi(u)$ does not take in $-1 \leq u \leq 1$ any value greater than 1 and not greater than 2.

$$\text{Now} \quad \frac{s(1) - s(-1)}{2} = \frac{1}{4} + \int_0^1 \sqrt{1+4u^2} du,$$

which is greater than 1 and less than 2 and is not a value of $\phi(u)$ in $-1 \leq u \leq 1$, so that (26.3) does not hold.

THE MEAN-VALUE OF THE ZETA-FUNCTION ON THE CRITICAL LINE

By E. C. TITCHMARSH.

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1. *Introduction.* The importance of mean-value theorems for the function $\zeta(\frac{1}{2}+it)$ is well known from the researches of Hardy and Littlewood*. Their original result is that when $T \rightarrow \infty$

$$(1.1) \quad \int_0^T |\zeta(\tfrac{1}{2}+it)|^2 dt \sim T \log T.$$

In their second paper the approximate functional equation for $\zeta(s)$ was used to give a more direct proof of the result, and to prove that

$$(1.2) \quad \int_0^T |\zeta(\tfrac{1}{2}+it)|^4 dt = O(T \log^4 T).$$

Later this was improved by Ingham† to

$$(1.3) \quad \int_0^T |\zeta(\tfrac{1}{2}+it)|^4 dt \sim \frac{1}{2\pi^2} T \log^4 T,$$

the proof depending on an approximate functional equation for the square of $\zeta(s)$, also due to Hardy and Littlewood.

In the present paper I give an independent proof of the equivalent theorem

$$(1.4) \quad \int_0^\infty |\zeta(\tfrac{1}{2}+it)|^4 e^{-\delta t} dt \sim \frac{1}{2\pi^2} \frac{1}{\delta} \log^4 \frac{1}{\delta}.$$

This is equivalent to (1.3) in the sense that either can be deduced from the other without further reference to the properties of $\zeta(s)$; but while

* G. H. Hardy and J. E. Littlewood, *Acta Math.*, 41 (1917), 119–196 (151); *Proc. London Math. Soc.* (2), 21 (1922), 29–74; *Proc. Royal Soc.*, (A) 103 (1923), 403–412.

† A. E. Ingham, *Proc. London Math. Soc.* (2), 22 (1923), xxix (*Records* for April 26, 1923).

(1.4) follows from (1.3) by integration by parts, the converse deduction requires the Tauberian argument of Hardy and Littlewood*. On the other hand (1.2) is an obvious consequence of (1.4). In what follows we shall not be concerned with these connexions, and shall consider only means of the form (1.4).

The method also gives similar results for the function

$$\eta(s) = \Sigma (-1)^{n-1} (2n-1)^{-s}.$$

2. *Theory of Fourier transforms.* The work is based on some well known formulae connected with Fourier integrals. Let

$$(2.1) \quad f(x) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} F(t) e^{ixt} dt,$$

where x is real; and it may be assumed for the purpose in hand that both $f(x)$ and $F(x)$ are analytic functions, regular and bounded on the real axis, and absolutely integrable over $(-\infty, \infty)$. Then Fourier's transformation gives the reciprocal formula

$$(2.2) \quad F(x) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(t) e^{-ixt} dt.$$

Hence

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) \bar{f}(x) dx &= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(x) dx \int_{-\infty}^{\infty} \bar{F}(t) e^{-ixt} dt \\ &= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \bar{F}(t) dt \int_{-\infty}^{\infty} f(x) e^{-ixt} dx = \int_{-\infty}^{\infty} \bar{F}(t) F(t) dt, \end{aligned}$$

or

$$(2.3) \quad \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(x)|^2 dx.$$

Again

$$\begin{aligned} (2.4) \quad \int_{-\infty}^{\infty} F(x) \bar{F}(x+t) dx &= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} F(x) dx \int_{-\infty}^{\infty} \bar{f}(u) e^{i(x+t)u} du \\ &= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \bar{f}(u) e^{itu} du \int_{-\infty}^{\infty} F(x) e^{ixu} dx = \int_{-\infty}^{\infty} |f(u)|^2 e^{itu} du. \end{aligned}$$

Hence

$$\frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} F(x) \bar{F}(x+t) dx$$

* See Lemma 2.413 of their paper in *Acta Math.*

is the Fourier transform of $|f(x)|^2$, and we may apply (2.3) to these functions, giving

$$(2.5) \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} F(x) \bar{F}(x+t) dx \right|^2 dt = \int_{-\infty}^{\infty} |f(t)|^4 dt.$$

Also the modulus of the right-hand side of (2.4) is unchanged when we change the sign of t . Hence

$$\left| \int_{-\infty}^{\infty} F(x) \bar{F}(x-t) dx \right| = \left| \int_{-\infty}^{\infty} F(x) \bar{F}(x+t) dx \right|,$$

and (2.5) may be written

$$(2.6) \quad \frac{1}{\pi} \int_0^{\infty} \left| \int_{-\infty}^{\infty} F(x) \bar{F}(x+t) dx \right|^2 dt = \int_{-\infty}^{\infty} |f(t)|^4 dt.$$

3. *Application to $\zeta(s)$.* We have

$$(3.1) \quad \frac{1}{e^x - 1} = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \Gamma(s) \zeta(s) x^{-s} ds \quad (k > 1, |\arg x| < \frac{1}{2}\pi).$$

Moving the line of integration to $\sigma = \frac{1}{2}$, we meet a pole at $s = 1$ with residue $1/x$. Hence

$$\frac{1}{e^x - 1} - \frac{1}{x} = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma(s) \zeta(s) x^{-s} ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma(\tfrac{1}{2}+it) \zeta(\tfrac{1}{2}+it) x^{-\frac{1}{2}-it} dt.$$

Putting $x = ie^{\xi-i\delta}$ ($\delta > 0$), we obtain

$$\frac{1}{\exp(i e^{\xi-i\delta}) - 1} - \frac{1}{i e^{\xi-i\delta}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma(\tfrac{1}{2}+it) \zeta(\tfrac{1}{2}+it) e^{-\frac{1}{2}\xi - i\xi t - i(\frac{1}{2}\pi - \delta)(\frac{1}{2}+it)} dt.$$

Hence in the formulae of § 2 we may take

$$(3.2) \quad f(t) = \frac{1}{\sqrt{(2\pi)}} \Gamma(\tfrac{1}{2}+it) \zeta(\tfrac{1}{2}+it) e^{-i(\frac{1}{2}\pi - \delta)(\frac{1}{2}+it)},$$

$$F(\xi) = e^{\frac{1}{2}\xi} \left\{ \frac{1}{\exp(i e^{\xi-i\delta}) - 1} - \frac{1}{i e^{\xi-i\delta}} \right\}.$$

4. As a preliminary application of the method, we prove

$$\text{THEOREM I.} \quad \int_0^{\infty} |\zeta(\tfrac{1}{2}+it)|^2 e^{-\delta t} dt \sim \frac{1}{\delta} \log \frac{1}{\delta}.$$

We have by (2.3) and (3.2)

$$\begin{aligned}
 (4.1) \quad & \frac{1}{2\pi} \int_{-\infty}^{\infty} |\Gamma(\tfrac{1}{2} + it) \zeta(\tfrac{1}{2} + it)|^2 e^{(\pi - 2\delta)t} dt \\
 &= \int_{-\infty}^{\infty} e^{\xi} \left| \frac{1}{\exp(i e^{\xi - i\delta}) - 1} - \frac{1}{i e^{\xi - i\delta}} \right|^2 d\xi \\
 &= \int_0^{\infty} \left| \frac{1}{\exp(iue^{-i\delta}) - 1} - \frac{1}{iue^{-i\delta}} \right|^2 du = \int_0^{\pi} + \int_{\pi}^{\infty} = O(1) + \int_{\pi}^{\infty},
 \end{aligned}$$

since the integrand is bounded for $0 < u < \pi$, $\delta < \delta_0$. The last integral is equal to

$$\begin{aligned}
 & \int_{\pi}^{\infty} \left\{ \frac{1}{\exp(iue^{-i\delta}) - 1} - \frac{1}{iue^{-i\delta}} \right\} \left\{ \frac{1}{\exp(-iue^{i\delta})} + \frac{1}{iue^{i\delta}} \right\} du \\
 &= \int_{\pi}^{\infty} \frac{du}{\{\exp(iue^{-i\delta}) - 1\} \{\exp(-iue^{i\delta}) - 1\}} \\
 & \quad + 2R \left\{ ie^{i\delta} \int_{\pi}^{\infty} \frac{1}{\exp(-iue^{i\delta}) - 1} \frac{du}{u} \right\} + \int_{\pi}^{\infty} \frac{du}{u^2}.
 \end{aligned}$$

Now by Cauchy's theorem, turning the line of integration from (π, ∞) to $(\pi, \pi + i\infty)$,

$$\begin{aligned}
 & \left| \int_{\pi}^{\infty} \frac{1}{\exp(-iue^{i\delta}) - 1} \frac{du}{u} \right| = \left| \int_0^{\infty} \frac{1}{\exp\{-i(\pi + iv)e^{i\delta}\} - 1} \frac{idv}{\pi + iv} \right| \\
 & < \int_0^{\infty} \frac{dv}{\exp(v \cos \delta + \pi \sin \delta) - 1} = \sec \delta \int_{\pi \sin \delta}^{\infty} \frac{dw}{e^w - 1} = O\left(\log \frac{1}{\delta}\right).
 \end{aligned}$$

So only the first term is of importance. It is equal to

$$\begin{aligned}
 & \int_{\pi}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \exp(-imue^{-i\delta} + inue^{i\delta}) du \\
 &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\exp\{- (m+n)\pi \sin \delta - i(m-n)\pi \cos \delta\}}{(m+n) \sin \delta + i(m-n) \cos \delta} \\
 &= \sum_{n=1}^{\infty} \frac{\exp(-2n\pi \sin \delta)}{2n \sin \delta} \\
 & \quad + 2 \sum_{m=2}^{\infty} \sum_{n=1}^{m-1} \frac{(m+n) \sin \delta \cos\{(m-n)\pi \cos \delta\}}{(m+n)^2 \sin^2 \delta + (m-n)^2 \cos^2 \delta} \exp\{-(m+n)\pi \sin \delta\} \\
 & \quad - 2 \sum_{m=2}^{\infty} \sum_{n=1}^{m-1} \frac{(m-n) \cos \delta \sin\{(m-n)\pi \cos \delta\}}{(m+n)^2 \sin^2 \delta + (m-n)^2 \cos^2 \delta} \exp\{-(m+n)\pi \sin \delta\} \\
 &= \Sigma_1 + \Sigma_2 + \Sigma_3,
 \end{aligned}$$

the series of imaginary parts vanishing identically. Now

$$\Sigma_1 \sim \frac{1}{2\delta} \int_1^\infty e^{-2x\pi \sin \delta} \frac{dx}{x} = \frac{1}{2\delta} \int_{\sin \delta}^\infty e^{-2x\pi} \frac{dx}{x} \sim \frac{1}{2\delta} \log \frac{1}{\delta},$$

$$|\Sigma_2| < 2 \sum_{m=2}^\infty \sum_{n=1}^{m-1} \frac{2m \sin \delta}{(m-n)^2 \cos^2 \delta} e^{-m\pi \sin \delta} = \sum_{m=2}^\infty m e^{-m\pi \sin \delta} O(\delta) = O\left(\frac{1}{\delta}\right),$$

and, since

$$|\sin \{(m-n)\pi \cos \delta\}| = |\sin \{2(m-n)\pi \sin^2 \frac{1}{2}\delta\}| \leq \frac{1}{2}(m-n)\pi \delta^2,$$

$$|\Sigma_3| < 2 \sum_{m=2}^\infty \sum_{n=1}^{m-1} \frac{\frac{1}{2}(m-n)^2 \pi \delta^2 \cos \delta}{(m-n)^2 \cos^2 \delta} e^{-m\pi \sin \delta} = \sum_{m=2}^\infty e^{-m\pi \sin \delta} O(m\delta) = O(1).$$

Also since $|\Gamma(\frac{1}{2}+it)|^2 = \pi \operatorname{sech} \pi t = 2\pi e^{-\pi|t|} + O(e^{-3\pi|t|})$, the left-hand side of (4.1) is equal to

$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^2 e^{-2\delta t} dt + O(1).$$

This proves the theorem.

5. THEOREM II.

$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^4 e^{-\delta t} dt \sim \frac{1}{2\pi^2} \frac{1}{\delta} \log^4 \frac{1}{\delta}.$$

5.1. *Preliminary.* From (2.6) and (3.2) we obtain

$$\begin{aligned} (5.11) \quad \frac{1}{4}\pi \int_{-\infty}^\infty |\zeta(\tfrac{1}{2}+it)|^4 \operatorname{sech}^2 \pi t e^{(2\pi-4\delta)t} dt \\ = \int_1^\infty \left| \int_0^\infty \left\{ \frac{1}{\exp(iue^{-i\delta})} - \frac{1}{iue^{-i\delta}} \right\} \left\{ \frac{1}{\exp(-iue^{i\delta})-1} + \frac{1}{iue^{i\delta}} \right\} du \right|^2 dv. \end{aligned}$$

The left-hand side is equal to

$$\pi \int_0^\infty |\zeta(\tfrac{1}{2}+it)|^4 e^{-4\delta t} dt + O(1).$$

In discussing the right-hand side we shall use repeatedly the fact that if

$$\int_1^\infty |f(v, \delta)|^2 dv \sim \phi(\delta), \quad \int_1^\infty |g(v, \delta)|^2 dv = o\{\phi(\delta)\},$$

then
$$\int_1^\infty |f(v, \delta) + g(v, \delta)|^2 dv \sim \phi(\delta),$$

which follows from the inequalities

$$\begin{aligned} \left(\int_1^\infty |f|^2 dv \right)^{\frac{1}{2}} - \left(\int_1^\infty |g|^2 dv \right)^{\frac{1}{2}} &\leq \left(\int_1^\infty |f+g|^2 dv \right)^{\frac{1}{2}} \\ &\leq \left(\int_1^\infty |f|^2 dv \right)^{\frac{1}{2}} + \left(\int_1^\infty |g|^2 dv \right)^{\frac{1}{2}}. \end{aligned}$$

The integral with respect to u in (5.11) may be written

$$\begin{aligned} &\int_1^\infty \frac{du}{\{\exp(iue^{-i\delta}) - 1\} \{\exp(-iuve^{i\delta}) - 1\}} + ie^{i\delta} \int_1^\infty \frac{1}{\exp(-iuve^{i\delta}) - 1} \frac{du}{u} \\ &\quad - \frac{ie^{-i\delta}}{v} \int_1^\infty \left\{ \frac{1}{\exp(iue^{-i\delta}) - 1} - \frac{1}{iue^{-i\delta}} \right\} \frac{du}{u} \\ &\quad + \int_0^1 \left\{ \frac{1}{\exp(iue^{-i\delta}) - 1} - \frac{1}{iue^{-i\delta}} \right\} \left\{ \frac{1}{\exp(-iuve^{i\delta}) - 1} + \frac{1}{iuve^{i\delta}} \right\} du \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

If, therefore, we can prove that

$$\int_1^\infty |I_1|^2 dv \sim \frac{1}{8\pi} \frac{1}{\delta} \log^4 \frac{1}{\delta},$$

while
$$\int_1^\infty |I_k|^2 dv = o\left(\frac{1}{\delta} \log^4 \frac{1}{\delta}\right)$$

for $k = 2, 3, 4$, then the theorem will follow.

Consider first I_2 . As in the case of the similar term in § 4,

$$\begin{aligned} |I_2| &< \frac{1}{v \cos \delta} \int_{v \sin \delta}^\infty \frac{dz}{e^z - 1} = O\left(\frac{1}{v} \log \frac{1}{v\delta}\right) \quad \left(v < \frac{1}{\delta}\right), \\ &= O\left(\frac{1}{\delta}\right) \quad \left(v > \frac{1}{\delta}\right). \end{aligned}$$

Hence

$$\int_1^\infty |I_2|^2 dv = O\left(\int_1^{1/\delta} \frac{1}{v^2} \log^2 \frac{1}{v\delta} dv\right) + O\left(\int_{1/\delta}^\infty \frac{dv}{v^2}\right) = O\left(\log^2 \frac{1}{\delta}\right).$$

Similarly in I_3 we turn the line of integration to $(1, 1-i\infty)$, and obtain

$$\begin{aligned} |I_3| &= \frac{1}{v} \left| \int_0^\infty \left\{ \frac{1}{\exp\{i(1-iy)e^{-i\delta}\}-1} - \frac{1}{i(1-iy)e^{-i\delta}} \right\} \frac{-i dy}{1-iy} \right| \\ &< \frac{1}{v} \left\{ \int_0^\infty \frac{dy}{\exp(y \cos \delta + \sin \delta) - 1} + 1 \right\} \\ &= \frac{1}{v \cos \delta} \int_{\sin \delta}^\infty \frac{dz}{e^z - 1} + \frac{1}{v} = O\left(\frac{1}{v} \log \frac{1}{\delta}\right). \end{aligned}$$

Hence
$$\int_1^\infty |I_3|^2 dv = O\left(\int_1^\infty \frac{1}{v^2} \log^2 \frac{1}{\delta} dv\right) = O\left(\log^2 \frac{1}{\delta}\right).$$

In I_4 we write

$$I_4 = \int_0^1 = \int_0^{i/\delta} + \int_{i/\delta}^{1+i/\delta} - \int_1^{1+i/\delta} = I_4^1 + I_4^2 + I_4^3.$$

The first factor in the integrand is bounded on the rectangle. Hence

$$\begin{aligned} I_4^1 &= O\left\{\int_0^{1/\delta} \left| \frac{1}{\exp(yve^{i\delta})-1} - \frac{1}{yve^{i\delta}} \right| dy\right\} \\ &= O\left\{\frac{1}{v} \int_0^{v/\delta} \left| \frac{1}{\exp(ze^{i\delta})-1} - \frac{1}{ze^{i\delta}} \right| dz\right\} \\ &= O\left(\frac{1}{v}\right) + O\left(\frac{1}{v} \int_1^\infty \frac{dz}{e^{z \cos \delta} - 1}\right) + O\left(\frac{1}{v} \int_1^{v/\delta} \frac{dz}{z}\right) \\ &= O\left(\frac{1}{v}\right) + O\left(\frac{1}{v}\right) + O\left(\frac{1}{v} \log \frac{v}{\delta}\right) = O\left(\frac{1}{v} \log \frac{v}{\delta}\right). \end{aligned}$$

A similar calculation gives the same result for I_4^3 . Lastly

$$\begin{aligned} I_4^2 &= O\left\{\int_0^1 \left| \frac{1}{\exp\{-i(i/\delta+x)ve^{i\delta}\}-1} + \frac{1}{i(i/\delta+x)ve^{i\delta}} \right| dx\right\} \\ &= O\left\{\int_0^1 O\left(\frac{\delta}{v}\right) dx\right\} = O\left(\frac{\delta}{v}\right). \end{aligned}$$

Hence
$$\int_1^\infty |I_4|^2 dv = O\left(\int_1^\infty \frac{1}{v^2} \log^2 \frac{v}{\delta} dv\right) = O\left(\log^2 \frac{1}{\delta}\right).$$

5.2. *The main term.* The problem is thus reduced to the discussion of I_1 . Now $1/\{\exp(iue^{-i\delta})-1\}$ has poles at $u = 2n\pi e^{i\delta}$ ($n = 1, 2, \dots$) with residues $-ie^{i\delta}$. Hence the calculus of residues gives

$$\begin{aligned} I_1 &= \sum_{n=1}^{\infty} \frac{2\pi e^{i\delta}}{\exp(-2in\pi v e^{2i\delta})-1} + \int_1^{1+i\infty} \frac{du}{\{\exp(iue^{-i\delta})-1\} \{\exp(-iuv e^{i\delta})-1\}} \\ &= 2\pi e^{i\delta} \sum_{n=1}^{\infty} d(n) \exp(2in\pi v e^{2i\delta}) + O\left\{\int_0^{\infty} \frac{dy}{\exp(vy \cos \delta + v \sin \delta)-1}\right\}, \end{aligned}$$

where $d(n)$ denotes the number of divisors of n ; and as before the integral of the square of the last term is $O\{\log^2(1/\delta)\}$. Also

$$\begin{aligned} &\int_1^{\infty} \left| 2\pi e^{i\delta} \sum_{n=1}^{\infty} d(n) \exp(2in\pi v e^{2i\delta}) \right|^2 dv \\ &= 4\pi^2 \int_1^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} d(m) d(n) \exp(2im\pi v e^{2i\delta} - 2in\pi v e^{-2i\delta}) dv \\ &= 4\pi^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} d(m) d(n) \frac{\exp\{-2\pi(m+n) \sin 2\delta + 2i\pi(m-n) \cos 2\delta\}}{2\pi(m+n) \sin 2\delta - 2i\pi(m-n) \cos 2\delta} \\ &= \frac{\pi}{\sin 2\delta} \sum_{n=1}^{\infty} \frac{\{d(n)\}^2}{n} \exp(-4\pi n \sin 2\delta) \\ &\quad + 4\pi \sum_{m=2}^{\infty} \sum_{n=1}^{m-1} d(m) d(n) \frac{(m+n) \sin 2\delta \cos\{2\pi(m-n) \cos 2\delta\}}{(m+n)^2 \sin^2 2\delta + (m-n)^2 \cos^2 2\delta} \\ &\quad \quad \quad \times \exp\{-2\pi(m+n) \sin 2\delta\} \\ &\quad - 4\pi \sum_{m=2}^{\infty} \sum_{n=1}^{m-1} d(m) d(n) \frac{(m-n) \cos 2\delta \sin\{2\pi(m-n) \cos 2\delta\}}{(m+n)^2 \sin^2 2\delta + (m-n)^2 \cos^2 2\delta} \\ &\quad \quad \quad \times \exp\{-2\pi(m+n) \sin 2\delta\} \\ &= \Sigma_1 + \Sigma_2 + \Sigma_3, \end{aligned}$$

the series of imaginary parts vanishing identically.

Now Ramanujan* proved that

$$(5.21) \quad \sum_1^n \{d(\nu)\}^2 \sim \frac{1}{\pi^2} n \log^3 n.$$

Hence by partial summation

$$(5.22) \quad \sum_1^n \frac{\{d(\nu)\}^2}{\nu} \sim \frac{1}{4\pi^2} \log^4 n,$$

* S. Ramanujan, *Messenger of Math.*, 45 (1916), 81-84. See also B. M. Wilson, *Proc. London Math. Soc.* (2), 21 (1922), 235-255.

while, if $k > 0$,

$$(5.23) \quad \sum_1^n \nu^k \{d(\nu)\}^2 = O(n^{k+1} \log^3 n).$$

Hence

$$\begin{aligned} \Sigma_1 &= \frac{\pi}{\sin 2\delta} (1 - e^{-4\pi \sin 2\delta}) \sum_{n=1}^{\infty} e^{-4\pi n \sin 2\delta} \sum_{\nu=1}^n \frac{\{d(\nu)\}^2}{\nu} \\ &\sim 4\pi^2 \sum_{n=1}^{\infty} e^{-4\pi n \sin 2\delta} \frac{1}{4\pi^2} \log^4 n \sim \int_0^{\infty} e^{-4\pi x \sin 2\delta} \log^4 x dx \\ &= \frac{1}{4\pi \sin 2\delta} \int_0^{\infty} e^{-y} \log^4 \left(\frac{y}{4\pi \sin 2\delta} \right) dy \sim \frac{1}{8\pi\delta} \log^4 \frac{1}{\delta}, \\ |\Sigma_2| &< 4\pi \sum_{m=2}^{\infty} \sum_{n=1}^{m-1} d(m) d(n) \frac{4m\delta}{(m-n)^2 \cos^2 2\delta} e^{-2\pi m \sin 2\delta} \\ &= \frac{16\pi\delta}{\cos^2 2\delta} \sum_{m=2}^{\infty} m d(m) e^{-2\pi m \sin 2\delta} \sum_{\nu=1}^{m-1} \frac{d(m-\nu)}{\nu^2} \\ &= \frac{16\pi\delta}{\cos^2 2\delta} \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \sum_{m=\nu+1}^{\infty} m d(m) d(m-\nu) e^{-2\pi m \sin 2\delta} \\ &< A\delta \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \left\{ \sum_{m=\nu+1}^{\infty} m^2 \{d(m)\}^2 e^{-2\pi m \sin 2\delta} \sum_{m=\nu+1}^{\infty} \{d(m-\nu)\}^2 e^{-2\pi m \sin 2\delta} \right\}^{\frac{1}{2}} \\ &< A\delta \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \left\{ \sum_{m=1}^{\infty} m^2 \{d(m)\}^2 e^{-2\pi m \sin 2\delta} \sum_{m=1}^{\infty} \{d(m)\}^2 e^{-2\pi m \sin 2\delta} \right\}^{\frac{1}{2}} \\ &< A\delta \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \left\{ \delta \sum_{m=1}^{\infty} m^3 \log^3 m e^{-2\pi m \sin 2\delta} \delta \sum_{m=1}^{\infty} m \log^3 m e^{-2\pi m \sin 2\delta} \right\}^{\frac{1}{2}} \\ &< A\delta \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \left\{ \delta \cdot \frac{1}{\delta^4} \log^3 \frac{1}{\delta} \cdot \delta \cdot \frac{1}{\delta^2} \log^3 \frac{1}{\delta} \right\}^{\frac{1}{2}} = O \left(\frac{1}{\delta} \log^3 \frac{1}{\delta} \right), \\ |\Sigma_3| &< 4\pi \sum_{m=2}^{\infty} \sum_{n=1}^{m-1} d(m) d(n) \frac{4\pi(m-n)^2 \delta^2}{(m-n)^2 \cos^2 2\delta} e^{-2\pi m \sin 2\delta} \\ &= O \left\{ \delta^2 \sum_{m=2}^{\infty} d(m) e^{-2\pi m \sin 2\delta} \sum_{n=1}^{m-1} d(n) \right\} \\ &= O \left\{ \delta^2 \sum_{m=2}^{\infty} m \log m d(m) e^{-2\pi m \sin 2\delta} \right\} \\ &= O \left\{ \delta^3 \sum_{m=2}^{\infty} m^2 \log^2 m e^{-2\pi m \sin 2\delta} \right\} = O \left(\log^2 \frac{1}{\delta} \right). \end{aligned}$$

We have thus proved that

$$\pi \int_0^\infty |\xi(\tfrac{1}{2} + it)|^4 e^{-4\delta t} dt \sim \frac{1}{8\pi\delta} \log^4 \frac{1}{\delta},$$

which is equivalent to the theorem stated.

6. *The function $\eta(s)$.* We have

$$(6.1) \quad \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \Gamma(s) \eta(s) x^{-s} ds = \tfrac{1}{2} \operatorname{sech} x \quad (|\arg x| < \tfrac{1}{2}\pi),$$

and we may in this case take $k = \tfrac{1}{2}$, since there is no pole at $s = 1$. Hence

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma(\tfrac{1}{2} + it) \eta(\tfrac{1}{2} + it) x^{-\frac{1}{2}-it} dt = \tfrac{1}{2} \operatorname{sech} x.$$

Hence, as in § 3, we obtain as a pair of Fourier transforms

$$(6.2) \quad f(t) = \frac{1}{\sqrt{(2\pi)}} \Gamma(\tfrac{1}{2} + it) \eta(\tfrac{1}{2} + it) e^{-i(\frac{1}{2}\pi - \delta)(\frac{1}{2} + it)},$$

$$F(\xi) = \tfrac{1}{2} e^{\frac{1}{2}\xi} \operatorname{sech}(ie^{\frac{1}{2}\xi - i\delta}).$$

The reader may now easily prove

$$\text{THEOREM III.} \quad \int_0^\infty |\eta(\tfrac{1}{2} + it)|^2 e^{-\delta t} dt \sim \frac{1}{2\delta} \log \frac{1}{\delta}.$$

A brief indication of the proof of

$$\text{THEOREM IV.} \quad \int_0^\infty |\eta(\tfrac{1}{2} + it)|^4 e^{-\delta t} dt \sim \frac{1}{24\pi^2} \frac{1}{\delta} \log^4 \frac{1}{\delta}$$

will be sufficient, since it is like the proof of Theorem II, only easier. The analogue of (5.11) is

$$\begin{aligned} \frac{1}{4}\pi \int_{-\infty}^{\infty} |\eta(\tfrac{1}{2} + it)|^4 \operatorname{sech}^2 \pi t e^{(2\pi - 4\delta)t} dt \\ = \frac{1}{16} \int_1^\infty \left| \int_0^\infty \operatorname{sech}(iue^{-i\delta}) \operatorname{sech}(-iue^{i\delta}) du \right|^2 dv. \end{aligned}$$

Using the calculus of residues as before, we find that the integral with respect to u is, apart from a negligible remainder, equal to

$$\begin{aligned} 2i\pi e^{i\delta} \sum_{n=1}^{\infty} (-1)^n \operatorname{sech} \left\{ -i(n - \tfrac{1}{2})\pi v e^{2i\delta} \right\} \\ = -4i\pi e^{i\delta} \sum_{n=1}^{\infty} (-1)^{n-1} d(2n-1) \exp \left\{ \tfrac{1}{2}i\pi(2n-1) v e^{2i\delta} \right\}. \end{aligned}$$

On integrating the squared modulus of this we obtain

$$16\pi^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{m+n} d(2m-1) d(2n-1) \\ \times \frac{\exp \{-(m+n-1)\pi \sin 2\delta + i(m-n)\pi \cos 2\delta\}}{(m+n-1)\pi \sin 2\delta - i(m-n)\pi \cos 2\delta} \\ \sim \frac{16\pi}{\sin 2\delta} \sum_{n=1}^{\infty} \{d(2n-1)\}^2 \frac{\exp \{-(2n-1)\pi \sin 2\delta\}}{2n-1},$$

Now

$$\sum_{n=1}^{\infty} \frac{\{d(2n-1)\}^2}{(2n-1)^s} = \prod_{p>2} \sum_{m=0}^{\infty} \frac{(m+1)^2}{p^{ms}} = \prod_{p>2} \frac{1+p^{-s}}{(1-p^{-s})^3} = \frac{(1-2^{-s})^3}{1+2^{-s}} \frac{\{\zeta(s)\}^4}{\xi(2s)},$$

and the classical method gives

$$\sum_{\nu=1}^n \{d(2\nu-1)\}^2 \sim \frac{1}{6\pi^2} n \log^3 n.$$

The result is now obtained by partial summation, as before.

7. THEOREM V.

$$\int_0^{\infty} |\xi(\tfrac{1}{2}+it) \eta(\tfrac{1}{2}+it)|^2 e^{-\delta t} dt \sim \frac{1}{128\pi} \frac{1}{\delta} \log^2 \frac{1}{\delta}.$$

7.1. To prove this we require an extension of the formulæ of § 2. Let $f(x)$ be the Fourier transform of $F(x)$, as in (2.1), (2.2), and similarly let $g(x)$ be the Fourier transform of $G(x)$. Then

$$\begin{aligned} \int_{-\infty}^{\infty} \bar{f}(t) g(t) e^{ixt} dt &= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \bar{f}(t) e^{ixt} dt \int_{-\infty}^{\infty} G(u) e^{itu} du \\ &= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} G(u) du \int_{-\infty}^{\infty} \bar{f}(t) e^{i(x+u)t} dt \\ &= \int_{-\infty}^{\infty} G(u) \bar{F}(x+u) du. \end{aligned}$$

Hence
$$\frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} G(t) \bar{F}(x+t) dt$$

is the transform of $\bar{f}(t) g(t)$, and (2.3) gives

$$(7.11) \quad \int_{-\infty}^{\infty} |f(t) g(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} G(x) \bar{F}(x+t) dx \right|^2 dt.$$

$$\begin{aligned} \text{But} \quad \int_{-\infty}^0 \left| \int_{-\infty}^{\infty} G(x) \bar{F}(x+t) dx \right|^2 dt &= \int_0^{\infty} \left| \int_{-\infty}^{\infty} G(x) \bar{F}(x-t) dx \right|^2 dt \\ &= \int_0^{\infty} \left| \int_{-\infty}^{\infty} G(x+t) \bar{F}(x) dx \right|^2 dt. \end{aligned}$$

Hence

$$\begin{aligned} (7.12) \quad & \int_{-\infty}^{\infty} |f(t)g(t)|^2 dt \\ &= \frac{1}{2\pi} \int_0^{\infty} \left| \int_{-\infty}^{\infty} G(x) \bar{F}(x+t) dx \right|^2 dt + \frac{1}{2\pi} \int_0^{\infty} \left| \int_{-\infty}^{\infty} G(x+t) \bar{F}(x) dx \right|^2 dt. \end{aligned}$$

Let f and F be defined by (3.2), and g and G by (6.2). Then

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \xi\left(\frac{1}{2}+it\right) \eta\left(\frac{1}{2}+it\right) \right|^2 \left| \Gamma\left(\frac{1}{2}+it\right) \right|^4 e^{(2\pi-4\delta)t} dt \\ &= \frac{1}{2\pi} \int_0^{\infty} \left| \int_{-\infty}^{\infty} \frac{1}{2} e^{\frac{1}{2}x} \operatorname{sech}(ie^{x-i\delta}) e^{\frac{1}{2}(x+t)} \left\{ \frac{1}{\exp(-ie^{x+t+i\delta})-1} + \frac{1}{ie^{x+t+i\delta}} \right\} dx \right|^2 dt \\ & \quad + \frac{1}{2\pi} \int_0^{\infty} \left| \int_{-\infty}^{\infty} \frac{1}{2} e^{\frac{1}{2}(x+t)} \operatorname{sech}(ie^{x+t-i\delta}) e^{\frac{1}{2}x} \left\{ \frac{1}{\exp(-ie^{x+i\delta})-1} + \frac{1}{ie^{x+i\delta}} \right\} dx \right|^2 dt \\ &= \frac{1}{8\pi} \int_1^{\infty} \left| \int_0^{\infty} \operatorname{sech}(iue^{-i\delta}) \left\{ \frac{1}{\exp(-iue^{i\delta})-1} + \frac{1}{iue^{i\delta}} \right\} du \right|^2 dv \\ & \quad + \frac{1}{8\pi} \int_1^{\infty} \left| \int_0^{\infty} \operatorname{sech}(iue^{-i\delta}) \left\{ \frac{1}{\exp(-iue^{i\delta})-1} + \frac{1}{iue^{i\delta}} \right\} du \right|^2 dv \\ &= I_1 + I_2. \end{aligned}$$

7.2. Discussion of I_1 . It may be proved, as in § 5.1, that

$$I_1 \sim \frac{1}{8\pi} \int_1^{\infty} \left| \int_1^{\infty} \operatorname{sech}(iue^{-i\delta}) \frac{du}{\exp(-iue^{i\delta})-1} \right|^2 dv.$$

Turning the line of integration in the u -plane from $(1, \infty)$ to $(1, 1+i\infty)$, and using the calculus of residues, we see, as in previous cases, that the u -integral is, apart from a negligible remainder, equal to

$$\begin{aligned} & 2i\pi e^{i\delta} \sum_{n=1}^{\infty} \frac{(-1)^n}{\exp\left\{-\frac{1}{2}i\pi(2n-1)ve^{2i\delta}\right\}-1} \\ &= 2i\pi e^{i\delta} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^n \exp\left\{\frac{1}{2}i\pi m(2n-1)ve^{2i\delta}\right\} \\ &= -\frac{1}{2}i\pi e^{i\delta} \sum_{n=1}^{\infty} r(n) \exp\left(\frac{1}{2}i\pi nve^{2i\delta}\right), \end{aligned}$$

where

$$r(n) = 4 \sum_{2q+1 \mid n} (-1)^q.$$

Hence

$$\begin{aligned} I_1 &\sim \frac{\pi}{32} \int_1^\infty \left| \sum_{n=1}^\infty r(n) \exp(\tfrac{1}{2} i \pi n v e^{2i\delta}) \right|^2 dv \\ &= \frac{\pi}{32} \sum_{m=1}^\infty \sum_{n=1}^\infty r(m) r(n) \frac{\exp\{\tfrac{1}{2} i \pi (m+n) \sin 2\delta + \tfrac{1}{2} i \pi (m-n) \cos 2\delta\}}{\tfrac{1}{2} \pi (m+n) \sin 2\delta - \tfrac{1}{2} i \pi (m-n) \cos 2\delta} \\ &= \frac{1}{32 \sin 2\delta} \sum_{n=1}^\infty \frac{\{r(n)\}^2}{n} e^{-\pi n \sin 2\delta} \\ &\quad + \frac{1}{8} \sum_{m=2}^\infty \sum_{n=1}^{m-1} r(m) r(n) \frac{(m+n) \sin 2\delta \cos\{\tfrac{1}{2} \pi (m-n) \cos 2\delta\}}{(m+n)^2 \sin^2 2\delta + (m-n)^2 \cos^2 2\delta} e^{-\tfrac{1}{2} (m+n) \pi \sin 2\delta} \\ &\quad - \frac{1}{8} \sum_{m=2}^\infty \sum_{n=1}^{m-1} r(m) r(n) \frac{(m-n) \cos 2\delta \sin\{\tfrac{1}{2} \pi (m-n) \cos 2\delta\}}{(m+n)^2 \sin^2 2\delta + (m-n)^2 \cos^2 2\delta} e^{-\tfrac{1}{2} (m+n) \pi \sin 2\delta} \\ &= \Sigma_1 + \Sigma_2 + \Sigma_3. \end{aligned}$$

Now Ramanujan also gives the result

$$\sum_{\nu=1}^n \{r(\nu)\}^2 \sim \tfrac{1}{4} n \log n.$$

Hence by partial summation

$$\Sigma_1 \sim \tfrac{1}{5 \cdot \tfrac{1}{2}} \frac{1}{\delta} \log^2 \frac{1}{\delta}.$$

The estimation of Σ_2 is exactly analogous to that of § 5. 2, and gives

$$\Sigma_2 = O\left(\frac{1}{\delta} \log \frac{1}{\delta}\right).$$

Σ_3 , however, is slightly more complicated, owing to the occurrence of $\tfrac{1}{2}\pi$ instead of 2π . Here we write

$$\frac{m-n}{(m+n)^2 \sin^2 2\delta + (m-n)^2 \cos^2 2\delta} = \frac{1}{(m-n) \cos^2 2\delta} + O\left\{\frac{m^2 \delta^2}{(m-n)^3}\right\}.$$

The O -term gives

$$\begin{aligned} O\left\{\sum_{m=2}^\infty \sum_{n=1}^{m-1} r(m) r(n) e^{-\tfrac{1}{2} m \pi \sin 2\delta} \frac{m^2 \delta^2}{(m-n)^3}\right\} \\ = O\left\{\delta^2 \sum_{\nu=1}^\infty \frac{1}{\nu^3} \sum_{m=\nu+1}^\infty m^2 r(m) r(m-\nu) e^{-\tfrac{1}{2} m \pi \sin 2\delta}\right\} = O\left(\frac{1}{\delta} \log \frac{1}{\delta}\right), \end{aligned}$$

as in the case of Σ_2 . Finally

$$\begin{aligned} & \left| \sum_{m=2}^{\infty} \sum_{n=1}^{m-1} r(m)r(n) \frac{\sin \left\{ \frac{1}{2}\pi(m-n) \cos 2\delta \right\}}{m-n} e^{-\frac{1}{2}(m+n)\pi \sin 2\delta} \right| \\ &= \left| \frac{1}{4} \int_{-\frac{1}{2}\pi \cos 2\delta}^{\frac{1}{2}\pi \cos 2\delta} \left| \sum_{m=1}^{\infty} r(m) e^{-\frac{1}{2}m\pi \sin 2\delta + im\theta} \right|^2 d\theta - \frac{1}{4}\pi \cos 2\delta \sum_{m=1}^{\infty} \{r(m)\}^2 e^{-m\pi \sin 2\delta} \right| \\ &< \frac{1}{4} \int_{-\pi}^{\pi} \left| \sum_{m=1}^{\infty} r(m) e^{-\frac{1}{2}m\pi \sin 2\delta + im\theta} \right|^2 d\theta + \frac{1}{4}\pi \cos 2\delta \sum_{m=1}^{\infty} \{r(m)\}^2 e^{-m\pi \sin 2\delta} \\ &= (\tfrac{1}{2}\pi + \tfrac{1}{4}\pi \cos 2\delta) \sum_{m=1}^{\infty} \{r(m)\}^2 e^{-m\pi \sin 2\delta} = O\left(\frac{1}{\delta} \log \frac{1}{\delta}\right). \end{aligned}$$

Hence

$$I_1 \sim \frac{1}{5\frac{1}{2}} \frac{1}{\delta} \log^2 \frac{1}{\delta}.$$

7.3. *Discussion of I_2 .* Here we turn the u -integral from $(0, \infty)$ to $(0, -i\infty)$, and obtain, apart from a negligible remainder,

$$\begin{aligned} & -2\pi e^{-i\delta} \sum_{n=1}^{\infty} \operatorname{sech}(2in\pi v e^{-2i\delta}) \\ &= -4\pi e^{-i\delta} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{m-1} \exp \{ -2in(2m-1)\pi v e^{-2i\delta} \} \\ &= \pi e^{-i\delta} \sum_{n=1}^{\infty} r(n) \exp(-2in\pi v e^{-2i\delta}). \end{aligned}$$

Hence

$$I_2 \sim \frac{\pi}{8} \int_1^{\infty} \left| \sum_{n=1}^{\infty} r(n) \exp(-2in\pi v e^{-2i\delta}) \right|^2 dv,$$

and the discussion of this proceeds on the lines of § 5.2. The result is

$$I_1 \sim \frac{1}{5\frac{1}{2}} \frac{1}{\delta} \log^2 \frac{1}{\delta},$$

and Theorem V follows.

[*Note.*—Theorem II of my paper “On integral functions with real negative zeros”, *Proc. London Math. Soc.*, (2), 26 (1927), 185–200, is due to G. Valiron, *Thèses présentées à la faculté des sciences de Paris*, 1914.]

ON THE SOLUTION OF HAMILTONIAN SYSTEMS OF DIFFERENTIAL EQUATIONS IN THE NEIGHBOURHOOD OF A SINGULAR POINT

By T. M. CHERRY.

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WE shall be concerned with the solution of systems of the form

$$\frac{dx_k}{dt} = \frac{\partial F}{\partial y_k}, \quad \frac{dy_k}{dt} = -\frac{\partial F}{\partial x_k} \quad (k = 1, \dots, n), \quad (0.1)$$

in which F does not involve t and is a convergent series of powers of the x_k, y_k which begins with quadratic terms. The origin is a singular point of the system, since the right-hand sides of the equations vanish there.

The equations (0.1) belong to the general type

$$\frac{dx_k}{dt} = X_k \quad (k = 1, \dots, m), \quad (0.2)$$

in which the X_k are power series in $x_1, \dots, x_m$ vanishing at the origin. As regards the general solution of these more general equations the following theorem is well known* :

Let $a_{k1}x_1 + \dots + a_{km}x_m$ be the linear terms in X_k , and let $\lambda_1, \dots, \lambda_m$ be the roots of the equation

$$\begin{vmatrix} a_{11} - \lambda, & a_{12}, & \dots \\ a_{21}, & a_{22} - \lambda, & \dots \\ \dots & \dots & \dots \end{vmatrix} = 0 \quad (0.3)$$

* See, for example, Picard, *Traité d'Analyse*, 3, ch. 1.

and suppose that these satisfy the three conditions :

(a) if, in the complex λ -plane, we mark the points $\lambda_1, \dots, \lambda_m$, it is possible to draw through the origin a straight line which leaves them all on the same side* ;

(b) there is no relation of the form

$$p_1\lambda_1 + \dots + p_m\lambda_m = \lambda_r$$

in which the p_k are non-negative integers whose sum is greater than 1 ;

(c) no two of $\lambda_1, \dots, \lambda_m$ are equal ;

then the equations (0.2) have a general solution of the form

$$x_k = P_k(c_1 e^{\lambda_1 t}, \dots, c_m e^{\lambda_m t}),$$

where $c_1, \dots, c_m$ are arbitrary constants, and $P_1, \dots, P_m$ are power series convergent when the moduli of their m arguments are sufficiently small.

If either or both of the conditions (b), (c) are not satisfied it is still possible to find a solution in which $x_1, \dots, x_m$ are analytic functions of t and m independent arbitrary constants†. If, however, the condition (a) is not satisfied, no such general solution appears to be known ; we can find solutions in the form of series having a finite domain of convergence, but they involve fewer than m arbitrary constants.

For equations of Hamiltonian form (0.1) it is well known that (0.3) is an equation in λ^2 , so that if in the λ -plane we mark the points $\lambda_1, \dots, \lambda_{2n}$ we obtain a figure symmetrical about the origin ; thus the condition (a) is never satisfied. The series for the x_k, y_k to which we are led by a straightforward process of solution by successive approximation are power series in $(2n+1)$ arguments

$$\alpha_1 e^{\lambda_1 t}, \beta_1 e^{-\lambda_1 t}, \dots, \alpha_n e^{\lambda_n t}, \beta_n e^{-\lambda_n t}, t,$$

where $\alpha_1, \dots, \beta_n$ are arbitrary constants, and $\pm\lambda_1, \dots, \pm\lambda_n$ are the roots of (0.3). The object of this paper is twofold, viz. :

(1) to show that, when $\lambda_1, \dots, \lambda_n$ are mutually incommensurable, these series may be rearranged so that t appears only as the argument of exponential functions ;

(2) to show, in addition, that, when $n = 2$ and λ_1/λ_2 is not real, these rearranged series have a finite domain of absolute convergence.

* When the a_{rs} are real this implies that $\lambda_1, \dots, \lambda_m$ are to have non-zero real parts all of the same sign.

† Horn, *Journal für Math.*, 116 (1896), 256 ; 117 (1896), 104, 254.

We thus have a case in which, in spite of the non-satisfaction of condition (a), the system (0.2) has a general solution in the form of *absolutely convergent* series of the multiple Fourier type.

A system of the form (0.1) is easily transformed into one belonging to the type met with in Celestial Mechanics, and the series in §§ 2, 3 below are analogous to those of Lindstedt; the methods of this paper differ, however, from those customary in Celestial Mechanics in that power series only are employed.

Contact transformations. A transformation under which an arbitrary Hamiltonian system remains of Hamiltonian form is called a *contact transformation*. We shall use the conditions that a transformation

$$x_r = f_r(z_k, u_k, t), \quad y_r = g_r(z_k, u_k, t) \quad (r = 1, \dots, n)$$

may be of this nature in the form involving Lagrange brackets*, viz.

$$\left. \begin{aligned} [z_r, u_r] &= 1 & (r = 1, \dots, n) \\ [z_r, u_s] &= 0 & (r, s = 1, \dots, n; r \neq s) \\ [z_r, z_s] &= [u_r, u_s] = 0 & (r, s = 1, \dots, n) \end{aligned} \right\};$$

this form of the conditions is more convenient than the customary form $\Sigma(x_r dy_r - z_r du_r) = dW$, when we are dealing with power series.

Transformation of the linear terms. Let $\pm\lambda_1, \dots, \pm\lambda_n$ be the roots of the equation (0.3): these are called the *exponents*† of the origin. If no two of these are equal there exist linear transformations which bring the equations (0.1), so far as their linear terms are concerned, to the form

$$\frac{dx'_k}{dt} = \lambda_k x'_k, \quad \frac{dy'_k}{dt} = -\lambda_k y'_k \quad (k = 1, \dots, n),$$

and included in the family of such transformations is a sub-family of *contact transformations*‡. In the case of unequal exponents—with which alone we shall be concerned—there is thus no loss of generality in supposing that the Hamiltonian function F of our equations (0.1) has as its second degree terms

$$\lambda_1 x_1 y_1 + \dots + \lambda_n x_n y_n. \quad (0.4)$$

* See, for example, Whittaker, *Analytical dynamics*, § 129. For the definition of a Lagrange bracket-expression, see Whittaker, *loc. cit.* or equation (1.4) below.

† Or *characteristic exponents* (Poincaré, *Méthodes nouvelles*, No. 59).

‡ Cherry, "On the transformation of Hamiltonian systems of linear differential equations with constant or periodic coefficients", *Proc. London Math. Soc.* (2), 26 (1927), 211.

1. *Formal solution of the equations : series involving secular terms.*

With a view to a subsequent generalization of the methods of this paper we shall suppose, in this section, that the coefficients of the cubic and higher terms in F are periodic functions of $(t+\epsilon)$, ϵ being an arbitrary parameter; we here, of course, include the case in which these coefficients are constants. Writing $2\pi i/\nu$ for the period of these functions and $\gamma = e^{\nu\epsilon}$, we suppose that the periodic functions in question are expanded as Laurent series in the argument $\gamma e^{\nu t}$, γ being an arbitrary parameter.

Substitute, in (0.1),

$$\left. \begin{aligned} x_k &= x_k^{(1)} + x_k^{(2)} + \dots \\ y_k &= y_k^{(1)} + y_k^{(2)} + \dots \end{aligned} \right\}, \quad (1.1)$$

and equate separately the sets of terms whose orders, as measured by the sum of the superscripts in their factors, are equal. For the first order terms we obtain

$$\frac{dx_k^{(1)}}{dt} = \lambda_k x_k^{(1)}, \quad \frac{dy_k^{(1)}}{dt} = -\lambda_k y_k^{(1)},$$

giving the general solution

$$x_k^{(1)} = \alpha_k e^{\lambda_k t}, \quad y_k^{(1)} = \beta_k e^{-\lambda_k t}, \quad (1.2)$$

where the α_k, β_k are arbitrary constants of integration.

The m -th order terms give $(dx_k^{(m)}/dt) - \lambda_k x_k^{(m)}$, $(dy_k^{(m)}/dt) + \lambda_k y_k^{(m)}$ as m -th order polynomials in the $x_k^{(1)}, y_k^{(1)}, \dots, x_k^{(m-1)}, y_k^{(m-1)}$, the coefficients being Laurent series in $\gamma e^{\nu t}$. We may solve in succession for the terms of orders 2, 3, ..., and it is easily seen that we may obtain the $x_k^{(m)}, y_k^{(m)}$ as homogeneous m -th degree polynomials in the arguments

$$\alpha_1 e^{\lambda_1 t}, \beta_1 e^{-\lambda_1 t}, \dots, \alpha_n e^{\lambda_n t}, \beta_n e^{-\lambda_n t}, \quad (A),$$

the coefficients being Laurent series in $\gamma e^{\nu t}$ with coefficients polynomial in t .

For suppose that the $x_k^{(r)}, y_k^{(r)}$ for $r = 1, 2, \dots, (m-1)$ have been obtained in this form. Then the equation for the $x_k^{(m)}, y_k^{(m)}$ have the form

$$\frac{dx_k^{(m)}}{dt} - \lambda_k x_k^{(m)} = f_k, \quad \frac{dy_k^{(m)}}{dt} + \lambda_k y_k^{(m)} = g_k,$$

where the f_k, g_k are known functions of t , being homogeneous m -th degree polynomials in the arguments (A), the coefficients being Laurent

series in $\gamma e^{\nu t}$ with coefficients polynomial in t . Now to the typical term

$$(\gamma e^{\nu t})^c (a_1 e^{\lambda_1 t})^{a_1} (\beta_1 e^{-\lambda_1 t})^{b_1} \dots (a_n e^{\lambda_n t})^{a_n} (\beta_n e^{-\lambda_n t})^{b_n} \times (\text{polynomial in } t)$$

in f_k corresponds for $x_k^{(m)}$ a particular integral of the same form, the degree of the polynomial being increased by 0 or 1 according as

$$\lambda_1(a_1 - b_1) + \dots + \lambda_n(a_n - b_n) + \nu c \neq \lambda_k \text{ or } = \lambda_k;$$

and similarly for $y_k^{(m)}$. Hence there is a solution for the $x_k^{(m)}$, $y_k^{(m)}$ of the form stated above, and, by induction, this form is proved generally valid.

So far the Hamiltonian form of the equations (0.1) has played no part; we now show that, as a consequence of this form, the series (1.1) may always be supposed such that the Lagrange brackets of the constants of integration have the values

$$\left. \begin{aligned} [a_r, \beta_r] &= 1 & (r = 1, \dots, n) \\ [a_r, \beta_s] &= 0 & (r, s = 1, \dots, n; r \neq s) \\ [a_r, a_s] &= [\beta_r, \beta_s] = 0 & (r, s = 1, \dots, n) \end{aligned} \right\}. \quad (1.3)$$

The importance of this property will appear in § 2 below.

$$\text{We have} \quad [a_r, \beta_s] = \sum_{k=1}^n \left(\frac{\partial x_k}{\partial a_r} \frac{\partial y_k}{\partial \beta_s} - \frac{\partial x_k}{\partial \beta_s} \frac{\partial y_k}{\partial a_r} \right); \quad (1.4)$$

from the form of the series (1.1) it follows that $a_r \beta_s [a_r, \beta_s]$ is a power series in the arguments (A), the coefficients being Laurent series in γe^{ν} with coefficients polynomial in t . Since the equations (0.1) are Hamiltonian, t must disappear identically from this series*, and since γ and the a_k, β_k are arbitrary constants the coefficient of $\gamma^c a_1^{a_1} \beta_1^{b_1} \dots a_n^{a_n} \beta_n^{b_n}$ must be independent of t for all values of the indices c, a_k, b_k . This coefficient is

$$e^{c\nu t + (a_1 - b_1)\lambda_1 t + \dots + (a_n - b_n)\lambda_n t} \times (\text{polynomial in } t),$$

and is independent of t only if (1) the polynomial reduces to a constant and

$$c\nu + (a_1 - b_1)\lambda_1 + \dots + (a_n - b_n)\lambda_n = 0, \quad (1.5)$$

or (2) the polynomial vanishes identically. Thus $a_r \beta_s [a_r, \beta_s]$ must reduce to a series of the form

$$\sum f_{rs} \gamma^c a_1^{a_1} \beta_1^{b_1} \dots a_n^{a_n} \beta_n^{b_n},$$

* Whittaker, *Analytical dynamics*, § 146.

where f_{rs} is a numerical* coefficient, and for each term the indices satisfy (1.5)†; this series, of course, must have $\alpha_r \beta_s$ as a factor. A similar argument applies to the series for the other Lagrange brackets; corresponding to a set of indices satisfying (1.5) there are possible terms

$$\left. \begin{aligned} \text{in } [\alpha_r, \alpha_s]: & \quad d_{rs} \gamma^c \alpha_1^{a_1} \beta_1^{b_1} \dots \alpha_n^{a_n} \beta_n^{b_n} / \alpha_r \alpha_s \\ \text{in } [\beta_r, \beta_s]: & \quad e_{rs} \gamma^n \alpha_1^{a_1} \beta_1^{b_1} \dots \alpha_n^{a_n} \beta_n^{b_n} / \beta_r \beta_s \\ \text{in } [\alpha_r, \beta_s]: & \quad f_{rs} \gamma^c \alpha_1^{a_1} \beta_1^{b_1} \dots \alpha_n^{a_n} \beta_n^{b_n} / \alpha_r \beta_s \end{aligned} \right\} \quad (1.6)$$

where the d, e, f are numerical coefficients, and no terms can occur which are not thus in correspondence with a set of indices satisfying (1.5).

From (1.2) the terms in the Lagrange brackets of zero order with respect to the α_k, β_k have the value 0 or 1, as shown in (1.3). We desire then to show that the d, e, f belonging to all terms of higher order are zero. The possibility of this being so arises from the fact that to the "particular integral" solutions for $x_k^{(m)}, y_k^{(m)}$ we may add the "complementary function" terms $A_k e^{\lambda_k t}, B_k e^{-\lambda_k t}$ respectively, where A_k, B_k are arbitrary constants; the vanishing of the d, e, f depends on the proper assignment of these arbitraries.

Consider a particular set of values of c and the α_k, β_k satisfying (1.5), a set, say, in which the sum of the α_k and β_k is $(m+1)$. Corresponding to this set we may add to $x_k^{(m)}$ a term

$$p_k \gamma^c \alpha_1^{a_1} \beta_1^{b_1} \dots \alpha_n^{a_n} \beta_n^{b_n} e^{\lambda_k t} / \beta_k, \quad (1.7)$$

where p_k is an arbitrary numerical constant, for such a term is of the type proper to the "complementary function" part of $x_k^{(m)}$; if $b_k = 0$ we put $p_k = 0$. In virtue of (1.5) we may write this term in the form

$$p_k (\gamma e^{\nu t})^c (\alpha_1 e^{\lambda_1 t})^{a_1} (\beta_1 e^{-\lambda_1 t})^{b_1} \dots (\alpha_n e^{\lambda_n t})^{a_n} (\beta_n e^{-\lambda_n t})^{b_n} / \beta_k e^{-\lambda_k t},$$

so that $x_k^{(m)}$ remains a homogeneous m -th degree polynomial in the arguments (A) with coefficients which are Laurent series in $\gamma e^{\nu t}$. We may

* To avoid confusion between the arbitrary constants $\gamma, \alpha_k, \beta_k$ and definite constants whose numerical values are determinate as soon as we know in F the numerical value of the coefficient of each term

$$x_1^{a_1} y_1^{b_1} \dots x_n^{a_n} y_n^{b_n} (\gamma e^{\nu t})^c,$$

we characterize the latter sort of constant as *numerical*.

† Of course in this relation the α_k, β_k must be non-negative integers and c must be an integer.

add to $y_k^{(m)}$ a corresponding term

$$q_k \gamma^c a_1^{a_1} \beta_1^{b_1} \dots a_n^{a_n} \beta_n^{b_n} e^{-\lambda_k t} / \alpha_k, \quad (1.8)$$

where q_k is an arbitrary numerical constant, which we put zero if $a_k = 0$. Now the terms in $[\alpha_r, \beta_s]$ of degree $(m-1)$ in the α_k, β_k arise from

$$\sum_{k=1}^n \left\{ \frac{\hat{C}(x_k^{(1)}, y_k^{(m)})}{\partial(\alpha_r, \beta_s)} + \frac{\hat{C}(x_k^{(2)}, y_k^{(m-1)})}{\partial(\alpha_r, \beta_s)} + \dots + \frac{\partial(x_k^{(m)}, y_k^{(1)})}{\partial(\alpha_r, \beta_s)} \right\},$$

which on use of (1.2) becomes

$$e^{\lambda_r t} \frac{\partial y_r^{(m)}}{\partial \beta_s} + e^{-\lambda_s t} \frac{\partial x_s^{(m)}}{\partial \alpha_r} + \sum_{k=1}^n \left\{ \frac{\partial(x_k^{(2)}, y_k^{(m-1)})}{\partial(\alpha_r, \beta_s)} + \dots + \frac{\partial(x_k^{(m-1)}, y_k^{(2)})}{\partial(\alpha_r, \beta_s)} \right\},$$

and from the terms in $x_s^{(m)}, y_r^{(m)}$ of the types (1.7), (1.8) respectively we obtain in $[\alpha_r, \beta_s]$ terms of the type (1.6); we find, in fact,

$$f_{rs} = b_s q_r + a_r p_s + F_{rs}, \quad (1.9)$$

where F_{rs} depends only on the numerical coefficients in the $x_k^{(2)}, y_k^{(2)}, \dots, x_k^{(m-1)}, y_k^{(m-1)}$. Similarly

$$\left. \begin{aligned} d_{rs} &= a_s q_r - a_r q_s + D_{rs} \\ e_{rs} &= b_r p_s - b_s p_r + E_{rs} \end{aligned} \right\}. \quad (1.10)$$

The possibility of so assigning the p, q that the d, e, f vanish is now manifest; but further investigation is necessary, since there are $3n^2$ of the d, e, f , and only $2n$ of the p, q . There are, in fact, certain linear relations connecting the d, e, f .

We have identically

$$\frac{\partial}{\partial \alpha_k} [a_r, a_s] + \frac{\partial}{\partial \alpha_r} [a_s, a_k] + \frac{\partial}{\partial \alpha_s} [a_k, a_r] = 0,$$

or, on substitution of the series of which the typical terms are given by (1.6),

$$(a_k a_r a_s)^{-1} \Sigma (a_k d_{rs} + a_r d_{sk} + a_s d_{kr}) \gamma^c a_1^{a_1} \beta_1^{b_1} \dots a_n^{a_n} \beta_n^{b_n} = 0.$$

This being true for all values of the arbitrary $\gamma, \alpha_k, \beta_k$, we deduce that

$$a_k d_{rs} + a_r d_{sk} + a_s d_{kr} = 0, \quad (1.11)$$

for all sets of values of the indices c, a_k, b_k satisfying (1.5). From the

similar identities involving two a 's and one β , etc., we obtain similarly

$$b_k d_{rs} + a_r f_{sk} - a_s f_{rk} = 0, \quad (1.12)$$

$$a_k e_{rs} - b_r f_{ks} + b_s f_{kr} = 0, \quad (1.13)$$

$$b_k e_{rs} + b_r e_{sk} + b_s e_{kr} = 0. \quad (1.14)$$

The identities $[a_r, a_s] = -[a_s, a_r]$, $[\beta_r, \beta_s] = -[\beta_s, \beta_r]$ lead similarly to

$$d_{rs} = -d_{sr}, \quad e_{rs} = -e_{sr}. \quad (1.15)$$

Vanishing of the d , e , f . From (1.2) the terms in the Lagrange brackets of degree 0 in the a_k, β_k have the values (1.3). We assign in succession the p 's and q 's occurring in the $x_k^{(2)}, y_k^{(2)}, \dots, x_k^{(m)}, y_k^{(m)}, \dots$, there being n p 's and n q 's corresponding to each relation of the form (1.5), so that the terms of degree 1, ... $(m-1)$, ... in the Lagrange brackets may vanish.

To show that such an assignment is possible, suppose that the process has been successful as far as the terms of degree $(m-2)$ in the Lagrange brackets; then in (1.9), (1.10) the D, E, F have definite numerical values. It is obviously sufficient to consider the p, q, d, e, f belonging to a typical set of indices satisfying (1.5).

First suppose $a_1 \neq 0$. We give q_1 any definite numerical value we please; then we may determine uniquely

$$p_k \text{ from the condition } f_{1k} = 0 \text{ for } k = 1, \dots, n, \quad (1.16)$$

$$q_k \text{ from the condition } d_{1k} = 0 \text{ for } k = 2, \dots, n,$$

for in each case the coefficient of the quantity to be determined is the non-zero a_1 . From (1.15) we then deduce

$$d_{k1} = 0 \quad (k = 1, \dots, n).$$

From (1.11) with $k = 1$ we then have

$$d_{rs} = 0 \quad (r, s = 1, \dots, n); \quad (1.17)$$

putting $r = 1$ in (1.12) and using (1.16), (1.17) we have

$$f_{sk} = 0 \quad (s, k = 1, \dots, n);$$

and finally putting $k = 1$ in (1.13)

$$e_{rs} = 0 \quad (r, s = 1, \dots, n).$$

A similar argument applies unless every a and b is zero, and this cannot be the case since their sum is positive.

Thus in the general solution (1.1) the right-hand sides may always be supposed to be series of non-negative powers of the arguments (A) and positive and negative powers of $\gamma e^{\nu t}$ with coefficients polynomial in t , these series being such that the Lagrange brackets of the constants of integration α_k, β_k have the values (1.3).

This theorem remains true if two or more of the exponents of the origin ($\pm\lambda_1, \dots, \pm\lambda_n$) are equal; only trivial modifications are required in the exposition*.

2. Transformation of the equations. Existence of a general solution free of secular terms when $\lambda_1, \dots, \lambda_n$ are incommensurable.

We suppose from now onward that the coefficients in the series F are constants. Then the method of § 1 gives for the general solution of the equations (0.1)

$$\left. \begin{aligned} x_r &= f_r(\alpha_k e^{\lambda_k t}, \beta_k e^{-\lambda_k t}, t) \\ y_r &= g_r(\alpha_k e^{\lambda_k t}, \beta_k e^{-\lambda_k t}, t) \end{aligned} \right\} \quad (r = 1, \dots, n), \quad (2.1)$$

where the f_r, g_r are power series in the arguments (A) with coefficients polynomial in t , these series being such that the Lagrange brackets of the constants of integration have the values (1.3).

This form of the general solution, even if convergent, is useless for many purposes; for instance, the fact is hidden that t occurs additively with an arbitrary constant, and no indication is given of the possible existence of periodic solutions. We desire, therefore, to obtain a form of the general solution in which t occurs only as the argument of exponential functions; in accordance with the nomenclature of Celestial Mechanics a term $At^m e^{\mu t}$ in which $m \neq 0$ will be called a *secular term*, and we desire a solution free of secular terms.

For this purpose we transform the system (0.1) into a simpler system in the following way. We put $t = 0$ in (2.1), and regard the resulting equations, say

$$\left. \begin{aligned} x_r &= F_r(\alpha_k, \beta_k) \\ y_r &= G_r(\alpha_k, \beta_k) \end{aligned} \right\} \quad (r = 1, \dots, n), \quad (2.2)$$

as a transformation to new variables $\alpha_1, \beta_1, \dots, \alpha_n, \beta_n$. By putting $t = 0$ we shall not alter the values of the Lagrange brackets $[a_r, \beta_s]$, etc., as

* Modifications are necessary because the quadratic terms in F can no longer be supposed to be of the form (0.4) (Cherry, *loc. cit.*).

given by (1.3); for since in forming these bracket-expressions we do not differentiate with respect to t we may as well put $t = 0$ after the differentiations with respect to the α_k, β_k as before, and the right-hand sides of (1.3) are already independent of t . Thus (2.2) is a contact transformation.

The Hamiltonian function of the transformed system is obtained from $F(x_k, y_k)$ by making the substitution (2.2). To find its form consider the effect of making in F the substitution (2.1). We obtain a power series in the arguments (A) with coefficients polynomial in t , and since F is an integral of the equations (0.1) and the α_k, β_k are constants of integration t must disappear identically from this series. By the argument previously used when considering the series for the Lagrange bracket $[a_r, \beta_s]$ it follows that F becomes a power series in the α_k, β_k , say

$$K(\alpha_k, \beta_k) \equiv \sum A \alpha_1^{a_1} \beta_1^{b_1} \dots \alpha_n^{a_n} \beta_n^{b_n}, \quad (2.3)$$

such that for each term the indices satisfy the relation

$$\lambda_1(a_1 - b_1) + \dots + \lambda_n(a_n - b_n) = 0.$$

To find the effect of the substitution (2.2) we have only to put $t = 0$ in the result (2.3); this leaves K unaltered, since it is already independent of t .

It is convenient to rewrite (2.2), calling the new variables z_k, u_k instead of α_k, β_k . We see that there is a contact transformation

$$x_r = F_r(z_k, u_k), \quad y_r = G_r(z_k, u_k) \quad (r = 1, \dots, n), \quad (2.4)$$

where the F_r, G_r are formal power series in their arguments, under which the system (0.1) becomes

$$\frac{dz_k}{dt} = \frac{\partial K}{\partial u_k}, \quad \frac{du_k}{dt} = -\frac{\partial K}{\partial z_k} \quad (k = 1, \dots, n), \quad (2.5)$$

where K is a formal power series

$$K \equiv \sum A z_1^{a_1} u_1^{b_1} \dots z_n^{a_n} u_n^{b_n}, \quad (2.6)$$

in which for each term the indices satisfy the relation

$$\lambda_1(a_1 - b_1) + \dots + \lambda_n(a_n - b_n) = 0. \quad (2.7)$$

The series in (2.4), (2.6) are called *formal power series* because we have not proved them convergent.

In (2.1) the terms linear in the α_k, β_k are given by (1.2); hence, as far as the linear terms, (2.4) is

$$x_r = z_r, \quad y_r = u_r,$$

and from (0.4) the terms of lowest degree in K are the quadratic terms

$$\lambda_1 z_1 u_1 + \dots + \lambda_n z_n u_n. \quad (2.8)$$

The comparative simplicity of the transformed system may be appreciated by considering in detail the case in which $\lambda_1, \dots, \lambda_n$ are mutually incommensurable, *i.e.* are connected by no relation

$$A_1 \lambda_1 + \dots + A_n \lambda_n = 0 \quad (2.9)$$

in which $A_1, \dots, A_n$ are integers (positive, negative, or zero), not all zero. Then (2.7) requires that

$$a_1 = b_1, \quad \dots, \quad a_n = b_n,$$

and K reduces to a power series in the n arguments $v_1, \dots, v_n$, where

$$v_k = z_k u_k.$$

The equations (2.5) may thus be written

$$\frac{dz_k}{dt} = z_k \frac{\partial K}{\partial v_k}, \quad \frac{du_k}{dt} = -u_k \frac{\partial K}{\partial v_k}, \quad (2.10)$$

whence

$$\frac{dv_k}{dt} = z_k \frac{du_k}{dt} + \frac{dz_k}{dt} u_k = 0.$$

Hence, for any solution, the v_k are constant, and therefore also $\partial K / \partial v_k = m_k$, say, is constant. The equations (2.10) accordingly have the general solution

$$z_k = \alpha_k e^{m_k t}, \quad u_k = \beta_k e^{-m_k t}, \quad (2.11)$$

where the α_k, β_k are constants of integration in terms of which the v_k are given by $v_k = \alpha_k \beta_k$; the exponents m_k are thus power series in the $\alpha_k \beta_k$, and, from the form (2.8) of the leading terms in K , it follows that, when the $\alpha_k \beta_k$ all vanish, m_k reduces to λ_k .

If we substitute from (2.11) in (2.4) we obtain the general solution of the equations (0.1) expressed in a form free from secular terms; the x_k, y_k appear as power series in the $\alpha_k e^{m_k t}, \beta_k e^{-m_k t}$:

$$x_r = F_r(\alpha_k e^{m_k t}, \beta_k e^{-m_k t}), \quad y_r = G_r(\alpha_k e^{m_k t}, \beta_k e^{-m_k t}) \quad (r = 1, \dots, n). \quad (2.12)$$

Bearing in mind the manner in which (2.4) was derived from (2.1) we have the curious result that the new form (2.12) may be derived from

the old form (2.1) simply by *omitting* the secular terms and replacing $\lambda_1, \dots, \lambda_n$, wherever they appear as multiples of t in arguments of exponential functions, by suitable power series in the $\alpha_k \beta_k^*$.

Results of a character somewhat similar but more complicated can be obtained in cases where $\lambda_1, \dots, \lambda_n$ are connected by relations of the form (2.9). As the resulting series for the general solution are apparently never convergent except in quite special cases, we shall not consider the matter here†.

There is one general case in which the series (2.12) may be proved convergent, viz., when $n = 2$ and the ratio λ_1/λ_2 is not a real number. This will now be shown by giving first a direct method for the construction of these series and then a direct proof of their convergence.

3. Formal solution of the equations by series free of secular terms.

The process of the present section is valid whenever $\lambda_1, \dots, \lambda_n$ are connected by no relation of the form (2.9). Since, however, we shall require that $n = 2$ in § 4, we simplify the exposition by taking $n = 2$. We write

$$w_1 = (\lambda_1 + p_1)t, \quad w_2 = (\lambda_2 + p_2)t, \quad (3.1)$$

where p_1, p_2 are constants which at present are arbitrary, and we regard the x_k, y_k as functions of w_1, w_2 . If we put

$$F = \lambda_1 x_1 y_1 + \lambda_2 x_2 y_2 + F',$$

the equations (0.1) become

$$\left. \begin{aligned} (\lambda_1 + p_1) \frac{\partial x_k}{\partial w_1} + (\lambda_2 + p_2) \frac{\partial x_k}{\partial w_2} &= \lambda_k x_k + \frac{\partial F'}{\partial y_k} \\ (\lambda_1 + p_1) \frac{\partial y_k}{\partial w_1} + (\lambda_2 + p_2) \frac{\partial y_k}{\partial w_2} &= -\lambda_k y_k - \frac{\partial F'}{\partial x_k} \end{aligned} \right\} \quad (k = 1, 2). \quad (3.2)$$

Substitute here

$$\left. \begin{aligned} x_k &= x_k^{(1)} + x_k^{(2)} + \dots \\ y_k &= y_k^{(1)} + y_k^{(2)} + \dots \end{aligned} \right\}, \quad (3.3)$$

* Cf. Poincaré, *Leçons de mécanique céleste*, Nos. 107, 108, 126–129.

† Readers familiar with Celestial Mechanics will note that under the contact transformation $x_k = r_k^{\frac{1}{2}} e^{i\theta_k}$, $y_k = i r_k^{\frac{1}{2}} e^{-i\theta_k}$ the equations (0.1) take the form there customary (see for example Poincaré, *Méthodes nouvelles*, No. 13). The series (2.12) are analogous to those of Lindstedt, and those to which we are led when there exist relations of the form (2.9) are analogous to those of Bohlin.

and separate out the terms of the same order, regarding p_1 , p_2 as of order 1 and the $x_k^{(r)}$, $y_k^{(r)}$ as of order r ; we obtain

$$\left. \begin{aligned} \lambda_1 \frac{\partial x_k^{(1)}}{\partial w_1} + \lambda_2 \frac{\partial x_k^{(1)}}{\partial w_2} &= \lambda_k x_k^{(1)} \\ \lambda_1 \frac{\partial y_k^{(1)}}{\partial w_1} + \lambda_2 \frac{\partial y_k^{(1)}}{\partial w_2} &= -\lambda_k y_k^{(1)} \end{aligned} \right\}, \quad (3.4)$$

$$\left. \begin{aligned} \lambda_1 \frac{\partial x_k^{(r)}}{\partial w_1} + \lambda_2 \frac{\partial x_k^{(r)}}{\partial w_2} - \lambda_k x_k^{(r)} \\ = \theta_k^{(r)}(x_s^{(1)}, y_s^{(1)}, \dots, x_s^{(r-1)}, y_s^{(r-1)}) - p_1 \frac{\partial x_k^{(r-1)}}{\partial w_1} - p_2 \frac{\partial x_k^{(r-1)}}{\partial w_2} \\ \lambda_1 \frac{\partial y_k^{(r)}}{\partial w_1} + \lambda_2 \frac{\partial y_k^{(r)}}{\partial w_2} + \lambda_k y_k^{(r)} \\ = \phi_k^{(r)}(x_s^{(1)}, y_s^{(1)}, \dots, x_s^{(r-1)}, y_s^{(r-1)}) - p_1 \frac{\partial y_k^{(r-1)}}{\partial w_1} - p_2 \frac{\partial y_k^{(r-1)}}{\partial w_2} \end{aligned} \right\} (r=2, 3, \dots), \quad (3.5)$$

where the $\theta_k^{(r)}$, $\phi_k^{(r)}$ are polynomials of degree r in the indicated arguments. For these equations we require a solution involving w_1 , w_2 only through integral powers of the exponentials $e^{\pm w_1}$, $e^{\pm w_2}$.

For the first approximation, the equations (3.4) give

$$x_k^{(1)} = \alpha_k e^{w_k}, \quad y_k^{(1)} = \beta_k e^{-w_k}, \quad (3.6)$$

where the α_k , β_k are arbitrary constants. The equations to integrate at succeeding approximations will be of the form

$$\begin{aligned} \lambda_1 \frac{\partial x_k^{(r)}}{\partial w_1} + \lambda_2 \frac{\partial x_k^{(r)}}{\partial w_2} - \lambda_k x_k^{(r)} &= P_k, \\ \lambda_1 \frac{\partial y_k^{(r)}}{\partial w_1} + \lambda_2 \frac{\partial y_k^{(r)}}{\partial w_2} + \lambda_k y_k^{(r)} &= Q_k, \end{aligned}$$

where P_k , Q_k are polynomials in the arguments

$$\alpha_1 e^{w_1}, \quad \beta_1 e^{-w_1}, \quad \alpha_2 e^{w_2}, \quad \beta_2 e^{-w_2}, \quad p_1, \quad p_2. \quad (3.7)$$

A solution for $x_k^{(r)}$, $y_k^{(r)}$ may be derived from P_k , Q_k respectively by introducing into each term an appropriate divisor, *provided that these divisors do not vanish*. For a term

$$A p_1^{c_1} p_2^{c_2} (\alpha_1 e^{w_1})^{a_1} (\beta_1 e^{-w_1})^{b_1} (\alpha_2 e^{w_2})^{a_2} (\beta_2 e^{-w_2})^{b_2} \quad (3.8)$$

in P_k the appropriate divisor is

$$\lambda_1 (a_1 - b_1) + \lambda_2 (a_2 - b_2) - \lambda_k;$$

for the same term in Q_k the divisor is

$$\lambda_1(a_1 - b_1) + \lambda_2(a_2 - b_2) + \lambda_k.$$

Terms for which the appropriate divisor vanishes we shall call *critical*; a critical term in P_k leads to a secular term in the solution for $x_k^{(r)}$. To avoid the occurrence of secular terms we therefore at each approximation omit the critical terms and solve the equations as thus modified.

In consequence of this omission of the critical terms the "solution" (3.3), as thus constructed step by step, will not satisfy the equations (3.2) in general. It contains, however, the six arbitrary constants $a_1, a_2, \beta_1, \beta_2, p_1, p_2$, and we shall see that the equations are satisfied by making p_1 and p_2 suitable functions of the a_k, β_k .

We are considering only the case in which λ_1, λ_2 are incommensurable, i.e. are connected by no relation of the form $A_1\lambda_1 + A_2\lambda_2 = 0$ in which A_1, A_2 are integers not both zero. Thus, in the equation for $x_1^{(r)}$, the term (3.8) is critical only if

$$a_1 - b_1 - 1 = 0, \quad a_2 - b_2 = 0; \quad (3.9)$$

the same term in the equation for $y_1^{(r)}$ is critical only if

$$a_1 - b_1 + 1 = 0, \quad a_2 - b_2 = 0;$$

and similarly as regards the equations for $x_2^{(r)}, y_2^{(r)}$.

Suppose that the $x_k^{(2)}, y_k^{(2)}, \dots, x_k^{(r-1)}, y_k^{(r-1)}$ have been determined as homogeneous polynomials in the arguments (3.7) of orders 2, ... (r-1) respectively; then the right-hand sides of (3.5) become known homogeneous polynomials of order r in these arguments, and these equations may be written

$$\left. \begin{aligned} \lambda_1 \frac{\partial x_k^{(r)}}{\partial w_1} + \lambda_2 \frac{\partial x_k^{(r)}}{\partial w_2} - \lambda_k x_k^{(r)} &= g_k^{(r)}(a_s e^{w_s}, \beta_s e^{-w_s}, p_s) + \bar{g}_k^{(r)}(a_s e^{w_s}, \beta_s e^{-w_s}, p_s) \\ \lambda_1 \frac{\partial y_k^{(r)}}{\partial w_1} + \lambda_2 \frac{\partial y_k^{(r)}}{\partial w_2} + \lambda_k y_k^{(r)} &= h_k^{(r)}(a_s e^{w_s}, \beta_s e^{-w_s}, p_s) + \bar{h}_k^{(r)}(a_s e^{w_s}, \beta_s e^{-w_s}, p_s) \end{aligned} \right\}, \quad (3.10)$$

where $\bar{g}_k^{(r)}, \bar{h}_k^{(r)}$ are aggregates of terms each of which is critical, and $g_k^{(r)}, h_k^{(r)}$ are aggregates of terms none of which is critical. We reject $\bar{g}_k^{(r)}, \bar{h}_k^{(r)}$ and obtain the solution for $x_k^{(r)}, y_k^{(r)}$ from $g_k^{(r)}, h_k^{(r)}$ respectively by introducing in each term the appropriate divisor. Thus the terms of the series (3.3) may be found in succession, the $x_k^{(r)}, y_k^{(r)}$ being homogeneous polynomials of order r in the arguments (3.7).

The series (3.3) as thus constructed do not satisfy the equations (3.2), but evidently when substituted in the equations they make the right- and left-hand sides identical, except as regards the critical terms. With the notation of (3.10) these remaining terms in the equations for x_k, y_k , when transferred to the right-hand side, form the respective series

$$\left. \begin{aligned} &\bar{g}_k^{(2)} + \bar{g}_k^{(3)} + \dots \\ &\bar{h}_k^{(2)} + \bar{h}_k^{(3)} + \dots \end{aligned} \right\}, \tag{3.11}$$

and from the conditions (3.9), etc., that a term may be critical, these series are of the respective forms

$$\begin{aligned} &a_k e^{u_k} P_k(a_1 \beta_1, a_2 \beta_2, p_1, p_2), \\ &\beta_k e^{-u_k} Q_k(a_1 \beta_1, a_2 \beta_2, p_1, p_2), \end{aligned}$$

where P_k, Q_k are power series in the indicated arguments.

The series (3.3) will satisfy the equations (3.2) [and therefore also (0.1) when we replace w_1, w_2 by their values (3.1)] provided that the six constants $a_1, \beta_1, a_2, \beta_2, p_1, p_2$, hitherto arbitrary, have such values that the four series P_1, P_2, Q_1, Q_2 vanish. From § 2 we know, however, that the equations (0.1) possess a solution of the form we have been constructing and depending on four arbitrary constants $a_1, \beta_1, a_2, \beta_2$. We infer that the four relations

$$P_1 = Q_1 = P_2 = Q_2 = 0 \tag{3.12}$$

must be equivalent to *two* relations only; they must all be satisfied provided that we put, with the notation of § 2,

$$p_1 = m_1 - \lambda_1 = \frac{\partial K}{\partial v_1} - \lambda_1, \quad p_2 = m_2 - \lambda_2 = \frac{\partial K}{\partial v_2} - \lambda_2,$$

where $v_1 = a_1 \beta_1, v = a_2 \beta_2$. If we replace the p_k by these power series in the $a_k \beta_k$, (3.3) become power series in the four arguments $a_1 e^{m_1 t}, \beta_1 e^{-m_1 t}, a_2 e^{m_2 t}, \beta_2 e^{-m_2 t}$:

$$x_r = F_r(a_k e^{m_k t}, \beta_k e^{-m_k t}), \quad y_r = G_r(a_k e^{m_k t}, \beta_k e^{-m_k t}), \tag{3.13}$$

where m_1, m_2 are power series in $a_1 \beta_1, a_2 \beta_2$, say

$$m_r = \lambda_r + p_r = \omega_r(a_1 \beta_1, a_2 \beta_2). \tag{3.14}$$

The divisors introduced in passing from the $g_k^{(r)}, h_k^{(r)}$ to the $x_k^{(r)}, y_k^{(r)}$ are of the form $N_1 \lambda_1 + N_2 \lambda_2$, where N_1, N_2 are integers which can take

any values positive or negative, except simultaneous zeros. The condition that λ_1, λ_2 are incommensurable is satisfied if λ_1/λ_2 is either real and irrational or unreal. If λ_1/λ_2 is real and irrational the moduli of such divisors have zero as their lower bound, but if λ_1/λ_2 is unreal they have a positive lower bound. If we confine our attention to cases in which the original F [before we transform its quadratic terms to the form (0.4)] is a series with real coefficients, such of the four exponents $\pm\lambda_1, \pm\lambda_2$ as are unreal must occur in conjugate pairs, and λ_1/λ_2 is unreal in the two cases :

(I) λ_1 real, λ_2 pure imaginary, or vice versa ;

(II) λ_1, λ_2 conjugate complex quantities.

In Case I the lower bound of the moduli of the divisors is the smaller of $|\lambda_1|, |\lambda_2|$; in Case II it is the smallest of $|\lambda_1| (= |\lambda_2|), |\lambda_1 + \lambda_2|, |\lambda_1 - \lambda_2|$.

4. Convergence of the series (3.13), (3.14) when λ_1/λ_2 is unreal.

We use the method of dominant functions.

Notation. Suppose that $f(x_1, x_2, \dots)$ is a polynomial or power series in a certain number of arguments $x_1, x_2, \dots$, and $F(X_1, X_2, \dots)$ a corresponding polynomial or power series in corresponding arguments $X_1, X_2, \dots$; then, if each coefficient in F is positive and not less than the modulus of the corresponding coefficient in f , we write*

$$f(x_1, x_2, \dots) \ll F(X_1, X_2, \dots). \quad (4.1)$$

Suppose that by means of relations

$$x_r = \phi_r(y_1, y_2, \dots), \quad X_r = \Phi_r(Y_1, Y_2, \dots),$$

where

$$\phi_r(y_1, y_2, \dots) \ll \Phi_r(Y_1, Y_2, \dots),$$

we express f and F in terms of arguments $y_1, y_2, \dots$, and $Y_1, Y_2, \dots$ respectively ; then it is easily seen that

$$f(y_1, y_2, \dots) \ll F(Y_1, Y_2, \dots).$$

It is obvious also that if (4.1) holds and if

$$|x_1| \leq X_1, \quad |x_2| \leq X_2, \dots,$$

then $|f| \leq F$.

* Cf. Poincaré, *Méthodes nouvelles*, Chap. II.

Let ϵ be the lower bound of the moduli of the divisors, as found at the end of the last section, and let ψ_k, χ_k be power series in variables X_1, X_2, Y_1, Y_2 such that

$$\frac{\partial F'(x_s, y_s)}{\partial y_k} \ll \psi_k(X_s, Y_s),$$

$$\frac{\partial F''(x_s, y_s)}{\partial x_k} \ll \chi_k(X_s, Y_s);$$

then for comparison-equations to (3.2) we take

$$\left. \begin{aligned} \{\epsilon - m(P_1 + P_2)\} X_k &= \epsilon A_k + \psi_k(X_s, Y_s) \\ \{\epsilon - m(P_1 + P_2)\} Y_k &= \epsilon B_k + \chi_k(X_s, Y_s) \end{aligned} \right\} \quad (k = 1, 2), \quad (4.2)$$

where m is a positive constant whose value will be specified presently, and P_k, A_k, B_k will correspond to $p_k, a_k e^{w_k}, \beta_k e^{-w_k}$ respectively. Substituting

$$\left. \begin{aligned} X_k &= X_k^{(1)} + X_k^{(2)} + \dots \\ Y_k &= Y_k^{(1)} + Y_k^{(2)} + \dots \end{aligned} \right\} \quad (4.3)$$

and equating terms of the same order as in § 3, we obtain

$$X_k^{(1)} = A_k, \quad Y_k^{(1)} = B_k, \quad (4.4)$$

$$\left. \begin{aligned} \epsilon X_k^{(r)} &= \Theta_k^{(r)}(X_s^{(1)}, Y_s^{(1)}, \dots, X_s^{(r-1)}, Y_s^{(r-1)}) + m(P_1 + P_2) X_k^{(r-1)} \\ \epsilon Y_k^{(r)} &= \Phi_k^{(r)}(X_s^{(1)}, Y_s^{(1)}, \dots, X_s^{(r-1)}, Y_s^{(r-1)}) + m(P_1 + P_2) Y_k^{(r-1)} \end{aligned} \right\} \quad (r = 2, 3, \dots), \quad (4.5)$$

where

$$\left. \begin{aligned} \theta_k^{(r)}(x_s^{(1)}, y_s^{(1)}, \dots, x_s^{(r-1)}, y_s^{(r-1)}) &\ll \Theta_k^{(r)}(X_s^{(1)}, Y_s^{(1)}, \dots, X_s^{(r-1)}, Y_s^{(r-1)}) \\ \phi_k^{(r)}(x_s^{(1)}, y_s^{(1)}, \dots, x_s^{(r-1)}, y_s^{(r-1)}) &\ll \Phi_k^{(r)}(X_s^{(1)}, Y_s^{(1)}, \dots, X_s^{(r-1)}, Y_s^{(r-1)}) \end{aligned} \right\}. \quad (4.6)$$

We may solve in succession for the $X_k^{(2)}, Y_k^{(2)}, \dots$ as polynomials in the arguments $A_1, A_2, B_1, B_2, P_1, P_2$; when the terms up to the order $r-1$ have been so determined, the right-hand sides of (4.5) become known polynomials, giving, say,

$$\left. \begin{aligned} \epsilon X_k^{(r)} &= G_k^{(r)}(A_s, B_s, P_s) \\ \epsilon Y_k^{(r)} &= H_k^{(r)}(A_s, B_s, P_s) \end{aligned} \right\}. \quad (4.7)$$

Then from $G_k^{(r)}, H_k^{(r)}$ we may pass to $X_k^{(r)}, Y_k^{(r)}$ respectively by dividing each term by ϵ .

Suppose that we have proved that

$$\left. \begin{aligned} x_k^{(i)}(a_s e^{w_s}, \beta_s e^{-w_s}, p_s) &\ll X_k^{(i)}(A_s, B_s, P_s) \\ y_k^{(i)}(a_s e^{w_s}, \beta_s e^{-w_s}, p_s) &\ll Y_k^{(i)}(A_s, B_s, P_s) \end{aligned} \right\} \quad (k = 1, 2), \quad (4.8)$$

$$\left. \begin{aligned} \frac{\partial x_k^{(i)}}{\partial w_j}(a_s e^{w_s}, \beta_s e^{-w_s}, p_s) &\ll m X_k^{(i)}(A_s, B_s, P_s) \\ \frac{\partial y_k^{(i)}}{\partial w_j}(a_s e^{w_s}, \beta_s e^{-w_s}, p_s) &\ll m Y_k^{(i)}(A_s, B_s, P_s) \end{aligned} \right\} \quad (k, j = 1, 2), \quad (4.9)$$

for $i = 1, 2, \dots (r-1)$. Then comparing (3.5), (3.10) with (4.5), (4.7) we have on use of (4.6)

$$\begin{aligned} g_k^{(r)}(a_s e^{w_s}, \beta_s e^{-w_s}, p_s) + \bar{g}_k^{(r)}(a_s e^{w_s}, \beta_s e^{-w_s}, p_s) &\ll G_k^{(r)}(A_s, B_s, P_s), \\ h_k^{(r)}(a_s e^{w_s}, \beta_s e^{-w_s}, p_s) + \bar{h}_k^{(r)}(a_s e^{w_s}, \beta_s e^{-w_s}, p_s) &\ll H_k^{(r)}(A_s, B_s, P_s). \end{aligned}$$

Now no term $p_1^{c_1} p_2^{c_2} (a_1 e^{w_1})^{a_1} (\beta_1 e^{-w_1})^{b_1} (a_2 e^{w_2})^{a_2} (\beta_2 e^{-w_2})^{b_2}$ occurs in both $g_k^{(r)}$ and $\bar{g}_k^{(r)}$, for a term cannot be both critical and non-critical. Hence from the relations just written we may infer that

$$\left. \begin{aligned} g_k^{(r)}(a_s e^{w_s}, \beta_s e^{-w_s}, p_s) &\ll G_k^{(r)}(A_s, B_s, P_s) \\ h_k^{(r)}(a_s e^{w_s}, \beta_s e^{-w_s}, p_s) &\ll H_k^{(r)}(A_s, B_s, P_s) \end{aligned} \right\}, \quad (4.10)$$

$$\left. \begin{aligned} \bar{g}_k^{(r)}(a_s e^{w_s}, \beta_s e^{-w_s}, p_s) &\ll G_k^{(r)}(A_s, B_s, P_s) \\ \bar{h}_k^{(r)}(a_s e^{w_s}, \beta_s e^{-w_s}, p_s) &\ll H_k^{(r)}(A_s, B_s, P_s) \end{aligned} \right\}. \quad (4.11)$$

Bearing in mind the definition of ϵ and the manner in which we pass from the $g_k^{(r)}$, $h_k^{(r)}$, $G_k^{(r)}$, $H_k^{(r)}$, to the $x_k^{(r)}$, $y_k^{(r)}$, $X_k^{(r)}$, $Y_k^{(r)}$ respectively, it follows from (4.10) that (4.8) holds for $i = r$.

We now have to establish (4.9) for $i = r$. The relations connecting the $x_k^{(r)}$, $y_k^{(r)}$ with the $g_k^{(r)}$, $h_k^{(r)}$ are

$$\begin{aligned} \lambda_1 \frac{\partial x_k^{(r)}}{\partial w_1} + \lambda_2 \frac{\partial x_k^{(r)}}{\partial w_2} &= \lambda_k x_k^{(r)} + g_k^{(r)}, \\ \lambda_1 \frac{\partial y_k^{(r)}}{\partial w_1} + \lambda_2 \frac{\partial y_k^{(r)}}{\partial w_2} &= -\lambda_k y_k^{(r)} + h_k^{(r)}, \end{aligned}$$

and using (4.7), (4.10), and (4.8) with $i = r$, we deduce

$$\left. \begin{aligned} \left(\lambda_1 \frac{\partial x_k^{(r)}}{\partial w_1} + \lambda_2 \frac{\partial x_k^{(r)}}{\partial w_2} \right) (a_s e^{w_s}, \beta_s e^{-w_s}, p_s) &\ll (|\lambda_k| + \epsilon) X_k^{(r)}(A_s, B_s, P_s) \\ \left(\lambda_1 \frac{\partial y_k^{(r)}}{\partial w_1} + \lambda_2 \frac{\partial y_k^{(r)}}{\partial w_2} \right) (a_s e^{w_s}, \beta_s e^{-w_s}, p_s) &\ll (|\lambda_k| + \epsilon) Y_k^{(r)}(A_s, B_s, P_s) \end{aligned} \right\}.$$

This means that, if

$$d(a_1 e^{w_1})^{a_1} (\beta_1 e^{-w_1})^{b_1} (a_2 e^{w_2})^{a_2} (\beta_2 e^{-w_2})^{b_2} p_1^{c_1} p_2^{c_2}$$

is a term in $x_k^{(r)}$ or $y_k^{(r)}$ and

$$DA_1^{a_1} B_1^{b_1} A_2^{a_2} B_2^{b_2} P_1^{c_1} P_2^{c_2}$$

is the corresponding term in $X_k^{(r)}$ or $Y_k^{(r)}$, then

$$|d\{\lambda_1(a_1 - b_1) + \lambda_2(a_2 - b_2)\}| \leq (|\lambda_k| + \epsilon) D. \quad (4.12)$$

Take the case in which one of λ_1, λ_2 is real and the other is a pure imaginary; then the arguments of the complex numbers $d\lambda_1(a_1 - b_1)$, $d\lambda_2(a_2 - b_2)$ differ by $\pi/2$, and each of $|d\lambda_1(a_1 - b_1)|$, $|d\lambda_2(a_2 - b_2)|$ is not less than $|d\{\lambda_1(a_1 - b_1) + \lambda_2(a_2 - b_2)\}|$. Hence (4.12) leads to

$$d(a_1 - b_1) \leq \frac{(|\lambda_k| + \epsilon) D}{|\lambda_1|},$$

$$|d(a_2 - b_2)| \leq \frac{(|\lambda_k| + \epsilon) D}{|\lambda_2|};$$

and thus

$$\left. \begin{aligned} \frac{\partial x_k^{(r)}}{\partial w_j}(a_s e^{w_s}, \beta_s e^{-w_s}, p_s) &\ll X_k^{(r)}(A_s, B_s, P_s) \frac{|\lambda_k| + \epsilon}{|\lambda_j|} \\ \frac{\partial y_k^{(r)}}{\partial w_j}(a_s e^{w_s}, \beta_s e^{-w_s}, p_s) &\ll Y_k^{(r)}(A_s, B_s, P_s) \frac{|\lambda_k| + \epsilon}{|\lambda_j|} \end{aligned} \right\} (j, k = 1, 2).$$

We choose m equal to the greatest of the quantities

$$\frac{|\lambda_1| + \epsilon}{|\lambda_1|}, \frac{|\lambda_1| + \epsilon}{|\lambda_2|}, \frac{|\lambda_2| + \epsilon}{|\lambda_1|}, \frac{|\lambda_2| + \epsilon}{|\lambda_2|}, \quad (4.13)$$

and have thus established (4.9) for $i = r$. When λ_1, λ_2 are conjugate complex quantities we may show similarly that (4.12) leads to (4.9) with $i = r$ provided that we take m equal to the greatest of the quantities (4.13) multiplied by $|\operatorname{cosec} \theta|$, where $\theta = \arg \lambda_1 - \arg \lambda_2$.

From (3.6), (4.4) we see that (4.8) is true for $i = 1$; and since $m > 1$ (4.9) also is true for $i = 1$. Hence by induction (4.8), (4.9) hold for all values of i . If then the comparison-series (4.3) are convergent it follows from (4.8) that the series (3.3) are absolutely and uniformly convergent provided that

$$|a_k e^{w_k}| \leq A_k, \quad |\beta_k e^{-w_k}| \leq B_k, \quad |p_k| \leq P_k;$$

and from (4.7), (4.11) it follows that the series (3.11) are convergent within the same range.

It remains only to find dominant functions ψ_k, χ_k such that the series (4.3) can be proved to have a finite domain of convergence. The series $F(x_k, y_k)$ is by hypothesis absolutely convergent when the moduli of the x_k, y_k are sufficiently small, less than a positive constant ρ , say. We may then take for each of ψ_k, χ_k the expansion of

$$M(X_1 + Y_1 + X_2 + Y_2)^2 / \left(1 - \frac{X_1 + Y_1 + X_2 + Y_2}{\rho}\right),$$

where M is a certain positive constant. With this choice of ψ_k, χ_k , the series (4.3) are easily proved to be convergent provided that the positive quantities A_k, B_k, P_k satisfy the inequalities

$$A_1 + B_1 + A_2 + B_2 \leq \frac{\rho}{3}, \quad A_1 + B_1 + A_2 + B_2 \leq \frac{\epsilon}{145M}, \quad P_1 + P_2 \leq \frac{\epsilon}{3m}.$$

Hence the series (3.3), (3.11) are absolutely and uniformly convergent provided that

$$|\alpha_1 e^{w_1}| + |\beta_1 e^{-w_1}| + |\alpha_2 e^{w_2}| + |\beta_2 e^{-w_2}| \leq \frac{\rho}{3} \quad \text{and} \quad \leq \frac{\epsilon}{145M}, \quad (4.14)$$

and

$$|p_1| + |p_2| \leq \frac{\epsilon}{3m}. \quad (4.15)$$

The power series (3.14) for p_1, p_2 in terms of $\alpha_1\beta_1, \alpha_2\beta_2$ are derived by solution of a suitably selected pair of the equations (3.12); hence these series also will be absolutely and uniformly convergent when $|\alpha_1\beta_1|, |\alpha_2\beta_2|$ are sufficiently small. Since p_1, p_2 vanish with $\alpha_1\beta_1, \alpha_2\beta_2$, the condition (4.15) will be satisfied provided that

$$|\alpha_1 e^{w_1}|, |\beta_1 e^{-w_1}|, |\alpha_2 e^{w_2}|, |\beta_2 e^{-w_2}|$$

are all sufficiently small. Thus under conditions of the form

$$|\alpha_k e^{m_k t}| \leq \eta, \quad |\beta_k e^{-m_k t}| \leq \eta \quad (k = 1, 2),$$

where η is a certain positive constant, the series (3.13), (3.14) giving the general solution are all absolutely and uniformly convergent.

GIBBS'S PHENOMENON IN FOURIER-BESSEL SERIES
AND INTEGRALS

By RICHARD G. COOKE.

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1. *Introduction.*

It is the object of the present paper to investigate the Gibbs's phenomenon* in connexion with the Fourier-Bessel series

$$(1.01) \quad \sum_{m=1}^{\infty} A_m J_{\nu}(j_m x),$$

where
$$A_m = \frac{2}{J_{\nu+1}^2(j_m)} \int_0^1 J_{\nu}(j_m t) t f(t) dt,$$

and j_m is the m -th positive zero of $J_{\nu}(z)$; and also in connexion with the Hankel integral

$$(1.02) \quad \int_0^{\infty} J_{\nu}(xu) u du \int_0^{\infty} J_{\nu}(tu) t f(t) dt.$$

It has been proved by Young† that, except at the end-points $x = 0$, $x = 1$ of the range for which the Fourier-Bessel series is defined, the

* For a general account of the subject, see H. S. Carslaw, *Amer. J. of Math.*, 39 (1917), 185; also E. W. Hobson, *The theory of functions of a real variable*, 2nd Edition, 2, 498.

† W. H. Young, "On series of Bessel functions," *Proc. London Math. Soc.* (2), 18 (1920), 163-200. Young's results were really proved for Dini series [G. N. Watson, *Theory of Bessel functions* (1922), 596], but the Dini and Fourier-Bessel series have identical convergence properties in the open interval $0 < x < 1$ (Watson, *loc. cit.*, 600).

convergence phenomena for the Fourier-Bessel series and for an ordinary Fourier series are *identical*, so that the properties of the Gibbs's phenomenon at internal points of the range $(0, 1)$ of the Fourier-Bessel series are known. It is also well known that a similar identity of convergence properties holds between Fourier series and Fourier integrals; and the Hankel integral reduces to a Fourier integral when $x > 0$ on substitution of the asymptotic form for the Bessel functions. Hence the properties of the Gibbs's phenomenon at points other than the origin are also known for the Hankel integral. It remains, then, to consider the phenomenon at the end-points $x = 0$, $x = 1$, in the case of the Fourier-Bessel series, and at the point $x = 0$ in the case of the Hankel integral.

Now a function represented by a Fourier series is defined over $(-\pi, \pi)$, and one represented by a Fourier-Bessel series is defined over $(0, 1)$. Outside these intervals, $f(x)$ may be defined as being the sum of the series (if it can be obtained). The result is that we define $f(x)$ outside the range $(-\pi, \pi)$ by periodicity in the Fourier case, so that there is really no difference between internal points and end-points; but it is not so simple in the Fourier-Bessel case, and in fact, as we shall see, there is a distinct problem for the end-points. It is convenient to deal with the integral before the series.

The Hankel integral.

2. *Definitions.* At a point of ordinary discontinuity, $x = a$, other than the origin, of a function $f(x)$ given by the Hankel integral, we define the *Gibbs ratio* to be

$$(2.01) \quad G = \lim_{n \rightarrow \infty} \frac{\text{Max}_x I_n(x) - \text{Min}_x I_n(x)}{|f(a+0) - f(a-0)|},$$

where
$$I_n(x) = \int_0^n J_\nu(xu)u du \int_0^\infty J_\nu(tu)tf(t)dt,$$

and $\text{Max}_x I_n(x)$ means the maximum, and $\text{Min}_x I_n(x)$ the minimum, of $I_n(x)$ for all x in the neighbourhood of $x = a$.

At the origin, $x = 0$, we define the Gibbs ratio to be

$$(2.02) \quad G = \lim_{n \rightarrow \infty} \frac{\text{Max}_x I_n(x)}{f(+0)}, \quad [f(+0) \neq 0].$$

Thus, in the case $\nu = \frac{1}{2}$, when the Hankel integral of $f(x)$ reduces to the Fourier integral of $\sqrt{x}f(x)$, either (2.01) or (2.02) gives

$$G = \frac{2}{\pi} \int_0^\pi \frac{\sin v}{v} dv.$$

In view of the remarks made in § 1 concerning the relation between the Hankel integral and the Fourier integral, it will be seen that, for points of ordinary discontinuity other than the origin, we have for the Hankel integral

$$G = \frac{2}{\pi} \int_0^\pi \frac{\sin v}{v} dv.$$

We take $\nu > -1$ throughout the paper, and it is, of course, assumed that $f(x)$ satisfies convergence conditions for the series or the integral.

3. Gibbs's phenomenon at the origin.

3.0. In addition to the consideration of the Hankel integral at the origin, it is interesting to consider a generalization, which we will call the *modified Hankel integral*, viz.,

$$f(x) = x^\alpha \int_0^\infty J_\nu(xu)u du \int_0^\infty J_\nu(tu)t^{1-\alpha}f(t)dt;$$

this clearly reduces to the Hankel integral when $\alpha = 0$, and to the Fourier sine integral when $\alpha = \nu = \frac{1}{2}$. At points other than the origin, the modification makes no difference to the results.

THEOREM. *The modified Hankel integral*

$$x^\alpha \int_0^\infty J_\nu(xu)u du \int_0^\infty J_\nu(tu)t^{1-\alpha}f(t)dt$$

has a Gibbs ratio at $x = 0$ equal to

$$2^{1-\alpha} \frac{\Gamma(1-\frac{1}{2}\alpha+\frac{1}{2}\nu)}{\Gamma(\frac{1}{2}\alpha+\frac{1}{2}\nu)} \int_0^{J_{\nu,1}} J_\nu(v)v^{\alpha-1}dv,*$$

* It should be noticed that this expression is a function not only of α , but of ν ; whereas the Gibbs ratio at points other than the origin is independent of ν .

where $j_{\nu,1}$ is the smallest positive zero of $J_\nu(v)$, provided that

$$-\frac{1}{2} < a \leq 1, \quad a - \nu < 2, \quad a + \nu > 0, \quad f(+0) \neq 0.$$

If $a + \nu = 0$, $a > -\frac{1}{2}$, there is no Gibbs's phenomenon at $x = 0$. As a special case, if $\nu = \frac{1}{2}$, we have that the Fourier integral

$$\frac{2x^{a-\frac{1}{2}}}{\pi} \int_0^\infty \sin xu \, du \int_0^\infty t^{\frac{1}{2}-a} \sin tu f(t) dt \quad [f(+0) \neq 0]$$

has a Gibbs ratio at $x = 0$ equal to

$$\frac{2^{\frac{3}{2}-a}}{\sqrt{\pi}} \frac{\Gamma(\frac{5}{4}-\frac{1}{2}a)}{\Gamma(\frac{1}{4}+\frac{1}{2}a)} \int_0^\pi v^{a-\frac{3}{2}} \sin v \, dv,$$

provided that $-\frac{1}{2} < a < \frac{3}{2}$; and if $a = \frac{1}{2}$, this further reduces to

$$\frac{2}{\pi} \int_0^\pi \frac{\sin v}{v} \, dv.$$

In the modified Hankel integral, take

$$f(t) = 1 \quad (0 \leq t < 1)$$

$$= 0 \quad (t > 1),$$

so that it becomes

$$x^a \int_0^\infty J_\nu(xu) u \, du \int_0^1 J_\nu(tu) t^{1-a} \, dt.$$

3.1. It is convenient first to dispose of the case $a + \nu = 0$.

In this case we have

$$\begin{aligned} I_n(x) &= x^{-\nu} \int_0^n J_\nu(xu) u \, du \int_0^1 J_\nu(tu) t^{\nu+1} \, dt \\ &= x^{-\nu} \int_0^n J_\nu(xu) J_{\nu+1}(u) \, du, \end{aligned}$$

since $\nu > -1$; and this tends uniformly to unity* in the neighbourhood of the origin.

This can be seen as follows.

Let us divide the range of integration $(0, n)$ into the parts

$$(0, 1), \quad (1, 1/x), \quad (1/x, n), \quad \text{or} \quad (0, 1), \quad (1, n),$$

according as $1/x < n$, or $1/x \geq n$.

* See G. N. Watson, *loc. cit.*, 406 (8).

Take the first case, and call the three integrals which arise I_1 , I_2 , I_3 respectively.

(i) I_1 is independent of n and is continuous at $x = 0$, and so it tends uniformly to its limit at $x = 0$.

(ii) In the range $(1, 1/x)$, we insert the power-series* for $J_\nu(xu)$ and the asymptotic form† for $J_{\nu+1}(u)$, so that

$$\begin{aligned} I_2 &= x^{-\nu} \int_1^{1/x} J_\nu(xu) J_{\nu+1}(u) du \\ &= x^{-\nu} \int_1^{1/x} \left[\frac{(\frac{1}{2}xu)^\nu}{\Gamma(\nu+1)} + O\{(xu)^{\nu+2}\} \right] \\ &\quad \times \left[\sqrt{\left(\frac{2}{\pi u}\right)} \sin(u - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) + O(u^{-\frac{3}{2}}) \right] du \\ &= O(x^{-\nu+\frac{1}{2}}), \end{aligned}$$

and this tends to zero uniformly in the neighbourhood of $x = 0$ if $-\nu + \frac{1}{2} > 0$, i.e. if $\alpha > -\frac{1}{2}$.

(iii) In the range $(1/x, n)$ we employ the asymptotic form for $J_{\nu+1}(u)$, and, in order to avoid the discussion of error terms containing x , for $J_\nu(xu)$ we employ the form‡

$$(3.11) \quad \sqrt{xu} J_\nu(xu) = \cos xu f_1(xu) + \sin xu f_2(xu),$$

where f_1 , f_2 are functions of bounded variation over $(0, \infty)$ for all values of ν , when $xu \geq 1$.

We then have

$$\begin{aligned} I_3 &= x^{-\nu} \int_{1/x}^n J_\nu(xu) J_{\nu+1}(u) du \\ &= x^{-\nu-\frac{1}{2}} \int_{1/x}^n \{ \cos xu f_1(xu) + \sin xu f_2(xu) \} \\ &\quad \times \{ \sin(u - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) + O(u^{-1}) \} \frac{du}{u} \\ &= O(x^{-\nu+\frac{1}{2}}), \end{aligned}$$

* See G. N. Watson, *loc. cit.*, 40 (8).

† *Ibid.*, 199 (1).

‡ This form is due to E. C. Titchmarsh. See R. G. Cooke, "The inversion formulae of Hardy and Titchmarsh," *Proc. London Math. Soc.* (2), 24 (1925), 381-420 (385).

and this tends to zero uniformly in the neighbourhood of $x = 0$ if $-\nu + \frac{1}{2} > 0$, i.e. if $\alpha > -\frac{1}{2}$.

We may deal similarly with the second case.

This disposes of the case $\alpha + \nu = 0$.

There is therefore no Gibbs's phenomenon in this case: *e.g.* there is no phenomenon in the case $\alpha = \frac{1}{2}$, $\nu = -\frac{1}{2}$, when Fourier's cosine integral is obtained.

3.2. Now (to take the general case) let $\alpha + \nu > 0$. Then

$$I_n(x) = x^\alpha \int_0^n J_\nu(xu) u du \int_0^1 J_\nu(tu) t^{1-\alpha} dt.$$

Also we have*

$$\begin{aligned} \int_0^1 J_\nu(tu) t^{1-\alpha} dt &= u^{\alpha-2} \int_0^u J_\nu(z) z^{1-\alpha} dz \\ &= u^{\alpha-2} [(\nu-\alpha)u J_\nu(u) S_{-\alpha, \nu-1}(u) - u J_{\nu-1}(u) S_{1-\alpha, \nu}(u) + \lambda_{\nu, \alpha}], \end{aligned}$$

where $S_{\mu, \nu}(u)$ is Lommel's function of the second kind, and

$$\lambda_{\nu, \alpha} = 2^{1-\alpha} \frac{\Gamma(1-\frac{1}{2}\alpha+\frac{1}{2}\nu)}{\Gamma(\frac{1}{2}\alpha+\frac{1}{2}\nu)},$$

arising from the lower limit of the integral, convergence being there secured by the condition $\alpha - \nu < 2$. Hence

$$(3.21) \quad I_n(x) = x^\alpha \int_0^n J_\nu(xu) \left\{ \Theta_{\nu, \alpha}(u) + \frac{\lambda_{\nu, \alpha}}{u} \right\} u^\alpha du,$$

where $\Theta_{\nu, \alpha}(u) = (\nu - \alpha) J_\nu(u) S_{-\alpha, \nu-1}(u) - J_{\nu-1}(u) S_{1-\alpha, \nu}(u)$.

Now the asymptotic expansion of $S_{\mu, \nu}(u)$ is†

$$\begin{aligned} (3.22) \quad S_{\mu, \nu}(u) &= u^{\mu-1} \sum_{m=0}^{p-1} \frac{(-1)^m \Gamma(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu+m) \Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu+m)}{(\frac{1}{2}u)^{2m} \Gamma(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu) \Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu)} \\ &\quad + O(u^{\mu-2p}), \end{aligned}$$

if $\mu \pm \nu$ are not odd positive integers. Hence, employing the asymptotic

* G. N. Watson, *loc. cit.*, 350 (3).

† *Ibid.*, 351.

form for $J_\nu(u)$, we derive the asymptotic formula

(3.23)
$$\Theta_{\nu,\alpha}(u) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{\sin(u - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)}{u^{\alpha+\frac{1}{2}}} + O(u^{-\alpha-\frac{3}{2}}),$$

taking $p = 1$ in (3.22).

Now consider

$$x^\alpha \int_0^n J_\nu(xu) \Theta_{\nu,\alpha}(u) u^\alpha du.$$

Let us divide the range of integration $(0, n)$ into the parts

$$(0, 1), \quad (1, 1/x), \quad (1/x, n), \quad \text{or} \quad (0, 1), (1, n),$$

according as $1/x < n$, or $1/x \geq n$.

Take the first case, and call the three integrals which arise I_1, I_2, I_3 respectively.

(i) I_1 is independent of n , and is continuous at $x = 0$, the integral being convergent at the lower limit if $\alpha + \nu > 0$; also $I_1 = 0$ at $x = 0$.

(ii) In the range $(1, 1/x)$ we insert the power series for $J_\nu(xu)$ and the asymptotic form (3.23) for $\Theta_{\nu,\alpha}(u)$, so that

$$\begin{aligned} I_2 &= x^\alpha \int_1^{1/x} J_\nu(xu) \Theta_{\nu,\alpha}(u) u^\alpha du \\ &= x^\alpha \int_1^{1/x} \left[\frac{(\frac{1}{2}xu)^\nu}{\Gamma(\nu+1)} + O\{(xu)^{\nu+2}\} \right] \\ &\quad \times \left[\sqrt{\left(\frac{2}{\pi}\right)} \frac{\sin(u - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)}{u^{\alpha+\frac{1}{2}}} + O(u^{-\alpha-\frac{3}{2}}) \right] u^\alpha du \\ &= O(x^{\alpha+\frac{1}{2}}), \end{aligned}$$

and this tends to zero uniformly in x if $\alpha > -\frac{1}{2}$.

(iii) In the range $(1/x, n)$ we insert the forms (3.11) for $J_\nu(xu)$, and (3.23) for $\Theta_{\nu,\alpha}(u)$, so that

$$\begin{aligned} I_3 &= x^\alpha \int_{1/x}^n J_\nu(xu) \Theta_{\nu,\alpha}(u) u^\alpha du \\ &= x^{\alpha-\frac{1}{2}} \int_{1/x}^n \{ \cos xu f_1(xu) + \sin xu f_2(xu) \} \\ &\quad \times \left\{ \sqrt{\left(\frac{2}{\pi}\right)} \frac{\sin(u - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)}{u^{\alpha+\frac{1}{2}}} + O(u^{-\alpha-\frac{3}{2}}) \right\} u^{\alpha-\frac{1}{2}} du \\ &= O(x^{\alpha+\frac{1}{2}}), \end{aligned}$$

and this tends to zero uniformly in x if $\alpha > -\frac{1}{2}$.

We may deal similarly with the second case.

3.3. It remains to consider

$$\lambda_{\nu, \alpha} x^\alpha \int_0^n J_\nu(xu) u^{\alpha-1} du = \lambda_{\nu, \alpha} \int_0^{nx} J_\nu(v) v^{\alpha-1} dv.$$

This integral clearly has maxima where

$$nx = j_{\nu, 2p+1}^*,$$

p being a positive integer; and in order to prove the theorem stated at the beginning of this section, it is now only necessary to prove that the *greatest* maximum of the integral occurs where $nx = j_{\nu, 1}$, and to this end we establish the following

THEOREM. *The sequence*

$$\left| \int_{j_p}^{j_{p+1}} J_\nu(v) v^{\alpha-1} dv \right| \quad (p = 1, 2, 3, \dots)$$

is a steadily decreasing sequence, provided that

$$(i) \quad -\frac{1}{2} < \alpha \leq 1 \quad \text{when} \quad \nu \geq -\frac{1}{2};$$

$$(ii) \quad \alpha - \nu < 2 \quad \text{when} \quad \nu < -\frac{1}{2}.$$

Dealing first with case (i), it is sufficient to prove the theorem in the case $\alpha = 1$, for we shall then have proved that

$$\left| \int_{j_p}^{j_{p+1}} J_\nu(v) dv \right| \quad (p = 1, 2, 3, \dots)$$

is a steadily decreasing sequence, and in that case we should have, if $\alpha \leq 1$,

$$\begin{aligned} \left| \int_{j_p}^{j_{p+1}} J_\nu(v) v^{\alpha-1} dv \right| &\geq \left| \int_{j_p}^{j_{p+1}} J_\nu(v) j_{p+1}^{\alpha-1} dv \right| \\ &\geq \left| \int_{j_{p+1}}^{j_{p+2}} J_\nu(v) j_{p+1}^{\alpha-1} dv \right| \geq \left| \int_{j_{p+1}}^{j_{p+2}} J_\nu(v) v^{\alpha-1} dv \right|. \end{aligned} \quad (3.31)$$

Let

$$I_p \equiv \int_{j_p}^{\infty} J_\nu(v) dv.$$

* We shall employ the symbol $j_{\nu, m}$ for the m -th positive zero of $J_\nu(x)$ when it is desired to draw attention to the order ν ; when no ambiguity arises we shall simply write j_m .

Then we have to prove that

$$|I_p - I_{p+1}| > |I_{p+1} - I_{p+2}|. \quad (3.32)$$

Now $I_p - I_{p+1}$, $I_{p+1} - I_{p+2}$ are of opposite signs.

Suppose p even. Then the inequality (3.32) is

$$I_p - I_{p+1} > I_{p+2} - I_{p+1},$$

or

$$I_p > I_{p+2}.$$

Suppose p odd. Then (3.32) becomes

$$I_{p+1} - I_p > I_{p+1} - I_{p+2},$$

or

$$-I_p > -I_{p+2}.$$

But we have*

$$I_p = \int_{j_p}^{\infty} J_{\nu}(v) dv = j_p J'_{\nu}(j_p) S_{0,\nu}(j_p),$$

and $J'_{\nu}(j_p) = -J_{\nu+1}(j_p)$ is alternately positive and negative for successive values of p †; hence it follows that I_p is positive if p is even, and negative if p is odd, provided that

$$S_{0,\nu}(j_p) > 0. \quad (3.33)$$

Thus in either case (p odd or even) it is sufficient to prove that

$$I_p^2 > I_{p+2}^2,$$

(3.33) being satisfied.

Hence sufficient conditions for the truth of the theorem are that

$$(a) \quad j_p^2 J_{\nu}^{\prime 2}(j_p) S_{0,\nu}^2(j_p) \quad (p = 1, 2, 3, \dots)$$

should be a steadily decreasing sequence; and

$$(b) \quad S_{0,\nu}(x) > 0 \quad \text{when } x \geq j_1.$$

3.4. In this sub-section we establish the condition (a).

To effect this, we employ a method similar to that given by Watson‡ for proving certain monotonic properties of cylinder functions in con-

* G. N. Watson, *loc. cit.*, 350 (3).

† *Ibid.*, 480.

‡ *Ibid.*, 488.

nexion with their stationary values. Let

$$\Theta(x) \equiv A(x) J_\nu^2(x) + 2B(x) J_\nu(x) J'_\nu(x) + C(x) J_\nu'^2(x),$$

where $A(x)$, $B(x)$, $C(x)$ are to be suitably chosen. Then, using Bessel's differential equation,

$$\begin{aligned} \Theta'(x) &= \left\{ A'(x) - 2(x^2 - \nu^2) \frac{B(x)}{x^2} \right\} J_\nu^2(x) \\ &\quad + 2 \left\{ B'(x) + A(x) - \frac{B(x)}{x} - (x^2 - \nu^2) \frac{C(x)}{x^2} \right\} J_\nu(x) J'_\nu(x) \\ &\quad + \left\{ C'(x) + 2B(x) - \frac{2C(x)}{x} \right\} J_\nu'^2(x) \\ &= \Lambda(x) J_\nu^2(x), \end{aligned}$$

where

$$\Lambda(x) = A'(x) - \frac{2(x^2 - \nu^2)}{x^2} B(x),$$

provided that $C(x)$ is chosen arbitrarily, and that $B(x)$ and $A(x)$ are then defined by

$$B(x) = \frac{C(x)}{x} - \frac{1}{2} C'(x),$$

$$A(x) = \frac{B(x)}{x} - B'(x) + \frac{x^2 - \nu^2}{x^2} C(x).$$

For brevity write $S_{0,\nu}(x) \equiv S$, and in the above result take

$$C(x) = x^2 S^2;$$

then

$$B(x) = -x^2 S S',$$

$$A(x) = x S S' + x^2 S'^2 + x^2 S S'' + (x^2 - \nu^2) S^2.$$

Hence

$$\begin{aligned} \Lambda(x) &= 2x S^2 + 3x S'^2 + (4x^2 - 4\nu^2 + 1) S S' \\ &\quad + 3x S S'' + 3x^2 S' S'' + x^2 S S'''. \end{aligned}$$

Now, by Lommel's differential equation*,

$$x^2 S'' + x S' + (x^2 - \nu^2) S = x. \quad (3.41)$$

Therefore, differentiating,

$$x^2 S''' + 3x S'' + (x^2 - \nu^2 + 1) S' + 2x S = 1. \quad (3.42)$$

* G. N. Watson, *loc. cit.*, 345 (1).

Hence, substituting for $x^2 S'''$ from (3.42), we have

$$\Lambda(x) = 3(x^2 - \nu^2)SS' + 3xS'^2 + S + 3x^2S'S''.$$

Finally, substituting for $x^2 S''$ from (3.41), we get

$$\Lambda(x) = S + 3xS'.$$

Thus

$$\Theta(j_\nu) \equiv j_\nu^2 J_\nu'^2(j_\nu) S^2$$

will be a decreasing sequence if $\Theta'(x) < 0$, and this will be the case if

$$S + 3xS' < 0. \quad (3.43)$$

Now we have*

$$S_{0,\nu}(x) = \frac{1}{x} \left[1 + \sum_{m=1}^q (\nu^2 - 1^2)(\nu^2 - 3^2) \dots \{\nu^2 - (2m-1)^2\} x^{-2m} \right] \\ - \frac{1}{2\pi i x} \int_{-\infty i - \frac{1}{2}}^{\infty i - \frac{1}{2}} \frac{\Gamma(\frac{1}{2} + \frac{1}{2}\nu - s) \Gamma(\frac{1}{2} - \frac{1}{2}\nu - s)}{\Gamma(\frac{1}{2} + \frac{1}{2}\nu) \Gamma(\frac{1}{2} - \frac{1}{2}\nu)} \frac{\pi(\frac{1}{2}x)^{2s}}{\sin s\pi} ds, \quad (3.44)$$

provided that $q > \pm \frac{1}{2}\nu - 1$. Hence

$$S + 3xS' = -\frac{2}{x} - \sum_{m=1}^q (6m+2)(\nu^2 - 1^2)(\nu^2 - 3^2) \dots \{\nu^2 - (2m-1)^2\} x^{-2m} \\ + \frac{1}{2\pi i x} \int_{-\infty i - \frac{1}{2}}^{\infty i - \frac{1}{2}} \frac{\Gamma(\frac{1}{2} + \frac{1}{2}\nu - s) \Gamma(\frac{1}{2} - \frac{1}{2}\nu - s)}{\Gamma(\frac{1}{2} + \frac{1}{2}\nu) \Gamma(\frac{1}{2} - \frac{1}{2}\nu)} \frac{\pi(\frac{1}{2}x)^{2s}(6s-2)}{\sin s\pi} ds.$$

Let $q = [\frac{1}{2}\nu + \frac{1}{2}]$, i.e. $2q-1 \leq \nu < 2q+1$.

Then all the terms preceding the integral are negative, and, calling the term involving the integral $F(x)$, it is sufficient to prove that†

$$|F(x)| < \frac{2}{x}. \quad (3.45)$$

Writing $s = it - q - \frac{1}{2}$, we have

$$|\Gamma(\frac{1}{2} + \frac{1}{2}\nu - s)| \leq \Gamma(\frac{1}{2}\nu + q + 1) < \Gamma(2q + \frac{3}{2}),$$

$$|\Gamma(\frac{1}{2} - \frac{1}{2}\nu - s)| \leq \Gamma(-\frac{1}{2}\nu + q + 1) \leq \Gamma(\frac{1}{2}),$$

$$|6s-2| = |6it-6q-5| \leq 6|t| + 6q+5,$$

$$|\Gamma(\frac{1}{2} + \frac{1}{2}\nu) \Gamma(\frac{1}{2} - \frac{1}{2}\nu)| = \left| \frac{\pi}{\cos \frac{1}{2}\nu\pi} \right| \geq \pi.$$

* G. N. Watson, *loc. cit.*, 347 (1), and 351.

† Note that, if ν is an odd positive integer, the series for $S_{0,\nu}(x)$ terminates, and the theorem is then obvious.

Hence

$$\begin{aligned} |F(x)| &\leq \frac{1}{2\pi x} \int_{-\infty}^{\infty} \frac{1}{\pi} \Gamma(2q + \tfrac{3}{2}) \Gamma(\tfrac{1}{2}) \frac{\pi}{\cosh \pi t} (6|t| + 6q + 5) (\tfrac{1}{2}x)^{-2q-1} dt \\ &< \frac{2^{2q}}{x^{2q+2}} \frac{\Gamma(2q + \tfrac{3}{2})}{\sqrt{\pi}} (6q + 8), \end{aligned}$$

since
$$\int_{-\infty}^{\infty} \frac{dt}{\cosh \pi t} = 1,$$

and
$$\int_{-\infty}^{\infty} \frac{|t| dt}{\cosh \pi t} = \frac{4}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(2m-1)^2} < \frac{4}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2} = \tfrac{1}{2}.$$

Hence it is sufficient to prove that

$$\frac{2^{2q}}{x^{2q+2}} \frac{\Gamma(2q + \tfrac{3}{2})}{\sqrt{\pi}} (6q + 8) < \frac{2}{x},$$

where $x > j_{\nu,1} \geq j_{2q-1,1}$; i.e. that*

$$2^{2q-1} \Gamma(2q + \tfrac{3}{2}) (6q + 8) < \sqrt{\pi} \cdot (j_{2q-1,1})^{2q+1}. \quad (3.46)$$

(i) Suppose $q \geq 6$. Then, since† $j_{\nu,1} > \sqrt{\{\nu(\nu+2)\}}$, it is sufficient to prove that

$$2^{2q-1} \Gamma(2q + \tfrac{3}{2}) (6q + 8) < \sqrt{\pi} \cdot (4q^2 - 1)^{q+\frac{1}{2}}.$$

Now‡
$$\Gamma(x) < x^{x-\frac{1}{2}} e^{-x} \sqrt{(2\pi)} e^{1/12x} < x^{x-\frac{1}{2}} e^{-x} \sqrt{(2\pi)} e^{1/60}$$

for $x > j_{11,1} > 10$.

Hence it is sufficient to prove that

$$2^{2q-1} (2q + \tfrac{3}{2})^{2q+1} e^{-2q-\frac{3}{2}+\frac{1}{60}} \sqrt{(2\pi)} (6q + 8) < \sqrt{\pi} \cdot (4q^2 - 1)^{q+\frac{1}{2}},$$

or
$$2^{2q+\frac{1}{2}} \left(1 + \frac{3}{4q}\right)^{2q+1} e^{-2q-\frac{3}{2}+\frac{1}{60}} (3q + 4) < \left(1 - \frac{1}{4q^2}\right)^{q+\frac{1}{2}}.$$

But
$$\left(1 + \frac{3}{4q}\right)^{2q+1} / \left(1 - \frac{1}{4q^2}\right)^{q+\frac{1}{2}}$$

steadily decreases for $q > 0$, and when $q = 6$, it has the value 4.8328.

* G. N. Watson, *loc. cit.*, 508, extended theorem.

† *Ibid.*, 486 (5).

‡ Whittaker and Watson, *Modern analysis*, 3rd ed. (1920), 253.

Hence it is sufficient to prove that

$$4.8328 \cdot 2^{2q+\frac{1}{2}} e^{-2q-\frac{3}{2}+1\frac{1}{2}0} (3q+4) < 1.$$

This is true for $q = 6$, and the function on the left decreases for $q > 0$. Hence it is true for $q \geq 6$.

(ii) Suppose $1 \leq q \leq 5$. We have

$$j_{1,1} > 3.8, \quad j_{3,1} > 6.3, \quad j_{5,1} > 8.7, \quad j_{7,1} > 11, \quad j_{9,1} > 13.$$

Substituting these values for $j_{2q-1,1}$ in (3.46) for $q = 1, 2, 3, 4, 5$ respectively, the inequalities so obtained are easily seen to be satisfied.

We have now established the inequality (3.46) for $q \geq 1$, *i.e.* for values of $\nu \geq 1$.

(iii) It remains only to examine the case $q = 0$.

We can here obtain a more refined inequality than (3.46) by noticing that in this case

$$|\Gamma(\frac{1}{2} + \frac{1}{2}\nu - s)| \leq \Gamma(1 + \frac{1}{2}\nu), \quad |\Gamma(\frac{1}{2} - \frac{1}{2}\nu - s)| \leq \Gamma(1 - \frac{1}{2}\nu),$$

so that

$$\left| \frac{\Gamma(\frac{1}{2} + \frac{1}{2}\nu - s) \Gamma(\frac{1}{2} - \frac{1}{2}\nu - s)}{\Gamma(\frac{1}{2} + \frac{1}{2}\nu) \Gamma(\frac{1}{2} - \frac{1}{2}\nu)} \right| \leq |\frac{1}{2}\nu \cot \frac{1}{2}\nu\pi|.$$

Hence, when $q = 0$, it is sufficient to prove that

$$\frac{4}{x^2} |\nu \cot \frac{1}{2}\nu\pi| < \frac{2}{x}.$$

But $\nu \cot \frac{1}{2}\nu\pi$ is never infinite when $-1 < \nu < 1$, and its maximum value is $2/\pi$, when $\nu = 0$.

Hence it is sufficient, when $q = 0$, to prove that

$$\frac{4}{\pi x} < 1. \tag{3.47}$$

But, in the case under consideration, $\nu \geq -\frac{1}{2}$; thus x is greater than the smallest positive zero of $J_{-\frac{1}{2}}(x)$, *viz.*, $\frac{1}{2}\pi$.

Thus when $-\frac{1}{2} \leq \nu < 1$, (3.47) is satisfied, since $8/\pi^2 < 1$. We have now proved that (3.45) is satisfied for all values of ν greater than or equal to $-\frac{1}{2}$.

3.5. In this subsection we will prove the condition (b) of § 3.3, *viz.*, that when $x \geq j_{\nu,1}$ and $\nu \geq -\frac{1}{2}$, then

$$S_{0,\nu}(x) > 0.$$

Take the expression (3.44) for $S_{0,\nu}(x)$, and denote the term involving the integral by $\psi(x)$. Proceeding exactly as in § 3.4, it is sufficient to prove that

$$|\psi(x)| < \frac{1}{x},$$

and this leads to

$$2^{2q} \Gamma(2q + \frac{3}{2}) < \sqrt{\pi} \cdot (j_{2q-1,1})^{2q+1} \quad (3.51)$$

as a sufficient condition.

(i) Suppose $q \geq 2$. We have

$$\Gamma(x) < x^{x-\frac{1}{2}} e^{-x} \sqrt{(2\pi)} e^{1/12x} < x^{x-\frac{1}{2}} \cdot e^{-x+\frac{1}{12}} \cdot \sqrt{(2\pi)}$$

for $x > j_{3,1} > 6.3$.

Then again proceeding as in § 3.4, and noting that the greatest value of

$$\left(1 + \frac{3}{4q}\right)^{2q+1} \left/ \left(1 - \frac{1}{4q^2}\right)^{q+\frac{1}{2}}\right.$$

for values of q under consideration is 5.7756 (when $q = 2$), we see that it is sufficient to prove that

$$5.7756 \cdot 2^{2q+\frac{1}{2}} e^{-2q-\frac{3}{2}+\frac{1}{12}} < 1.$$

This is true for $q = 2$ and the function decreases for $q > 0$; hence it is true for $q \geq 2$.

(ii) Suppose $q = 1$. Put $q = 1$, and $j_{1,1} > 3.8$ in (3.51), and the inequality is at once seen to be satisfied.

(iii) Suppose $q = 0$.

In this case we have, as in § 3.4,

$$|\psi(x)| < \frac{1}{x^2} \left| \frac{1}{2}\nu \cot \frac{1}{2}\nu\pi \right| < \frac{2}{\pi^2 x} < \frac{1}{x},$$

if $\nu \geq -\frac{1}{2}$.

This proves the result.

Accordingly, the theorem is now proved completely for all values of ν greater than or equal to $-\frac{1}{2}$.

3.6. Case (ii).

We cannot extend the above argument to cover the range $-1 < \nu < -\frac{1}{2}$, because the smallest positive zero of $J_\nu(x)$ tends steadily to zero as we approach the value -1 of ν .

Instead, we proceed as follows. With the previous notation let

$$C(x) = x^{2+2\nu};$$

then $B(x) = -\nu x^{1+2\nu}, A(x) = x^{2\nu}(\nu^2 + x^2);$

hence $\Lambda(x) = 2(2\nu+1)x^{2\nu+1} < 0, \text{ if } \nu < -\frac{1}{2}.$

Now $\Theta(j_p) = j_p^{2+2\nu} J_\nu'^2(j_p);$

this is therefore a decreasing sequence when $\nu < -\frac{1}{2}.$

But we have

$$\int_0^{j_p} J_\nu(v) v^{\nu+1} dv = -j_p^{\nu+1} J_\nu'(j_p).$$

The required result therefore follows for this integral, and hence, by the argument of (3.31), for the integral

$$\int_0^{j_p} J_\nu(v) v^{a-1} dv,$$

provided that $a-\nu < 2.$

This proves the theorem completely.

3.7. It only remains to point out that in the special case $\nu = \frac{1}{2},$ cited in the enunciation of the theorem at the beginning of this section, the result holds for $-\frac{1}{2} < a < \frac{3}{2},$ since the greatest maximum of

$$\int_0^{nx} v^{a-\frac{3}{2}} \sin v dv$$

obviously occurs at $nx = \pi,$ whenever the integral is convergent.

The Fourier-Bessel series.

4. *Definitions.* We define the *Gibbs ratio* for the Fourier-Bessel series for points x such that $0 \leq x < 1$ in the same way as for the Hankel Integral, except that, in the formulae (2.01) and (2.02), $I_n(x)$ is replaced by $S_n(x),$ this being the n -th partial sum of the Fourier-Bessel series for the given function $f(x).$

Consider the end-point $x = 1.$ In the special case $\nu = \frac{1}{2},$ the Fourier-Bessel series for the function x^ν reduces to the Fourier series

$$\begin{aligned} 2 \sum_{m=1}^{\infty} \frac{(-1)^{m-1} \sin m\pi x}{m\pi} &= x \quad (-1 < x < 1) \\ &= 0 \quad (x = \pm 1), \end{aligned}$$

and this is defined by periodicity elsewhere. Here $f(1-0) = 1$, $f(1+0) = -1$, so that (2.01) gives

$$G = \frac{2}{\pi} \int_0^\pi \frac{\sin v}{v} dv.$$

We therefore choose a definition of the Gibbs ratio for the Fourier-Bessel series at $x = 1$ which is consistent with this particular case, viz.,

$$(4.01) \quad G = \lim_{n \rightarrow \infty} \frac{\text{Max } S_n(x)}{f(1-0)} \quad [f(1-0) \neq 0].$$

In view of Young's work, alluded to in § 1, it will be seen that for points of ordinary discontinuity internal to the range $(0, 1)$, we have for the Fourier-Bessel series

$$G = \frac{2}{\pi} \int_0^\pi \frac{\sin v}{v} dv.$$

5. Gibbs's phenomenon at the point $x = 1$.

THEOREM. *The Fourier-Bessel series for a given function $f(x)$ has, at the point $x = 1$, a Gibbs ratio equal to*

$$\frac{2}{\pi} \int_0^\pi \frac{\sin v}{v} dv \quad [f(1-0) \neq 0].$$

Take $f(t) = t^\nu$ ($0 \leq t \leq 1$). Then

$$\begin{aligned} S_n(x) &= \sum_{m=1}^n \frac{2J_\nu(j_m x)}{J_{\nu+1}^2(j_m)} \int_0^1 J_\nu(j_m t) t^{\nu+1} dt \\ &= \sum_{m=1}^n \frac{2J_\nu(j_m x) J_{\nu+1}(j_m)}{j_m J_{\nu+1}^2(j_m)} \\ &= \frac{2}{\sqrt{x}} \sum_{m=1}^n \left[\frac{\cos(j_m x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi) \sin(j_m - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)}{j_m} + O(m^{-2}) \right] \\ &= \frac{1}{\sqrt{x}} \sum_{m=1}^n \left[\frac{\sin j_m(1-x) - \cos \{j_m(1+x) - \nu\pi\}}{j_m} + O(m^{-2}) \right] \\ &= \frac{1}{\sqrt{x}} \sum_{m=1}^n \left[\frac{2 \sin \{(m + \frac{1}{2}\nu - \frac{1}{4}) \pi(1-x)\}}{j_m} + O(m^{-2}) \right], \end{aligned} \tag{5.01}$$

since $j_m = (m + \frac{1}{2}\nu - \frac{1}{4})\pi + O(m^{-1})^*$; hence the series (5.01) is equal to

$$\begin{aligned} & \frac{2}{\sqrt{x}} \sum_{m=1}^n \frac{\sin \{ (m + \frac{1}{2}\nu - \frac{1}{4}) \pi (1-x) \}}{(m + \frac{1}{2}\nu - \frac{1}{4}) \pi} + \phi_n(x) \\ &= \frac{2}{\sqrt{x}} \int_0^{1-x} \left\{ \sum_{m=1}^n \cos (m + \frac{1}{2}\nu - \frac{1}{4}) \pi \alpha \right\} d\alpha + \phi_n(x) \\ &= \frac{2}{\sqrt{x}} \left[\int_0^{1-x} \frac{\sin (n + \frac{1}{2}\nu - \frac{1}{4}) \pi \alpha}{2 \sin \frac{1}{2} \pi \alpha} d\alpha - \int_0^{1-x} \frac{\sin (\frac{1}{4} + \frac{1}{2}\nu) \pi \alpha}{2 \sin \frac{1}{2} \pi \alpha} d\alpha \right] + \phi_n(x), \end{aligned}$$

where $\phi_n(x)$ tends uniformly to its limit in the neighbourhood of $x=1$.

Now the second of these integrals is continuous at $x=1$; but as $n \rightarrow \infty$, the first integral tends to the same limit as

$$\frac{2}{\pi \sqrt{x}} \int_0^{(1-x)(n + \frac{1}{2}\nu - \frac{1}{4})\pi} \frac{\sin v}{v} dv,$$

by the Riemann-Lebesgue Lemma, and this has its greatest maximum when

$$x = 1 - \frac{1}{n + \frac{1}{2}\nu - \frac{1}{4}},$$

its limit as $n \rightarrow \infty$ being

$$\frac{2}{\pi} \int_0^\pi \frac{\sin v}{v} dv.$$

We have proved that

$$S_n(x) = \psi_n(x) + \frac{2}{\pi \sqrt{x}} \int_0^{(1-x)(n + \frac{1}{2}\nu - \frac{1}{4})\pi} \frac{\sin v}{v} dv,$$

where $\psi_n(x)$ is continuous at $x=1$.

Put $x=1$, then $\psi_n(1) = S_n(1) = 0$ for all integral n , since

$$J_\nu(j_m) = 0 \quad \text{for all } m.$$

Hence

$$\lim_{n \rightarrow \infty} \text{Max}_x S_n(x) = \frac{2}{\pi} \int_0^\pi \frac{\sin v}{v} dv,$$

and $f(1-0) = 1$; so that[†], by (4.01),

$$G = \frac{2}{\pi} \int_0^\pi \frac{\sin v}{v} dv.$$

* G. N. Watson, *loc. cit.*, 506.

† Results concerning the convergence of Dini series in the neighbourhood of $x=1$ have been obtained by W. H. Young ["On non-harmonic Fourier series", *Proc. London Math. Soc.* (2), 18, (1920), 333-335]; but since the behaviour of Dini series near $x=1$ is different from that of Fourier-Bessel series (Watson, *loc. cit.*, 594 and 602-3), we cannot infer the Gibbs ratio at $x=1$ for the latter series from Young's work.

6. *Gibbs's phenomenon at the origin.*

By analogy with the Hankel Integral, it is interesting to consider the *modified Fourier-Bessel series*

$$x^a \sum_{m=1}^{\infty} \frac{2J_{\nu}(j_m x)}{J_{\nu+1}^2(j_m)} \int_0^1 J_{\nu}(j_m t) t^{1-a} f(t) dt.$$

This reduces to the Fourier-Bessel series when $a=0$, and to an ordinary Fourier series when $a=\nu=\frac{1}{2}$.

THEOREM. *The modified Fourier-Bessel series*

$$x^a \sum_{m=1}^{\infty} \frac{2J_{\nu}(j_m x)}{J_{\nu+1}^2(j_m)} \int_0^1 J_{\nu}(j_m t) t^{1-a} f(t) dt$$

has a Gibbs ratio at $x=0$ equal to

$$2^{1-a} \frac{\Gamma(1-\frac{1}{2}a+\frac{1}{2}\nu)}{\Gamma(\frac{1}{2}a+\frac{1}{2}\nu)} \int_0^{j_{\nu,1}} J_{\nu}(v) v^{a-1} dv,$$

where $j_{\nu,1}$ is the smallest positive zero of $J_{\nu}(v)$, provided that

$$-\frac{1}{2} < a \leq 1, \quad a-\nu < 2, \quad a+\nu > 0, \quad f(+0) \neq 0.$$

If $a+\nu=0$, $a > -\frac{1}{2}$, there is no Gibbs's phenomenon at $x=0$. As a special case, if $\nu=\frac{1}{2}$, we have the result that the Fourier series

$$\frac{2x^{a-\frac{1}{2}}}{\pi} \sum_{m=1}^{\infty} \sin m\pi x \int_0^1 t^{\frac{1}{2}-a} f(t) \sin m\pi t dt \quad [f(+0) \neq 0],$$

has a Gibbs ratio at $x=0$ equal to

$$\frac{2^{\frac{3}{2}-a}}{\sqrt{\pi}} \frac{\Gamma(\frac{5}{4}-\frac{1}{2}a)}{\Gamma(\frac{1}{4}+\frac{1}{2}a)} \int_0^{\pi} v^{a-\frac{3}{2}} \sin v dv,$$

provided that $-\frac{1}{2} < a < \frac{3}{2}$; and if $a=\frac{1}{2}$, this further reduces to

$$\frac{2}{\pi} \int_0^{\pi} \frac{\sin v}{v} dv.$$

In the modified Fourier-Bessel series, take

$$f(t) = 1 \quad (0 \leq t \leq 1),$$

so that it becomes

$$x^a \sum_{m=1}^{\infty} \frac{2J_{\nu}(j_m x)}{J_{\nu+1}^2(j_m)} \int_0^1 J_{\nu}(j_m t) t^{1-a} dt.$$

We will first deal with the general case $\alpha + \nu > 0$.

We have

$$\begin{aligned} S_n(x) &= x^\alpha \sum_{m=1}^n \frac{2J_\nu(j_m x)}{J_{\nu+1}^2(j_m)} \int_0^1 J_\nu(j_m t) t^{1-\alpha} dt \\ &= x^\alpha \sum_{m=1}^n \frac{2J_\nu(j_m x)}{J_{\nu+1}^2(j_m)} j_m^{\alpha-1} \Theta_{\nu, \alpha}(j_m) + \lambda_{\nu, \alpha} x^\alpha \sum_{m=1}^n \frac{2J_\nu(j_m x)}{J_{\nu+1}^2(j_m)} j_m^{\alpha-2}, \quad (6.01) \end{aligned}$$

where $\Theta_{\nu, \alpha}(u)$ and $\lambda_{\nu, \alpha}$ have been defined in § 3, and $\alpha - \nu < 2$ gives convergence at the lower limit of the integral.

Consider the first of these sums.

Let $[y]$ denote the greatest integer contained in y , and, if $[x^{-1}] < n$, let the range of summation be divided into the two parts

$$(1, [x^{-1}]), \quad ([x^{-1}], n);$$

no such subdivision is required if $[x^{-1}] \geq n$.

Take the first case, and let the two sums which arise be called Σ_1, Σ_2 .

(i) In the range $(1, [x^{-1}])$, we employ the power series for $J_\nu(j_m x)$, the form (3.23) for $\Theta_{\nu, \alpha}(j_m)$, and for $J_{\nu+1}^2(j_m)$ the asymptotic form

$$(6.02) \quad J_{\nu+1}^2(j_m) = \frac{2}{\pi} j_m^{-1} \{1 + O(m^{-1})\}.$$

Then Σ_1 converges uniformly to zero in the neighbourhood of $x = 0$ if $\alpha > -\frac{1}{2}$, since $\alpha + \nu > 0$; this may be shown precisely as in § 3.

(ii) In the range $([x^{-1}], n)$, we employ asymptotic forms for all the functions concerned. Then Σ_2 converges uniformly to zero in the neighbourhood of $x = 0$ if $\alpha > -\frac{1}{2}$; this may also be shown as in § 3.

We may deal similarly with the second case.

It remains to examine the second sum in (6.01). We shall now prove that, uniformly in x in the neighbourhood of $x = 0$,

$$(6.03) \quad x^\alpha \left[\sum_{m=1}^n \frac{2j_m^{\alpha-2} J_\nu(j_m x)}{J_{\nu+1}^2(j_m)} - \int_0^{j_{n-1}} J_\nu(xu) u^{\alpha-1} du \right] \rightarrow 0$$

as $n \rightarrow \infty$.

We divide the range of summation of the series into the two parts

$$(1, [x^{-\nu}]), \quad ([x^{-\nu}], n),$$

where $\frac{1}{2} < p < 1$, if $[x^{-p}] < n$; no such subdivision is required if $[x^{-p}] \geq n$.

(i) Inserting the power-series for $J_\nu(j_m x)$, and the asymptotic form (6.02) for $J_{\nu+1}^2(j_m)$, we have

$$x^\alpha \sum_{m=1}^{[x^{-p}]} \frac{2j_m^{\alpha-2} J_\nu(j_m x)}{J_{\nu+1}^2(j_m)} = x^{\alpha+\nu} \sum_{m=1}^{[x^{-p}]} O(m^{\alpha+\nu-1}) = O\{x^{(\alpha+\nu)(1-p)}\},$$

which tends to zero uniformly in the neighbourhood of $x=0$, since $\alpha+\nu > 0$, $p < 1$. Similarly

$$\begin{aligned} x^\alpha \int_0^{j_{[x^{-p}]-1}} J_\nu(xu) u^{\alpha-1} du &= x^{\alpha+\nu} \int_0^{j_{[x^{-p}]-1}} O(u^{\alpha+\nu-1}) du, \\ &= O\{x^{(\alpha+\nu)(1-p)}\}, \end{aligned}$$

and this tends to zero uniformly in the neighbourhood of $x=0$, as before.

(ii) Turning now to the range $([x^{-p}], n)$, we have

$$\begin{aligned} \frac{1}{\pi} \int_{j_{m-1}}^{j_m} J_\nu(xu) u^{\alpha-1} du \\ &= J_\nu\{x(j_m - \eta_m)\} (j_m - \eta_m)^{\alpha-1} \{1 + O(m^{-1})\}, \quad 0 \leq \eta_m \leq \pi; \\ &= J_\nu(j_m x) j_m^{\alpha-1} \{1 + O(m^{-1})\} \\ &\quad + [J_\nu\{x(j_m - \eta_m)\} - J_\nu(j_m x)] j_m^{\alpha-1} \{1 + O(m^{-1})\}. \end{aligned}$$

Also, by (6.02),

$$\frac{2j_m^{\alpha-2} J_\nu(j_m x)}{J_{\nu+1}^2(j_m)} = \pi j_m^{\alpha-1} J_\nu(j_m x) \{1 + O(m^{-1})\}.$$

Hence

$$\begin{aligned} x^\alpha \left[\frac{2j_m^{\alpha-2} J_\nu(j_m x)}{J_{\nu+1}^2(j_m)} - \int_{j_{m-1}}^{j_m} J_\nu(xu) u^{\alpha-1} du \right] \\ &= \pi x^\alpha j_m^{\alpha-1} J_\nu(j_m x) O(m^{-1}) \\ &\quad - \pi x^\alpha j_m^{\alpha-1} [J_\nu\{x(j_m - \eta_m)\} - J_\nu(j_m x)] \{1 + O(m^{-1})\}. \end{aligned}$$

Consider

$$\pi x^\alpha \sum_{m=[x^{-p}]}^n j_m^{\alpha-1} J_\nu(j_m x) O(m^{-1});$$

writing

$$J_\nu(j_m x) = O\{(j_m x)^{-\frac{1}{2}}\},$$

$$\text{this becomes } x^{\alpha+\frac{1}{2}} \sum_{m=[x^{-p}]}^n O(m^{\alpha-\frac{3}{2}}) = O\{x^{\alpha+\frac{1}{2}-p(\alpha-\frac{3}{2})}\},$$

since $\alpha < \frac{3}{2}$.

This tends to zero uniformly in the neighbourhood of $x=0$, since $\alpha > -\frac{1}{2}$, $p > \frac{1}{2}$. Again,

$$\begin{aligned} (6.04) \quad \pi x^\alpha j_m^{\alpha-1} [J_\nu\{x(j_m - \eta_m)\} - J_\nu(j_m x)] \{1 + O(m^{-1})\} \\ = -\pi x^\alpha j_m^{\alpha-1} \eta_m x J'_\nu(j_m x + \theta \eta_m x) \{1 + O(m^{-1})\}, \\ \text{where } 0 < \theta < 1; \\ = \pi x^{\alpha+1} j_m^{\alpha-1} \eta_m O\{(j_m x)^{-\frac{1}{2}}\} \\ = x^{\alpha+\frac{1}{2}} O(m^{\alpha-\frac{3}{2}}), \quad \text{if } j_m x \geq K. \end{aligned} \quad (6.05)$$

We accordingly consider the summation of the expression (6.04) over the range $([x^{-p}], n)$ in the two parts

$$([x^{-p}], [x^{-1}]), \quad ([x^{-1}], n),$$

in case $[x^{-1}] < n$; otherwise no such subdivision is necessary.

From (6.05) this summation over $([x^{-1}], n)$ is

$$x^{\alpha+\frac{1}{2}} \sum_{m=[x^{-1}]}^n O(m^{\alpha-\frac{3}{2}}) = O(x),$$

which tends to zero uniformly in the neighbourhood of $x=0$.

In the range $([x^{-p}], [x^{-1}])$, we employ the power-series for J_ν , and we see that the required sum is

$$\begin{aligned} x^{\alpha+\nu} \sum_{m=[x^{-p}]}^{[x^{-1}]} O(m^{\alpha+\nu-2}) = O(x), \quad \text{if } \alpha+\nu \geq 1; \\ = O\{x^{(\alpha+\nu)(1-p)+p}\}, \quad \text{if } \alpha+\nu < 1; \end{aligned}$$

and both expressions tend to zero uniformly in the neighbourhood of the origin, since $\frac{1}{2} < p < 1$, and $\alpha+\nu > 0$.

Hence (6.03) is proved.

Now

$$\lambda_{\nu, \alpha} x^\alpha \int_0^{j_{n-1}} J_\nu(xu) u^{\alpha-1} du = \lambda_{\nu, \alpha} \int_0^{j_{n-1}x} J_\nu(v) v^{\alpha-1} dv,$$

and by the second theorem of § 3, and under the conditions there stated, this integral has its greatest maximum when $j_{n-1}x = j_{\nu,1}$, and hence the theorem stated at the beginning of this section is proved in the case $\alpha + \nu > 0$.

In the case $\alpha + \nu = 0$ we have

$$\begin{aligned} S_n(x) &= x^{-\nu} \sum_{m=1}^n \frac{2J_\nu(j_m x)}{J_{\nu+1}^2(j_m)} \int_0^1 J_\nu(j_m t) t^{\nu+1} dt \\ &= x^{-\nu} \sum_{m=1}^n \frac{2J_\nu(j_m x)}{j_m J_{\nu+1}(j_m)}, \end{aligned}$$

since $\nu > -1$; and this tends uniformly to unity* in the neighbourhood of the origin, as may be shown by a process exactly similar to that given at the beginning of § 3 in the corresponding case for the Hankel integral.

Hence there is no Gibbs's phenomenon at $x = 0$ in this case.

It remains only to point out that in the special case $\nu = \frac{1}{2}$, cited in the enunciation of the theorem at the beginning of this section, the result holds for $-\frac{1}{2} < \alpha < \frac{3}{2}$, since the greatest maximum of

$$\int_0^{(n-1)\pi x} v^{\alpha-\frac{3}{2}} \sin v \, dv$$

obviously occurs at $(n-1)\pi x = \pi$ whenever the integral is convergent.

* See G. N. Watson, *loc. cit.*, 586 (4).

DOUBLE BINARY PERPETUANTS

By H. W. TURNBULL.

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Introduction.

The double binary form

$$f = \sum_{r,s} \alpha_{rs} x_1^{m-r} x_2^r y_1^{n-s} y_2^s$$

of orders m, n in two sets of binary variables x and y respectively, can be transformed into another such (m, n) form by independent binary linear transformations applied simultaneously to x and y . This leads to an invariant theory of double binary forms analogous to the well known ordinary binary theory.

Symbolic methods* lead successfully to a proof† of Gordan's theorem on the finiteness of the number of irreducible concomitants of a given system of such ground forms.

But the part of the binary theory concerning forms of infinite order, whose irreducible seminvariants are called *perpetuants*‡, has not yet been extended to double binary forms. I propose to show how such an extension is possible, and to obtain the generating function of double and of multiple binary perpetuants involving any number of ground

* Cf. Peano, *Battaglini Giorn. di Mat.*, 20 (1882), 79–101.

† H. W. Turnbull, *Proc. Royal Soc. Edinburgh*, 43 (1922), 43–50; *Proc. Edinburgh Math. Soc.*, 41 (1923), 116–127.

‡ MacMahon, *Proc. London Math. Soc.* (1), 26 (1895), 262–284; *American J. of Math.*, 7 (1884), 26–47, 259–263; 8 (1885), 1–18. *Trans. Cambridge Phil. Soc.*, 19 (1901), 234–248. Stroh, *Math. Annalen*, 36 (1890), 262–303. Grace, *Proc. London Math. Soc.* (1), 35 (1902), 107–111 and 319–331. Grace and Young, *Algebra of Invariants* (Cambridge, 1903), 326–330.

forms, §§ 6, 7, and involving a single double binary ground form, §§ 8-12.

1. In the symbolic notation let the (m, n) form be written

$$f = a_x^m a_y'^n. \quad (1)$$

If we drop the accent and make y equal to x we have the ordinary binary $(m+n)$ -ic

$$a_x^{m+n}. \quad (2)$$

Now just as a symbolic product of bracket factors

$$\Pi(ab)^\lambda = (ab)^\lambda (ac)^\mu (ad)^\nu (bc)^{\lambda_1} \dots \quad (3)$$

represents the seminvariant of a binary form of infinite order, without the use of $a_1, b_1, \dots$ as factors besides, so also the double binary seminvariant is typified by

$$\Pi(ab)^\lambda \Pi(a'b')^\rho \quad (4)$$

with its accented and unaccented bracket factors.

In the binary theory for different ground forms, each symbol $a, b, \dots$ refers to a separate form. None are equivalent. The seminvariant of degree k is linear in k different forms. Hence the only fundamental reduction process is the identity

$$(a_2 a_3) + (a_3 a_1) + (a_1 a_2) = 0, \quad (5)$$

where a_1, a_2, a_3 are any three of the k symbols $a, b, c, \dots$. So also for k double binary forms the only reduction processes are this identity together with

$$(a'_2 a'_3) + (a'_3 a'_1) + (a'_1 a'_2) = 0. \quad (6)$$

Let us write $n+1$ for k , and single out one quantic f , with symbol a , for special prominence. Let us arrange the other quantics

$$f_1, f_2, \dots, f_n \quad (7)$$

in a definite order, and attach the double binary symbols a_i, a'_i to the quantic f_i . Then the products

$$P = (aa_1)^{\lambda_1} (aa_2)^{\lambda_2} \dots (aa_n)^{\lambda_n} \quad (\lambda_i > 0), \quad (8)$$

and $Q = (aa_1)^{\mu_1} (a'a'_1)^{\nu_1} (aa_2)^{\mu_2} (a'a'_2)^{\nu_2} \dots (aa_n)^{\mu_n} (a'a'_n)^{\nu_n}, \quad (9)$

where $\mu_i + \nu_i > 0$, represent respectively the typical binary and double binary seminvariant of this degree $n+1$. For by identities (5) and (6) the bracket factors containing neither symbol a nor a' may be replaced by such as do so. For example, $(a_1 a_2)$ would be replaced by $(aa_2) - (aa_1)$.

We now recall the conditions that the product P should be a perpetuant, namely

$$\lambda_1 \geq 2^{n-1}, \quad \lambda_2 \geq 2^{n-2}, \quad \dots, \quad \lambda_{n-1} \geq 2, \quad \lambda_n \geq 1. \quad (10)$$

If one of these fails, then P is a reducible seminvariant. To the question, what are the corresponding double binary conditions, an answer can be obtained by use of a device like polarization.

Polarization of identities.

2. Let there be any rational integral homogeneous function $\phi(x, y, z)$ which vanishes identically when x, y, z are connected by the relation

$$x + y + z = 0. \quad (1)$$

Then if ξ, η, ζ be another such trio satisfying

$$\xi + \eta + \zeta = 0, \quad (2)$$

we may derive a new identity

$$\left(\xi \frac{\partial}{\partial x}\right) \phi \equiv \left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \zeta \frac{\partial}{\partial z}\right) \phi(x, y, z) = 0 \quad (3)$$

from the identity $\phi(x, y, z) = 0. \quad (4)$

For taking x and y as independent variables, with $z = -x - y$, we have

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z} \frac{\partial z}{\partial x} = 0,$$

$$\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} \frac{\partial z}{\partial y} = 0.$$

Multiplying these respectively by ξ and η , and using (2), the result (3) follows.

Evidently this process, which is analogous to polarizing the ternary form ϕ with regard to the new set ξ, η, ζ , can be repeated. If we write

ϕ symbolically as

$$\phi_x^w$$

where ϕ_x is a ternary symbol, the new identity (3) may be written

$$w\phi_x^{w-1}\phi_\xi = 0. \quad (5)$$

Dropping the inevitable numerical factor, the r -th polar identity may be written

$$\phi_x^{w-r}\phi_\xi^r = 0. \quad (6)$$

We can also have mixed polars, such as

$$\phi_x^{w-2}\phi_\xi\phi_{\xi'} = 0 \quad (7)$$

where $\xi' + \eta' + \zeta' = 0$ is a new relation.

To illustrate, let $w = 2$. From

$$2yz = x^2 - y^2 - z^2$$

we deduce

$$y\zeta + \eta z = x\xi - y\eta - z\zeta.$$

From

$$3xyz = x^3 + y^3 + z^3,$$

where $w = 3$, we deduce

$$\xi yz + \eta zx + \zeta xy = x^2\xi + y^2\eta + z^2\zeta,$$

$$x\eta\zeta + y\zeta\xi + z\xi\eta = \xi^2x + \eta^2y + \zeta^2z.$$

From a given identity, $\phi(x, y, z) = 0$, of degree w in x, y, z , we deduce a set of $w+1$ such identities, if $\phi(x, y, z)$ and $\phi(\xi, \eta, \zeta)$ are regarded as giving the first and the last of the set.

In this we have a basis for a theory of double binary perpetuants.

3. More generally, if we have a set of $\frac{1}{2}n(n-1)$ variables x_{ij} such that

$$x_{ij} = -x_{ji}, \quad (1)$$

$$(i = 1, 2, 3, \dots, n, \quad j = 1, 2, 3, \dots, n),$$

satisfying identities of the form

$$x_{ik} + x_{kl} + x_{ij} = 0 \quad (2)$$

for all such values of i, j, k , then, from any consequent identity $\phi(x_{ij}) = 0$ involving some or all of the variables, we may deduce a further identity

$$\left(\sum \xi_{ij} \frac{\partial}{\partial x_{ij}}\right) \phi(x_{ij}) = 0. \quad (3)$$

In this the summation has $\frac{1}{2}n(n-1)$ terms, and the coefficients ξ_{ij} satisfy the same conditions as do x_{ij} .

To establish this result observe that the conditions imply that $n-1$ variables are independent, for example those for which $i = 1$. Differentiating ϕ with regard to these, we obtain $n-1$ equations

$$\frac{\partial \phi}{\partial x_{1j}} + \sum_{l=2}^n \frac{\partial \phi}{\partial x_{jl}} \frac{\partial x_{jl}}{\partial x_{1j}} = 0 \quad (j = 2, 3, \dots, n),$$

which may be written

$$\frac{\partial \phi}{\partial x_{1j}} - \sum_{l=2}^n \frac{\partial \phi}{\partial x_{jl}} = 0,$$

since $x_{jl} + x_{kl} + x_{lj} = 0$. Hence, multiplying by ξ_{1j} and summing,

$$\sum_{j=2}^n \xi_{1j} \frac{\partial \phi}{\partial x_{1j}} - \sum_j \sum_k \xi_{1j} \frac{\partial \phi}{\partial x_{jk}} = 0,$$

$$\text{i.e.} \quad \sum_j \xi_{1j} \frac{\partial \phi}{\partial x_{1j}} - \sum_{j \neq k} \sum (\xi_{1j} - \xi_{1k}) \frac{\partial \phi}{\partial x_{jk}} = 0,$$

$$\text{i.e.} \quad \sum_{(ij)} \xi_{ij} \frac{\partial \phi}{\partial x_{ij}} = 0.$$

It follows that any binary syzygy between the seminvariants may be polarized in the manner explained, so leading to a corresponding syzygy among double binary forms, or even multiple binary forms.

4. Let us write x_i for the bracket factor (aa_i) in a symbolic product such as P , so that

$$x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n} \quad (1)$$

now means exactly the same thing as

$$(aa_1)^{\lambda_1} (aa_2)^{\lambda_2} \dots (aa_n)^{\lambda_n}. \quad (2)$$

Let x'_i accented denote $(a'a'_i)$, so that the double binary seminvariant Q is now written

$$Q_{\mu\nu} = x_1^{\mu_1} x_1^{\nu_1} x_2^{\mu_2} x_2^{\nu_2} \dots \quad (\mu_i \geq 0, \nu_i \geq 0). \quad (3)$$

If $\mu_i + \nu_i = \lambda_i$ ($i = 1, 2, \dots, n$), every such Q is a term derivable from P by polarization with regard to

$$\left(x' \frac{\partial}{\partial x}\right)$$

$\nu_1 + \nu_2 + \dots + \nu_n$ times. Thus if P is of degree w in $x_1, x_2, \dots, x_n$ it leads to $w+1$ polar forms

$$P_0, P_1, P_2, \dots, P_w, \quad (4)$$

each of which may be written as a sum of terms Q ,

$$P_i = \sum \rho Q_{\mu\nu}, \quad (5)$$

where ρ is a positive numerical coefficient. Owing to the possible values of μ_i , from λ_i to zero inclusive, the number of separate terms Q involved in the $w+1$ forms P_i is

$$(\lambda_1 + 1)(\lambda_2 + 1) \dots (\lambda_n + 1). \quad (6)$$

As an example, let $n = 2$, $w = 3$, $\lambda_1 = 2$, $\lambda_2 = 1$. Then

$$P = x^2y$$

symbolizes a binary seminvariant. This leads to four polars containing $(2+1)(1+1) = 6$ terms:

$$\left. \begin{aligned} P_0 &= x^2y, \\ P_1 &= 2xx'y + x^2y', \\ P_2 &= 2x'^2y + 4xx'y', \\ P_3 &= 6x'^2y', \end{aligned} \right\} \quad (7)$$

representing six double binary seminvariants.

Again, if P, P' are two binary seminvariants of the same degree and weight, but with a different set λ , none of the derived forms P_i, P'_i and terms $Q_{\mu\nu}, Q'_{\mu\nu}$ can be equal. For the polar process carries with it the difference in the two sets of indices.

Reducibility of a double binary seminvariant.

5. For given values of n and w we consider a seminvariant $Q_{\mu\nu}$ to be reducible only if it is derived from a reducible binary seminvariant P . Also in this case only one term $Q_{\mu\nu}$ out of each polar form P_i can be considered reducible. In fact, if P itself is irreducible, no derived polar P_i , or term $Q_{\mu\nu}$, can be reducible. For otherwise the mere suppression of accents in P_i , or $Q_{\mu\nu}$, would reduce P . But if P is reducible it may be written

$$P = \Sigma A + \Sigma BC, \quad (1)$$

where the terms A , if any, are intrinsically irreducible, but satisfy conditions (10), § 1, while the terms BC are products of actual seminvariants of lower degree and weight than P . Also the indices λ of P do not satisfy all the conditions (10), § 1. For example, xy^2 is reducible because

$$xy^2 = -x^2y - \frac{1}{3}x^3 - \frac{1}{3}y^3 - \frac{1}{3}z^3,$$

where $x+y+z=0$; and here ΣA consists of the one term $-x^2y$, to be retained in preference to xy^2 , since the order of symbols of x, y is fixed beforehand.

If we polarize this equation (1) of reducibility, we obtain reducible terms entirely on the right-hand side. For each polar process can only affect one factor, B or C , of BC at a time, while it also does not alter the conditions (10), § 1, which determine A .

Hence, using (4), (6), § 4, each reducible binary seminvariant of weight w leads to $(w+1)$ reducible double binary seminvariants of weight w , together with

$$(\lambda_1+1)(\lambda_2+1) \dots (\lambda_n+1) - (w+1) \quad (2)$$

irreducible seminvariants.

Generating function of perpetuants.

6. This last result (2) gives us the number of double binary perpetuants of degree $n+1$, weight w , and indices $\lambda_1, \dots, \lambda_n$. These indices satisfy

$$\lambda_1 \geq 2^{n-1}, \quad \lambda_2 \geq 2^{n-2}, \quad \dots, \quad \lambda_{n-1} \geq 2, \quad \lambda \geq 1. \quad (1)$$

This leads to a *generating function*

$$\sum_{w=0}^{\infty} c_w t^w;$$

this name is given when the coefficient c_w of the typical term is the number of perpetuants of degree $n+1$ and weight w . For let

$$\left((t^w) \right) (t+t^2+\dots)^n$$

denote the coefficient of t^w in the expansion of $(t+t^2+\dots)^n$. Manifestly this equals the total number of binary seminvariants of degree $n+1$ and weight w , for it is the total number of terms such as

$$P = x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}, \quad \Sigma \lambda = w. \quad (2)$$

Now the number of binary perpetuants (irreducible seminvariants) is known* to be

$$\left((t^w) \right) \frac{t^{2n-1}}{(1-t)^n}. \quad (3)$$

Hence the number of reducible binary seminvariants of degree $n+1$ and weight w is the difference of these two coefficients of t^w . In other words, it is

$$\left((t^w) \right) \left\{ \frac{t^n}{(1-t)^n} - \frac{t^{2n-1}}{(1-t)^n} \right\} = B_w, \quad (4)$$

say.

Since each form P leads [(6), § 4] to

$$(\lambda_1+1)(\lambda_2+1) \dots (\lambda_n+1)$$

double binary forms, reducible and irreducible, the total number of such forms is

$$\begin{aligned} & \left((t^w) \right) \{ [2t+\dots+(\lambda_1+1)t^{\lambda_1}+\dots][2t+\dots+(\lambda_2+1)t^{\lambda_2}+\dots] \times \dots \\ & \dots \times [2t+\dots+(\lambda_n+1)t^{\lambda_n}+\dots] \} \end{aligned} \quad (5)$$

$$= \left((t^w) \right) \left((1-t)^{-2} - 1 \right)^n. \quad (6)$$

But of these there are $(w+1)$ reducible for each reducible P , and therefore $(w+1)B_w$ reducible in all.

* *Algebra of Invariants*, loc. cit.

Now

$$\begin{aligned}
 (w+1)B_w &= \left((t^w) \right) \Sigma(w+1)B_w t^w \\
 &= \left((t^w) \right) \frac{d}{dt} \Sigma B_w t^{w+1} \\
 &= \left((t^w) \right) \frac{d}{dt} t \Sigma B_w t^w \\
 &= \left((t^w) \right) \frac{d}{dt} t \frac{t^n - t^{2n-1}}{(1-t)^n}.
 \end{aligned} \tag{7}$$

If this number is deducted from the total, (6) gives the number of double binary perpetuants of degree $n+1$ and weight w . Hence

The generating function of double binary perpetuants is

$$\left(\frac{2t-t^2}{(1-t)^2} \right)^n - \frac{d}{dt} \frac{t^{n+1}-t^{2n}}{(1-t)^n}, \tag{8}$$

wherewith the coefficient of t^w , in the expansion by ascending powers of t , is the number of perpetuants of degree $n+1$ and weight w .

Multiple binary perpetuants.

7. The same argument applies to multiple binary forms involving m independent pairs of homogeneous binary coordinates, and therefore m kinds of bracket factor

$$(ab), (a'b'), (a''b''), \dots$$

for symbolic expression of seminvariants. In this case one binary seminvariant,

$$P = x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n},$$

can be polarized with $m-1$ different operators*

$$\omega_1, \omega_2, \dots, \omega_{m-1},$$

where $\omega_i = \{ \Sigma x^{(i)} (\partial / \partial x) \}$. If we adjoin the identical operator $\omega_0 = 1$, we obtain all multiple binary polars of P , each in the form

$$\omega_0^i \omega_1^j \dots \omega_{m-1}^k P,$$

* As, for example, in (7), § 2.

where $i+j+\dots+k=w$. The number of different polars is therefore

$$\left((t) \right)^w (1-t)^{-m} = \binom{m+w-1}{w} = \binom{m+w-1}{m-1}.$$

Also if P is reducible, so is each such polar. Hence the number of reducibles is

$$\begin{aligned} & \binom{m+w-1}{m-1} B_w \\ &= \left((t^w) \right) \binom{m+w-1}{m-1} \Sigma B_w t^{wv} \\ &= \left((t^w) \right) \frac{d^{m-1}}{dt^{m-1}} \frac{\Sigma B_w t^{wv+m-1}}{(m-1)!} \\ &= \left((t^w) \right) \frac{d^{m-1}}{dt^{m-1}} \frac{t^{m-1}}{(m-1)!} \Sigma B_w t^{wv}. \end{aligned}$$

But $\Sigma B_w t^{wv} = (t^n - t^{2^n-1})(1-t)^{-n}.$

Again the total number of terms involved in all polars derived from binary forms P , of degree $n+1$ and weight w , will now be

$$\left((t^w) \right) \left\{ \frac{1}{(1-t)^n} - 1 \right\}^n.$$

Hence the generating function of perpetuants of binary forms of multiplicity m is

$$\left(\frac{1}{(1-t)^n} - 1 \right)^n - \frac{d^{m-1}}{dt^{m-1}} \left\{ \frac{t^{m-1}}{(m-1)!} \cdot \frac{t^n - t^{2^n-1}}{(1-t)^n} \right\}.$$

This reduces to the ordinary binary generating function, (3), § 6, if $m=1$.

Perpetuants of one quantic.

8. It has been proved* by J. H. Grace that, if the symbols

$$a_1, a_2, \dots, a_{n+1}$$

all refer to *one* quantic, the general perpetuant is

$$(a_1 a_2)^{2\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_n a_{n+1})^{\lambda_n}, \quad (1)$$

* *Proc. London Math. Soc.* (1), 35 (1902), 319-331.

wherein

$$2\lambda_1 \geq \lambda_2, \lambda_2 \geq \lambda_3 + 2^{n-3}, \lambda_3 \geq \lambda_4 + 2^{n-4}, \dots, \lambda_{n-1} \geq \lambda_n, \lambda_n > 0.$$

Also from these conditions it follows that the number of perpetuants of a single binary form is given by

$$\left((t) \right)^w \frac{t^{2^n-1}}{(1-t^2)(1-t^3)\dots(1-t^n)(1-t^{n+1})}. \quad (2)$$

We may deduce a generating function for double binary perpetuants by the preceding methods. Let the series in (2) be written

$$\sum C_w t^w, \quad (3)$$

and let the typical perpetuant (1) be

$$y_1^{2\lambda_1} y_2^{\lambda_2} \dots y_n^{\lambda_n}. \quad (4)$$

As before, each such term will lead to

$$(2\lambda_1+1)(\lambda_2+1) \dots (\lambda_n+1)$$

double binary perpetuants. Hence the total number of reducible binary seminvariants of this weight and degree is

$$\begin{aligned} \left((t^{10}) \right) \left[\frac{t^n}{(1-t)^n} - \frac{t^{2^n-1}}{(1-t^2)(1-t^3)\dots(1-t^{n+1})} \right] \\ = D_w, \text{ say.} \end{aligned} \quad (5)$$

9. Now each such binary reducible form may be polarized $w+1$ times leading to $w+1$ double binary reducibles. But owing to the interchange of equivalent symbols the number of irreducibles furnished by these $w+1$ polars will be less than in the preceding case. For if

$$P = y_1^{\lambda_1} y_2^{\lambda_2} \dots y_n^{\lambda_n} \quad (1)$$

is the reducible binary form in question, it leads to more than the $w+1$ double binary reducibles in every case in which the indices $\lambda_1, \dots, \lambda_n$ are *symmetrical about the centre of their row, i.e.* when

$$\lambda_1 = \lambda_n, \lambda_2 = \lambda_{n-1}, \dots, \lambda_i = \lambda_{n-i}, \quad (2)$$

$$i = 1, 2, \dots, \frac{n}{2} \text{ or } \frac{n-1}{2}.$$

This is made clear by considering the case $n = 3$. If a, b, c, d are equivalent symbols

$$P = (ab)^{\lambda_1}(bc)^{\lambda_2}(cd)^{\lambda_3};$$

and also by interchange

$$\begin{aligned} P &= (dc)^{\lambda_1}(cb)^{\lambda_2}(ba)^{\lambda_3}, \\ &= (-)^{\lambda_1+\lambda_2+\lambda_3}(ab)^{\lambda_3}(bc)^{\lambda_2}(cd)^{\lambda_1}, \\ &= (-)^{\Sigma \lambda} P \end{aligned} \quad (3)$$

provided that $\lambda_1 = \lambda_3$. Clearly when the λ 's are symmetrical this interchange applies to P and also to all its $w+1$ polars

$$P_0, P_1, \dots, P_w. \quad (4)$$

Let us write $\{\lambda_1\lambda_2\lambda_3\}$ to denote P , and $\{\mu_1\mu_2\mu_3\}$ to denote

$$(ab)^{\mu_1}(a'b')^{\lambda_1-\mu_1}(bc)^{\mu_2}(b'c')^{\lambda_2-\mu_2}(cd)^{\mu_3}(c'd')^{\lambda_3-\mu_3}, \quad (5)$$

which typifies a term in a polar P_i . Each such term of $P_0, P_1, \dots, P_w$ is so typified in general by $\{\mu_1\mu_2\mu_3 \dots \mu_n\}$, where

$$\mu_1 = 0, 1, \dots, \lambda_1, \quad \mu_2 = 0, 1, \dots, \lambda_2, \quad \mu_i = 0, 1, \dots, \lambda_i, \quad (6)$$

$$i = 1, 2, \dots, n.$$

10. For degree $n+1$ and weight w , the number of double binary perpetuants will be given by three terms

$$A_w - (w+1)D_w - E_w \quad (1)$$

say, where A_w denotes the number of all such seminvariants, $(w+1)D_w$ denotes the reducible forms obtained from the $w+1$ polars of a binary reducible, and E_w denotes the symmetrical reducibles. It is this last set which is new and did not arise in the more general case of § 6. We take these in turn, and first obtain

$$A_w = (t^w) \left((1-t)^{-2} - 1 \right)^n \quad (2)$$

as in § 6 (6). Next

$$\begin{aligned} (w+1)D_w &= (t^w) \frac{d}{dt} \Sigma D_w t^{w+1} \\ &= (t^w) \frac{d}{dt} \left[\frac{t^{n+1}}{(1-t)^n} - \frac{t^{2n}}{(1-t^2)(1-t^3)\dots(1-t^{n+1})} \right] \end{aligned} \quad (3)$$

by § 8 (5). Lastly, we calculate E_w . There are two cases arising from odd and even degree, as indicated by

$$P = \{\lambda_1 \lambda_2 \dots \lambda_k \lambda_k \dots \lambda_2 \lambda_1\}, \quad n = 2k$$

and $Q = \{\lambda_1 \lambda_2 \dots \lambda_{k-1} \lambda_k \lambda_{k-1} \dots \lambda_2 \lambda_1\}, \quad n = 2k-1 \quad [\S 9 (2)]. \quad (4)$

The characteristic numbers λ are symmetrical about the centre of the row. First consider P . This symmetrical reducible binary form leads to $w+1$ polars containing N terms, where

$$\begin{aligned} N &= (\lambda_1+1)(\lambda_2+1) \dots (\lambda_k+1) \dots (\lambda_1+1) \\ &= (\lambda_1+1)^2 (\lambda_2+1)^2 \dots (\lambda_k+1)^2. \end{aligned}$$

Among these, there are $(\lambda_1+1)(\lambda_2+1) \dots (\lambda_k+1)$ symmetrical terms, *i.e.* terms of type

$$\{\mu_1 \mu_2 \dots \mu_k \mu_k \dots \mu_2 \mu_1\}$$

as in § 9 (5) and (6). Hence the number of unsymmetrical terms for P is N' , where

$$N' = (\lambda_1+1)^2 \dots (\lambda_{k-1}+1)^2 (\lambda_k+1)^2 - (\lambda_1+1) \dots (\lambda_{k-1}+1) (\lambda_k+1). \quad (5)$$

Likewise for Q this number is M' , where

$$M' = (\lambda_1+1)^2 \dots (\lambda_{k-1}+1)^2 (\lambda_k+1) - (\lambda_1+1) \dots (\lambda_{k-1}+1) (\lambda_k+1). \quad (6)$$

Now the interchange of equivalent symbols indicated by the substitution

$$\begin{pmatrix} a_1 a_2 & \dots & a_n \\ a_n a_{n-1} & \dots & a_1 \end{pmatrix},$$

results in the relation

$$\{\mu_1 \mu_2 \dots \mu_n\} = (-)^{\Sigma \lambda} \{\mu_n \dots \mu_2 \mu_1\} \quad (7)$$

for the double, just as for the single binary case of § 9 (3). We infer that half the total number of unsymmetrical terms for P or Q may be rejected. Also from (7) we infer that if $\Sigma \lambda$ is even, the symmetrical terms are unaffected by the interchange substitution; but if $\Sigma \lambda$ is odd they all vanish. This last case occurs only when λ_k is odd in Q . It does not affect P .

Further, the reducibles already enumerated in $(w+1)D_w$ are unaffected by rejecting half the number of unsymmetricals in the case of

P ; hence, when $n = 2k$, *i.e.* when the degree is odd, E_w is given by $\Sigma \frac{1}{2} N'$. Thus, by (5),

$$\begin{aligned} E_w &= \frac{1}{2} \left((t^{\lambda_1 + \lambda_2 + \dots + \lambda_k}) \right) \left[\left(\Sigma (\lambda_1 + 1)^2 t^{\lambda_1} \right) \dots \left(\Sigma (\lambda_k + 1)^2 t^{\lambda_k} \right) \right. \\ &\quad \left. - \left(\Sigma (\lambda_1 + 1) t^{\lambda_1} \right) \dots \left(\Sigma (\lambda_k + 1) t^{\lambda_k} \right) \right], \\ &= \frac{1}{2} \left((t^{\frac{1}{2}w}) \right) \left\{ \left[\Sigma \frac{2(\lambda_1 + 1)(\lambda_1 + 2) - 2(\lambda_1 + 1)}{2} t^{\lambda_1} \right]^k - \left[\Sigma (\lambda_1 + 1) t^{\lambda_1} \right]^k \right\} \\ &= \frac{1}{2} \left((t^{\frac{1}{2}w}) \right) \left\{ \left(\frac{2}{(1-t)^3} - \frac{1}{(1-t)^2} - 1 \right)^k - \left(\frac{1}{(1-t)^2} - 1 \right)^k \right\} \\ &= \left((t^w) \right) \frac{1}{2} \left\{ \left(\frac{2}{(1-t^2)^3} - \frac{1}{(1-t^2)^2} - 1 \right)^k - \left(\frac{1}{(1-t^2)^2} - 1 \right)^k \right\}, \quad (8) \end{aligned}$$

where $n+1 = 2k+1$.

This gives a generating function for perpetuants of odd degree $n+1$ by combining results (2), (3), (8), as in (1).

11. Finally, consider the even degree case in which $n+1 = 2k$. In this we deal with Q , which, in turn, yields two cases according as λ_k is even or odd, as already noticed. Thus, if λ_k is even, and, therefore, also if the weight w is even, the expression for E_w is $\frac{1}{2} M'$.

Thus, by § 10 (6),

$$\begin{aligned} E_w &= \frac{1}{2} \left((t^{w + \frac{1}{2}\lambda_k}) \right) \left[\left(\Sigma (\lambda_1 + 1)^2 t^{\lambda_1} \right) \dots \left(\Sigma (\lambda_k + 1) t^{\lambda_k} \right) \right. \\ &\quad \left. - \left(\Sigma (\lambda_1 + 1) t^{\lambda_1} \right) \dots \left(\Sigma (\lambda_k + 1) t^{\lambda_k} \right) \right]. \end{aligned}$$

Replace t by t^2 , and divide throughout by t^{λ_k} . Hence

$$\begin{aligned} E_w &= \left((t^w) \right) \frac{1}{2} \left\{ \left(\frac{2}{(1-t^2)^3} - \frac{1}{(1-t^2)^2} - 1 \right)^{k-1} - \left(\frac{1}{(1-t^2)^2} - 1 \right)^{k-1} \right\} \\ &\quad \times \left(\frac{1}{(1-t^2)^2} - 1 \right), \quad (1) \end{aligned}$$

where $n+1 = 2k$, and w is even.

If, however, λ_k is odd, so that w is odd, then half the aggregate of unsymmetrical, and all the symmetrical, terms for Q are rejected. This

total is

$$M'' = \frac{1}{2} \{ (\lambda_1 + 1)^2 \dots (\lambda_{k-1} + 1)^2 (\lambda_k + 1) + (\lambda_1 + 1) \dots (\lambda_{k-1} + 1) (\lambda_k + 1) \}. \quad (2)$$

But of these we have enumerated $w+1$ in the expression $(w+1)D_w$, § 10 (1). Hence for each form

$$Q = \{ \lambda_1 \dots \lambda_{k-1} \lambda_k \lambda_{k-1} \dots \lambda_1 \}$$

(λ_k odd), we enumerate $M'' - (w+1)$ reducible double binary forms. Let E_w denote the result of this, corresponding to E_w in (1). Then if we rewrite E_w in (1) as

$$E_w = (t^w) \left[\frac{1}{2} (f^{k-1} - g^{k-1}) \frac{2t - t^2}{(1-t)^2} \right] = (t^w) \phi(t), \quad (3)$$

where
$$f = \frac{2}{(1-t^2)^3} - \frac{1}{(1-t^2)^2} - 1 = \frac{4t^2 - 3t^4 + t^6}{(1-t^2)^3}, \quad (4)$$

$$g = \frac{1}{(1-t^2)^2} - 1 = \frac{2t^2 - t^4}{(1-t^2)^2}, \quad (5)$$

we may write

$$\begin{aligned} E'_w &= (t^w) \left[\frac{1}{2} (f^{k-1} + g^{k-1}) \frac{2t - t^2}{(1-t)^2} - \frac{d}{dt} \frac{t^2}{(1-t^2)^k} \right] \\ &= (t^w) \psi(t) \end{aligned} \quad (6)$$

For M'' , as given in (2), provides this first term in the square brackets $[\]$, after the manner in which E_w was formed, while the second enumerates the number of possible forms Q , with λ_k odd, multiplied by $w+1$. Since the even powers of t in $\phi(t)$ alone are required, w being even, while only odd powers are required in $\psi(t)$, it follows that both cases are included in

$$E_w = (t^w) \frac{1}{2} \{ \psi(t) - \psi(-t) + \phi(t) + \phi(-t) \}, \quad (7)$$

$$= (t^w) \left[f^{k-1} \frac{2t - t^2}{2(1-t)^2} + g^{k-1} \frac{2t + t^2}{2(1+t)^2} - \frac{d}{dt} \frac{t^2}{(1-t^2)^k} \right]. \quad (8)$$

This gives the generating function for perpetuants of even degree, where $n+1 = 2k$, by combining the results of § 10 (2), (3) with this last (8), as in § 10 (1).

12. Summing up, the generating function for a single double binary perpetuant is

$$\frac{2t-t^2}{(1-t)^{2n}} - \frac{d}{dt} \left[\frac{t^{n+1}}{(1-t)^n} - \frac{t^{2n}}{(1-t^2)(1-t^3)\dots(1-t^{n+1})} \right] - E,$$

where $E = \frac{1}{2}(f^{2n}-g^{2n})$, if n is even, and

$$E = f^{\frac{1}{2}(n-1)} \frac{2t-t^2}{2(1-t)^2} + g^{\frac{1}{2}(n-1)} \frac{2t+t^2}{2(1+t)^2} - \frac{d}{dt} \frac{t^2}{(1-t^2)^{\frac{1}{2}(n+1)}},$$

if n is odd. Also

$$f = \frac{2}{(1-t^2)^3} - \frac{1}{(1-t^2)^2} - 1, \quad g = \frac{1}{(1-t^2)^2} - 1.$$

The coefficient of t^w in the generating function enumerates perpetuants of degree $n+1$.

Similar methods apply to two or more quantities.

THE SUMMATION OF DENJOY-FOURIER SERIES

By S. POLLARD.

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The coefficients.

1. It is supposed that $f(x)$ is periodic in 2π and integrable in the general Denjoy sense[†] in $(-\pi, \pi)$. Since $\cos nt$ and $\sin nt$ are of bounded variation, the integrals

$$(1.1) \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt dt, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt dt$$

[†] The terminology is that of the *Encyklopädie*. The *special* Denjoy integral is in the integral given by Denjoy in his early *Comptes rendus* note and it is described by Hobson as a *D*-integral, while the *general* Denjoy integral is the integral given by Denjoy in his papers in the *Annales Sci. de l'Ecole Normale* and it is described by Hobson as a *DKY*-integral.

The peculiarity of the special integral lies in the fact that there are associated with it major and minor functions with derivatives finite on one side, so that the special Denjoy integral is essentially a solution of the differential equation

$$\frac{d\eta}{dx} = f(x).$$

If $f(x)$ has a special Denjoy integral, then

$$\frac{d}{dx} \int_a^x f(t) dt \sim f(x),$$

where $\sim$ is used in the sense of "equal almost everywhere." The special Denjoy integral is identical with the integral defined by Perron.

The distinction between the general and the special integral may be illustrated by the relations of these integrals to the theory of trigonometrical series. As is well known, this theory is divided into two branches concerned respectively with trigonometrical series associated with a function, *i.e.* Fourier series, and trigonometrical series given by their coefficients. In the elementary part of the theory the general integral is associated with the former and the special integral with the latter.

exist as general Denjoy integrals, and are subject to the rule for integration by parts.

$$\text{If} \quad F_1(x) = \int_0^x \{f(t) - \tfrac{1}{2}a_0\} dt,$$

then, as is easily verified, $F_1(x)$ is continuous and periodic in 2π .

If $n \geq 1$, integration by parts gives

$$(1.2) \quad a_n = \frac{n}{\pi} \int_{-\pi}^{\pi} F_1(t) \sin nt dt, \quad b_n = -\frac{n}{\pi} \int_{-\pi}^{\pi} F_1(t) \cos nt dt.$$

These formulae show that

$$(1.3) \quad a_n = o(n), \quad b_n = o(n).$$

Notation.

2. Write

$$A_n(x) = a_n \cos nx + b_n \sin nx,$$

$$\sigma^k(\omega) = \tfrac{1}{2}a_0 + \sum_{1 \leq n < \omega} \left(1 - \frac{n}{\omega}\right)^k A_n(x),$$

$$\psi(t) = f(x+t) + f(x-t), \quad \phi(t) = f(x+t) + f(x-t) - 2s,$$

$$\Phi_0(t) = \phi(t), \quad \Phi_r(t) = \int_0^t \Phi_{r-1}(u) du \quad (r = 1, 2, \dots),$$

$$\Phi(t) = \Phi_1(t), \quad \phi_r(t) = \frac{r! \Phi_r(t)}{t^r} \quad (r = 1, 2, \dots),$$

$$\phi_\lambda(t) = \frac{\lambda}{t} \int_0^t \left(1 - \frac{u}{t}\right)^{\lambda-1} \phi(u) du \quad (\lambda > 0),$$

$$\phi_r^*(t) = \frac{1}{t} \int_0^t |\phi_{r-1}(u)| du \quad (r = 1, 2, \dots),$$

$$C_p(t) = \sum_{\nu=0}^{\infty} (-)^{\nu} \frac{t^{p+2\nu}}{\Pi(p+2\nu)} \quad (t \geq 0),$$

$$\gamma_\lambda(t) = \int_0^1 (1-u)^{\lambda-1} \cos tu du \quad (\lambda > 0).$$

That the two definitions of $\phi_r(x)$ are consistent is a well known property of integrals of general order which may be established by integration by parts.

The function $C_p(t)$.

3. The function $C_p(t)$ has been discussed by a number of writers. A compact account of it is given by Hobson†. It is assumed, in order to avoid the introduction of complex functions, that $t \geq 0$.

The following properties, all of which are elementary, are required in the sequel :

(3.1) $|C_p(t)| \leq A \quad (0 \leq p \leq 2),$

(3.2) $C_p(t) \sim \frac{t^{p-2}}{\Pi(p-2)} \quad (p > 2),$

(3.3) $\frac{d}{dt} C_p(t) = C_{p-1}(t),$

(3.4) $C_1(t) = \sin t.$

The kernel $\gamma(t)$.

4. If k is the index of summability, the kernel $\gamma(t)$ is defined by the formula

$$\gamma(t) = \gamma_{k+1}(t) = \int_0^1 (1-u)^k \cos t u du.$$

It will be assumed generally that $k > 0$, the paper being concerned primarily only with results which may, under reasonable conditions, hold almost everywhere. Actually, as is shown in § 8, any restriction less stringent than $k \geq 1$ is futile. The case $0 < k < 1$ is, however, discussed first in order to show the effect of the restriction $k \geq 1$.

By expanding $\cos tu$ in powers of tu and integrating term by term, it is readily shown that, with $t \geq 0$,

$$\gamma(t) = \Pi(k) \frac{C_{k+1}(t)}{t^{k+1}}.$$

Obviously $\gamma(t)$ is an even function.

From (3.1) and (3.2) it follows that

(4.1) $|\gamma(t)| \leq \frac{A}{t^{k+1}} \quad (0 \leq k \leq 1),$

(4.2) $\gamma(t) \sim \frac{k}{t^2} \quad (k \geq 1),$

† *Theory of functions of a real variable*, 2 (1926), 564-567.

p 2

where, with the usual notation, A is constant relative to each function considered, but may vary from function to function.

If r is any positive integer not greater than $k+1$, (3.3) gives, with $t \geq 0$,

$$\frac{1}{\Pi(k)} \gamma^{(r)}(t) = \frac{C_{k-r+1}(t)}{t^{k+1}} - r(k+1) \frac{C_{k-r+2}(t)}{t^{k+2}} + \dots \\ + (-)^r (k+1) \dots (k+r) \frac{C_{k+1}(t)}{t^{k+r+1}},$$

so that, by (3.1) and (3.3), as $|t| \rightarrow \infty$,

$$(4.3) \quad \gamma^{(r)}(t) = O\left(\frac{1}{t^{r+2}}\right) \quad (r \leq k-1),$$

$$(4.4) \quad = O\left(\frac{1}{t^{k+1}}\right) \quad (r \geq k-1).$$

In the special case $r = k$,

$$(4.5) \quad \frac{1}{\Pi(k)} \gamma^{(k)}(t) = \frac{C_1(t)}{t^{k+1}} + O\left(\frac{1}{t^{k+2}}\right) \\ = \frac{\sin t}{t^{k+1}} + O\left(\frac{1}{t^{k+2}}\right).$$

If $k \geq 1$,

$$(4.6) \quad \gamma'(t) = O\left(\frac{1}{t^2}\right).$$

If $k > 1$ and κ is the smaller of $k-1$ and 1,

$$(4.7) \quad \gamma'(t) = O\left(\frac{1}{t^{2+\kappa}}\right).$$

Since
$$\gamma(t) = \int_0^1 (1-u)^k \cos tu \, du,$$

and
$$\gamma^{(r)}(t) = \int_0^1 (1-u)^k u^r \cos^{(r)} tu \, du,$$

it follows that

$$(4.8) \quad |\gamma(t)| \leq A, \quad |\gamma^{(r)}(t)| \leq A.$$

These, together with (4.3) and (4.4), show that

$$\int_{-\infty}^{\infty} |\gamma'(t)| \, dt < \infty,$$

so that $\gamma(t)$ is of bounded variation in $(-\infty, \infty)$.

The fundamental formula.

5. The main argument is based on the formula

$$(5.1) \quad \sigma^k(\omega) = \frac{1}{\pi} \int_0^\infty \psi\left(\frac{t}{\omega}\right) \gamma(t) dt = \frac{\omega}{\pi} \int_0^\infty \psi(t) \gamma(\omega t) dt,$$

which will be established on the hypothesis $k > 0$.

There are several possible proofs†. Here the formula will be obtained from a general theorem which follows the lines of well known theorems due to Hardy and Young.

An extension of some theorems due to Hardy and Young.

6. If $f(t)$ is periodic in 2π , integrable in $(-\pi, \pi)$ in the general Denjoy sense, and such that

$$(6.1) \quad \int_{-\pi}^{\pi} f(t) dt = 0;$$

if, further, $g(t)$ is of bounded variation in $(-\infty, \infty)$ and, as $|t| \rightarrow \infty$,

$$(6.2) \quad g(t) \rightarrow 0;$$

then

$$(6.3) \quad \sum_{n=1}^{\infty} (a_n \alpha_n + b_n \beta_n) = \int_{-\infty}^{\infty} f(t) g(t) dt \quad (R, k),$$

for every $k > 0$, where a_n, b_n are as defined and

$$(6.4) \quad \alpha_n = \int_{-\infty}^{\infty} g(t) \cos nt dt, \quad \beta_n = \int_{-\infty}^{\infty} g(t) \sin nt dt.$$

If $g'(t)$ also is of bounded variation, then summability in the above may be replaced by convergence.

The case of convergence is not required in the sequel, but it is inserted for the sake of interest.

Write

$$F(t) = \int_0^t f(u) du.$$

Then $F(t)$ is continuous. Also, in virtue of (6.1), it is periodic in 2π . Thus, by a known theorem, the Fourier series of $F(t)$ is uniformly

† See Hobson, *ibid.*, 2 (1926), 583, for a method which can be adapted to the present case.

summable (R, k) to $F(t)$ for each $k > 0$, i.e. as $t \rightarrow \infty$,

$$F_{\omega}(t) \rightarrow F(t),$$

where

$$F_{\omega}(t) = \sum_{1 \leq n < \omega} \left(1 - \frac{n}{\omega}\right)^k \left(-\frac{b_n}{n} \cos nt + \frac{a_n}{n} \sin nt\right).$$

Since $g(t)$ is of bounded variation in $(-\infty, \infty)$,

$$(6.5) \quad \int_{-\infty}^{\infty} F_{\omega}(t) dg(t) \rightarrow \int_{-\infty}^{\infty} F(t) dg(t).$$

Now, by Stieltjes' theorem on integration by parts,

$$\begin{aligned} \int_{-\lambda}^{\lambda} F(t) dg(t) &= \left[F(t) g(t) \right]_{-\lambda}^{\lambda} - \int_{-\lambda}^{\lambda} g(t) dF(t) \\ &= \left[F(t) g(t) \right]_{-\lambda}^{\lambda} - \int_{-\lambda}^{\lambda} g(t) f(t) dt. \end{aligned}$$

Make $\lambda \rightarrow \infty$. By reason of (6.2) the term between the limits tends to 0, so that

$$\int_{-\infty}^{\infty} F(t) dg(t) = - \int_{-\infty}^{\infty} f(t) g(t) dt,$$

and (6.5) takes the form

$$(6.6) \quad \sum_{1 \leq n < \omega} \left(1 - \frac{n}{\omega}\right)^k \int_{-\infty}^{\infty} \left(\frac{b_n}{n} \cos nt - \frac{a_n}{n} \sin nt\right) dg(t) \rightarrow \int_{-\infty}^{\infty} f(t) g(t) dt.$$

But

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{b_n}{n} \cos nt dg(t) &= \lim_{\lambda \rightarrow \infty} \int_{-\lambda}^{\lambda} \frac{b_n}{n} \cos nt dg(t) \\ &= \lim_{\lambda \rightarrow \infty} \left\{ \left[\frac{b_n}{n} \cos nt g(t) \right]_{-\lambda}^{\lambda} + b_n \int_{-\lambda}^{\lambda} g(t) \sin nt dt \right\} \\ &= b_n \beta_n. \end{aligned}$$

Similarly

$$\int_{-\infty}^{\infty} -\frac{a_n}{n} \sin nt dg(t) = a_n \alpha_n.$$

These show that (6.6) takes the form

$$\sum_{1 \leq n < \omega} \left(1 - \frac{n}{\omega}\right)^k (a_n \alpha_n + b_n \beta_n) \rightarrow \int_{-\infty}^{\infty} f(t) g(t) dt,$$

which is (6.3).

In order to obtain convergence, it is sufficient to prove further that

$$a_n a_n + b_n \beta_n = O\left(\frac{1}{n}\right).$$

Since $g'(t)$ is of bounded variation, it is easily shown by integration by parts that

$$a_n = O\left(\frac{1}{n^2}\right), \quad \beta_n = O\left(\frac{1}{n^2}\right).$$

Together with (1.3) these give what is required.

If $f(t)$ and $g(t)$ are both even, the theorem still holds with the whole range $(-\infty, \infty)$ replaced in (6.3) and (6.4) by the half range $(0, \infty)$.

Deduction of the fundamental formula.

7. Replace $f(t)$ by $\psi(t) - a_0$, and take

$$g(t) = \frac{\omega}{\pi} \gamma(\omega t) = \frac{1}{\pi} \int_0^\omega \left(1 - \frac{u}{\omega}\right) \cos t u du.$$

On the hypothesis $k > 0$, $g(t)$ is continuous and of bounded variation in $(-\infty, \infty)$ and tends to 0 as $|t| \rightarrow \infty$.

Fourier's integral formula gives

$$(7.1) \quad \int_0^\infty g(t) \cos nt dt = \frac{1}{2} \left(1 - \frac{n}{\omega}\right)^k \quad (n < \omega),$$

$$= 0 \quad (n \geq \omega).$$

Since, as is easily verified,

$$\frac{1}{\pi} \int_{-\pi}^\pi [\psi(t) - a_0] \cos nt dt = 2A_n(x) \quad (n \geq 1),$$

$$= 0 \quad (n = 0),$$

the general theorem gives

$$(7.2) \quad \sum_{1 \leq n < \omega} \left(1 - \frac{n}{\omega}\right)^k A_n(x) = \frac{\omega}{\pi} \int_0^\infty [\psi(t) - a_0] \gamma(\omega t) dt.$$

Now the case $n = 0$ of (7.1) gives

$$(7.3) \quad \int_0^\infty g(t) dt = \frac{1}{2},$$

so that (7.2) takes the form (5.1).

It should be noticed that there is no question of summability in (7.2) because the series on the left contains only a finite number of terms.

The necessary and sufficient condition for summability to s .

8. It follows at once from the above that

$$(8.1) \quad \sigma_{\omega}^k - s = \frac{\omega}{\pi} \int_0^{\infty} \phi(t) \gamma(\omega t) dt,$$

so that the necessary and sufficient condition for summability to s is, if $k > 0$,

$$(8.2) \quad \lim_{\omega \rightarrow \infty} \omega \int_0^{\infty} \phi(t) \gamma(\omega t) dt = 0.$$

There are two cases to consider. If $k \geq 1$, a "Riemann" reduction is possible, *i.e.* (8.2) can be replaced by

$$(8.3) \quad \lim_{\omega \rightarrow \infty} \omega \int_0^{\eta} \phi(t) \gamma(\omega t) dt = 0,$$

where η is an arbitrary positive number.

If $k < 1$ this reduction is, in general, not possible. For if it were, then there would be summability to s almost everywhere in any interval of absolute integrability of $f(x)$. This would give, if such an interval existed,

$$A_n(x) = o(n^k)$$

for a set of values of x of positive measure, and so, by a theorem due to Lebesgue,

$$a_n = o(n^k), \quad b_n = o(n^k).$$

Titchmarsh has, however, shown[†] that functions exist with only a single point of non-absolute integrability, for which these results are not true if $k < 1$. In the sequel it will be assumed that $k \geq 1$.

The Riemann reduction of the necessary and sufficient condition for summability.

9. By integration by parts, in view of (9.2) below,

$$(9.1) \quad \omega \int_{\eta}^{\infty} \phi(t) \gamma(\omega t) dt = -\omega \Phi(\eta) \gamma(\omega \eta) - \omega^2 \int_{\eta}^{\infty} \Phi(t) \gamma'(t) dt.$$

[†] Hobson, *ibid.*, 2 (1926), 573. Titchmarsh, *Proc. London Math. Soc. Records* (2), 21 (1923), (iii), (iv).

Since

$$(9.2) \quad \gamma(\omega t) \sim \frac{k}{\omega^2 t^2}$$

the first term on the right in (9.1) tends to 0 as $\omega \rightarrow \infty$.

Now, if α and β are finite and greater than η ,

$$\omega^2 \int_{\alpha}^{\beta} \gamma'(\omega t) dt = \left[\omega \gamma(\omega t) \right]_{\alpha}^{\beta} \rightarrow 0,$$

by (9.2). Since, as is readily shown from (4.6),

$$\omega^2 \int_{\alpha}^{\beta} |\gamma'(\omega t)| dt$$

is uniformly bounded, we have, by Lebesgue's convergence theorem,

$$\omega^2 \int_{\eta}^{\zeta} \Phi(t) \gamma'(\omega t) dt \rightarrow 0$$

for all finite values of ζ , and (9.1) may be written

$$\omega \int_{\eta}^{\infty} \phi(t) \gamma(\omega t) dt = o(1) - \omega^2 \int_{\zeta}^{\infty} \Phi(t) \gamma'(\omega t) dt.$$

Now in view of (7.3) the Riemann reduction is independent of the particular value of s . We may therefore suppose s to be chosen so that $\Phi(t)$ is periodic and is consequently bounded. If M is the upper bound of $|\Phi(t)|$, then, by (4.6),

$$\left| \int_{\zeta}^{\infty} \Phi(t) \gamma'(\omega t) dt \right| \leq \int_{\zeta}^{\infty} \frac{MA}{\omega^2 t^2} = \frac{MA}{\omega^2 \zeta},$$

where A is a constant.

This gives

$$\omega^2 \int_{\eta}^{\infty} \Phi(t) \gamma'(\omega t) dt = o(1) + O\left(\frac{1}{\zeta}\right).$$

Since ζ is arbitrary,

$$(9.3) \quad \omega^2 \int_{\eta}^{\infty} \Phi(t) \gamma'(\omega t) dt \rightarrow 0,$$

and this shows that (8.2) is equivalent to

$$(9.4) \quad \lim_{\omega \rightarrow \infty} \omega \int_0^{\eta} \phi(t) \gamma(\omega t) dt = 0$$

for every positive η .

If $k > 1$ the proof of (9.3) can be shortened, and the appeal to Lebesgue's convergence theorem is not required. For

$$\begin{aligned} \left| \omega^2 \int_{\eta}^{\infty} \Phi(t) \gamma'(\omega t) dt \right| &\leq M \omega^2 \int_{\eta}^{\infty} |\gamma'(\omega t)| dt \\ &\leq M \omega^2 \int_{\eta}^{\infty} \frac{A}{(\omega t)^{2+\kappa}} dt \\ &= \frac{M}{\omega^{\kappa}} \int_{\eta}^{\infty} \frac{A}{t^{2+\kappa}} dt \rightarrow 0, \end{aligned}$$

by (4.7),

since κ is positive.

A convenient practical form of (9.4) is

$$\lim_{\eta \rightarrow 0} \overline{\lim}_{\omega \rightarrow \infty} \left| \omega \int_0^{\eta} \phi(t) \gamma(\omega t) dt \right| = 0,$$

a condition which is investigated in detail elsewhere†. For the sake of completeness a summary of the results will now be given (§ 10-13).

Transformation of the necessary and sufficient condition for summability to s.

10. Take $r < k+1$ and integrate repeatedly by parts. There results

$$\begin{aligned} \omega \int_0^{\eta} \phi(t) \gamma(\omega t) dt &= \omega \Phi_1(\eta) \gamma(\omega \eta) - \omega^2 \Phi_2(\eta) \gamma'(\omega \eta) + \dots + (-)^{r-1} \omega^r \Phi_r(\eta) \gamma^{(r-1)}(\omega \eta) \\ &\quad + (-)^r \omega^{r+1} \int_0^{\eta} \Phi_r(t) \gamma^{(r)}(\omega t) dt. \end{aligned}$$

By (4.3) the terms on the right, other than the integral, tend to 0, so that (8.2) is equivalent to

$$(10.1) \quad \lim_{\eta \rightarrow 0} \overline{\lim}_{\omega \rightarrow \infty} \left| \omega^{r+1} \int_0^{\eta} \Phi_r(t) \gamma^{(r)}(\omega t) dt \right| = 0 \quad (r < k+1).$$

The analogue of Lebesgue's criterion.

11. A sufficient condition for summability (R, k) to s is

$$(11.1) \quad \phi_r^*(t) \rightarrow 0 \quad (r < k+1);$$

the particular case $k=1$, $r=1$ of this has already been obtained by

† *Proc. Camb. Phil. Soc.*, 23 (1926), 373-382 (373-382).

Lebesgue†. The reasoning required to prove (11.1) is on the same lines as that employed by Lebesgue.

A special case of (11.1) is

$$(11.2) \quad \phi_{r-1}(t) \rightarrow 0 \quad (r < k+1).$$

A sufficient condition is

$$\phi_1(t) \rightarrow 0,$$

as obtained by Privaloff‡.

If k is integral, and so greater than or equal to 1,

$$\phi_k^*(t) \rightarrow 0$$

is a sufficient condition.

A particular case of this is

$$\phi_{k-1}(t) \rightarrow 0,$$

so that, for summability to s by *some* Rieszian mean, it is sufficient that

$$\phi_r(t) \rightarrow 0$$

for some positive integral value of r , *i.e.*

$$\phi(t) \rightarrow 0 \quad (C).$$

Reduction of summability to convergence.

12. Let k be integral and suppose that

$$(12.1) \quad \phi_k(t) \rightarrow 0.$$

By means of (4.5) it can be shown that (10.1), with $r = k$, is equivalent to

$$\lim_{\eta \rightarrow 0} \overline{\lim}_{\omega \rightarrow \infty} \left| \int_{1/\omega}^{\eta} \phi_k(t) \frac{\sin \omega t}{t} dt \right| = 0,$$

and this is known to be a necessary and sufficient condition for the convergence to 0 at $t = 0$ of the Fourier series of $\phi_k(t)$.

Thus, if (12.1) holds, summability can be reduced to convergence.

Generalization of the analogues of Lebesgue's criterion.

13. A convenient test for convergence which can now be applied is that due to W. H. Young. It shows that a sufficient condition for the

† *Math. Annalen*, 61 (1905), 251-280.

‡ See § 14.

convergence required is

$$\int_0^t |d\{t\phi_k(t)\}| = O(t),$$

from which there results the following generalization of the analogues of Lebesgue's criterion. A sufficient condition for summability (R, k) , where k is a positive integer, to s is

$$\phi_k(t) = o(1), \quad \phi_k^*(t) = O(1).$$

Privaloff's results.

14. An investigation on somewhat different lines has been given by Privaloff†. In this the starting point is Fejér's integral for $\sigma^1(n)$,

$$\sigma^1(n) = \frac{1}{2n\pi} \int_0^\pi \psi(t) \left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t} \right)^2 dt,$$

so that

$$\sigma^1(n) - s = \frac{1}{2n\pi} \int_0^\pi \phi(t) \left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t} \right)^2 dt.$$

By integration by parts this is expressed in terms of $\Phi(t)$, and Privaloff virtually shows that

$$\sigma^1(n) - s \rightarrow 0 \quad (C, \epsilon),$$

for every positive ϵ whenever

$$(14.1) \quad \phi_1(t) \rightarrow 0.$$

In view of an equivalence theorem due to Hausdorff‡, Privaloff's analysis shows that

$$\sigma^k(\omega) \rightarrow s \quad (k > 1)$$

whenever (14.1) holds.

The mean actually employed by Privaloff is one of Hölder's type, and his result could be expressed in the form

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = s \quad (H, 1+\epsilon),$$

whenever (14.1) holds.

† *Rend. di Palermo*, 41 (1916), 202-205 (205).

‡ *Math. Zeitschrift*, 9 (1921), 74-109 (90).

The case $k = 1$.

15. There is a fundamental distinction between the cases $k = 1$ and $k > 1$, largely due to the fact that

$$\epsilon^{k-1} \int_{\epsilon}^{\eta} \frac{dt}{t^k} = O(1),$$

as $\epsilon \rightarrow 0$, if $k > 1$, while, if $k = 1$,

$$\epsilon^{k-1} \int_{\epsilon}^{\eta} \frac{dt}{t^k} \sim \log \left(\frac{1}{\epsilon} \right).$$

The presence of the logarithm introduces complications which render results easily established in the case $k > 1$ invalid in the case $k = 1$. Thus, while (14.1) is a sufficient condition for summability to s when $k > 1$, Hahn† has shown that it is definitely not sufficient in the case $k = 1$.

It may be observed that, in the case $k = 1$, Fejér's form of the integral for $\sigma^1(n)$ has certain advantages over the form considered in § 5.

In the case of a genuine Denjoy-Fourier series, with

$$\int_{-\pi}^{\pi} |f(x)| dx = \infty,$$

the case $k = 1$ of the criteria, given in § 13, does not in general apply, since $\phi(t)$ is not, in general, absolutely integrable.

Other criteria can be obtained by replacing Young's criterion for convergence by any of the other standard criteria. Thus Dini's criterion for convergence shows that a sufficient condition for summability ($k = 1$) to s is

$$\int_0^{\eta} \left| \frac{\phi_1(t)}{t} \right| dt < \infty.$$

A particular case of this is

$$\phi_1(t) = O(t^{\epsilon}) \quad (\epsilon > 0).$$

Thus, suppose that $f(x)$ is defined by periodicity outside $(-\pi, \pi)$ and by the formulae

$$\begin{aligned} f(x) &= x \cos \frac{1}{x^2} + \frac{1}{x} \sin \frac{1}{x^2} \quad (0 < |x| < \pi), \\ &= 0 \quad (x = 0). \end{aligned}$$

Then $f(x)$ is periodic in 2π and Cauchy integrable—but not absolutely

† *Jahresbericht d. Deutschen Math. Vereinigung*, 25 (1916), 359.

integrable—in $(-\pi, \pi)$. For $x = 0$, with $s = 0$,

$$\phi(t) = 2 \left(t \cos \frac{1}{t^2} + \frac{1}{t} \sin \frac{1}{t^2} \right) \quad (t \neq 0),$$

$$\Phi(t) = t^2 \cos \frac{1}{t^2} \quad (t \neq 0),$$

$$\phi_1(t) = t \cos \frac{1}{t^2} = O(t),$$

so that there is summability ($k = 1$) to 0 at $x = 0$.

The case of a special Denjoy integral.

16. When $f(x)$ is integrable in the special Denjoy sense it is known that, if

$$F(x) = \int_0^x f(t) dt,$$

then

$$F'(x) = f(x)$$

for almost all values of x .

For these values of x ,

$$\frac{F(x+t) - F(x-t)}{t} \rightarrow 2f(x),$$

so that, if

$$s = f(x),$$

$$\phi_1(t) \rightarrow 0,$$

since

$$\Phi(t) = F(x+t) - F(x-t) - 2st.$$

Thus, in view of Privaloff's result, the Fourier series of a function which is integrable in the special Denjoy sense is summable almost everywhere to its generating function by means of order greater than 1.

AN EXTENSION OF HADAMARD'S MULTIPLICATION THEOREM

By U. S. HASLAM-JONES.

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1. The problem here considered is that of establishing a theorem in the theory of functions of two complex variables analogous to the well known "multiplication of singularities" theorem proved by Hadamard* for functions of one variable. In order to simplify the argument, the class of functions considered is restricted to functions whose singular loci are of a regular type. For brevity, it will be said that a function $p(u, v)$ is of type Z , when, for any constant u_1 , $p(u_1, v)$ is *either* identically zero *or* has isolated zeros in the v -plane, and also, for any constant v_1 , $p(u, v_1)$ is *either* identically zero *or* has isolated zeros in the u -plane. We shall prove the following

THEOREM: *Let $f(u, v)$ be an analytic function of u and v having a branch regular in the neighbourhood of $u = 0, v = 0$, and having as its Taylor series for this branch*

$$(1) \quad f(u, v) = \sum \sum a_{m,n} u^m v^n;$$

that $g(u', v')$ is a similar function with Taylor series

$$(2) \quad g(u', v') = \sum \sum b_{m,n} u'^m v'^n;$$

and that $f(u, v)$ and $g(u', v')$ have singularities consisting of points satisfying the relations $p(u, v) = 0$ and $q(u', v') = 0$ respectively, where $p(u, v)$ and $q(u', v')$ are analytic functions of type Z . Then the function $F(x, y)$ whose Taylor series is

$$(3) \quad F(x, y) = \sum \sum a_{m,n} b_{m,n} x^m y^n$$

* Hadamard, *Acta Math.*, 22 (1898), 55. Faber, *Jahresbericht d. Deutschen Math.-Vereinigung*, 16 (1907), 285.

has a branch which is regular near $x = 0$, $y = 0$, and has elsewhere no singularities other than points given by $x = uu'$, $y = vv'$, where $p(u, v) = q(u', v') = 0$ and

$$uv' \frac{\partial p}{\partial u} \frac{\partial q}{\partial v'} = u'v \frac{\partial p}{\partial u'} \frac{\partial q}{\partial v}.$$

The method employed is an adaptation of the method used by Hadamard to prove the one-variable theorem, namely, to express the function $F(x, y)$ as a double contour integral

$$I(x, y) = -\frac{1}{4\pi^2} \int_C \int_{C'} f(u, v) g\left(\frac{x}{u}, \frac{y}{v}\right) \frac{dv}{v} \frac{du}{u},$$

and to make a double application of the principle which underlies Hadamard's proof, that a contour passing between two variable singularities may be deformed so as to avoid them unless they tend to coincidence.

If, in (2), we write $u' = x/u$, $v' = y/v$, where (x, y) lies in a region R_0 defined by $|x| < r$, $|y| < s$, then, by a suitable choice of r and s , we can find a Laurent ring in either plane, inside which the series (1) and (2) are uniformly convergent with respect to u and v for every value of (x, y) in R_0 . Let C and C' be contours lying in these rings respectively; then, by virtue of the absolute and uniform convergence of the series (1) and (2) on these contours, we may multiply the series formally and integrate term by term, so obtaining

$$I(x, y) = \Sigma \Sigma a_{m,n} b_{m,n} x^m y^n = F(x, y).$$

We now proceed to consider the analytic continuation of the function defined by the integral.

2. From the conditions imposed on $p(u, v)$ it follows that, if u_1 is constant, $p(u_1, v) = 0$ only for isolated values of v , unless this relation is an identity, that is, unless the point (u_1, v) is a singularity of $f(u, v)$ for every value of v . We denote these isolated values of v , depending on u_1 , by (Γu_1) . Let $\theta_1, \theta_2, \dots$ be the values of u for which $p(u, v)$ vanishes identically; in particular, the θ 's all satisfy $p(u, 0) = 0$. Similarly, from $q(x/u, y/v) = 0$, we derive certain isolated points $(\Gamma' u)$ in the v -plane, and certain points $\phi_1, \phi_2, \dots$ in the u -plane at which this function vanishes identically; in particular, $\phi_1, \phi_2, \dots$ all satisfy $q(x/u, 0) = 0$.

When (x, y) lies in R_0 and u describes C , the points (Γu) describe contours Γ lying wholly outside C' , and $(\Gamma' u)$ describe contours Γ' lying

wholly inside C' . But when (x, y) does not lie in R_0 it is no longer always possible to draw a contour C' separating the Γ and Γ' figures; we therefore proceed to replace the contour C' by a contour C_u whose form depends upon the variable u .

Let C_u be the contour consisting of a large circle $|v| = R$, where $R > |(\Gamma'u)|$, indented to exclude the points $(\Gamma'u)$. Since the function under the integral sign is regular in the region between C' and C_u , it follows that, when (x, y) lies in R_0 ,

$$\int_C f(u, v) g\left(\frac{x}{u}, \frac{y}{v}\right) \frac{dv}{v} = \int_{C_u} f(u, v) g\left(\frac{x}{u}, \frac{y}{v}\right) \frac{dv}{v},$$

and hence

$$(4) \quad F(x, y) = -\frac{1}{4\pi^2} \int_C \int_{C_u} f(u, v) g\left(\frac{x}{u}, \frac{y}{v}\right) \frac{dv}{v} \frac{du}{u}.$$

Assuming that C is any contour in the u -plane, we shall prove two lemmas concerning the integral (4).

3. LEMMA I. *The integral (4) is an analytic function of x and y regular in a region R_c defined by the condition that none of the points of the u -plane determined by either $p(u, v) = q(x/u, y/v) = 0$ or $q(x/u, 0) = 0$ shall lie within some assigned positive distance δ from C .*

By Weierstrass's theorem on functions defined by integrals it is sufficient to show (i) that for any point u on C the function

$$(5) \quad \int_{C_u} f(u, v) g\left(\frac{x}{u}, \frac{y}{v}\right) \frac{dv}{v}$$

is an analytic function of x and y regular in R_c , (ii) that (5) is a continuous function of x, y and u when (x, y) lies in R_c and u on C . Now, when u is fixed, (5) is an analytic function of x and y provided that the points $(\Gamma'u)$ do not lie within some assigned positive distance δ_1 from C_u ; also, if we deform C_u into a contour $\bar{C}_u$ by a continuous deformation without passing over any singular point of the integrand, we shall have

$$\int_{C_u} f(u, v) g\left(\frac{x}{u}, \frac{y}{v}\right) \frac{dv}{v} = \int_{\bar{C}_u} f(u, v) g\left(\frac{x}{u}, \frac{y}{v}\right) \frac{dv}{v}.$$

Provided that (x, y) is a point such that $(\Gamma'u)$ does not coincide with any (Γu) or with ∞ , the deformation may be chosen so that the latter integral remains regular at (x, y) . Hence (5) is an analytic function of x and y ,

regular so long as no point given by

$$p(u, v) = q\left(\frac{x}{u}, \frac{y}{v}\right) = 0 \quad \text{or by} \quad q\left(\frac{x}{u}, 0\right) = 0$$

lies within the distance δ from u ; and so, *a fortiori*, for any u on C , the function (5) is an analytic function of x and y regular in R_c .

Let u be a point on C , and consider a small circle s of radius ρ with centre at u ; the representations of s on the Γ and Γ' figures are σ and σ' respectively. We may choose ρ so small that σ and σ' have no point in common; if this were not possible, u would necessarily satisfy

$$p(u, v) = q\left(\frac{x}{u}, \frac{y}{v}\right) = 0.$$

If we denote by C_σ a contour of the same type as C_u which excludes the whole region σ and includes the whole of σ' , we have

$$\int_{C_u} f(u, v) g\left(\frac{x}{u}, \frac{y}{v}\right) \frac{dv}{v} = \int_{C_\sigma} f(u, v) g\left(\frac{x}{u}, \frac{y}{v}\right) \frac{dv}{v}.$$

The latter integral is an analytic function of x , y , and u , regular when u lies in s and (x, y) in R_c ; hence we conclude that the function (5) is continuous in all three variables when u lies on C and (x, y) in R_c . By Weierstrass's theorem the lemma follows.

4. LEMMA II. If C_1 is a second contour such that the region between C and C_1 contains no points of the set A determined by any of the relations

$$p(u, v) = q\left(\frac{x}{u}, \frac{y}{v}\right) = 0, \quad u = 0,$$

$$q\left(\frac{x}{u}, 0\right) = 0 \quad \text{or} \quad p(u, 0) = 0,$$

then

$$(7) \quad \int_C \int_{C_u} f(u, v) g\left(\frac{x}{u}, \frac{y}{v}\right) \frac{dv}{v} \frac{du}{u} = \int_{C_1} \int_{C_u} f(u, v) g\left(\frac{x}{u}, \frac{y}{v}\right) \frac{dv}{v} \frac{du}{u}.$$

If we denote the region between C and C_1 by D , and its representations on the Γ and Γ' figures by Δ and Δ' , we have to consider two possible cases.

(i) If Δ and Δ' have no point in common, we may replace C_u by C_Δ for every point u lying inside or on the boundary of D , where C_Δ is a

contour of the same type as C_u but excluding the whole region Δ , and including the whole region Δ' . Now, if we consider x and y to be fixed,

$$\frac{1}{u} \int_{C_\Delta} f(u, v) g\left(\frac{x}{u}, \frac{y}{v}\right) \frac{dv}{v}$$

is an analytic function of u regular in D . Hence, in this case, the relation (7) is true.

(ii) If Δ and Δ' have points in common we may, nevertheless, divide D into a *finite* number of smaller regions d_s in such a way that their representations δ_s and δ'_s have no points in common. For, if we take any point u lying inside or on the boundary of D , and we consider a small circle s of radius ρ and centre u , then, as in Lemma I, by taking ρ sufficiently small we can ensure that σ and σ' have no points in common. By the Heine-Borel theorem on sets of points, we may select a *finite* number of these circles to cover the whole region D ; hence we can divide D into a finite number of smaller areas d_s whose representations δ_s and δ'_s have no point in common. Applying the argument of (i) above to each area d_s in succession we obtain the general result stated in the lemma.

5. We are now in a position to apply Hadamard's principle to the contour C . If x and y are interior points of the region R_0 , the function $F(x, y)$ is regular, since its Taylor series is here uniformly convergent. Moreover, we have seen in Lemma I that the function is, in fact, regular in the wider region R_c , and that its analytic continuation in this region is afforded by the integral (4). As (x, y) approaches the boundary of R_c certain of the singular points in the u -plane approach C , and the representation of $F(x, y)$ by this integral ceases to be valid. But, by Lemma II, we may replace the integral (4) by the integral (7) provided only that C_1 may be obtained from C by a continuous deformation without passing over any points of the set A . Now the integral over C_1 is an analytic function of x and y regular in a region R_{c_1} , which includes a neighbourhood of the point $x=0, y=0$; hence this integral is the analytic continuation of $F(x, y)$ in R_{c_1} . The process of continuation, by deforming the contour, can fail only if points of the set A lying on opposite sides of C tend to coincide: that is to say, if the contour is "nipped" between two singular points in the u -plane.

We conclude, therefore, that $F(x, y)$ has a principal branch which is regular in any (x, y) region in which no two points of the set A coincide.

6. The condition of coincidence among points of the set A will yield various (x, y) loci as possible singularities; certain of these we may eliminate by the aid of the following lemma and the fundamental theorem of Hurwitz and Hartogs* that an analytic function of two variables can have no isolated singularities.

LEMMA III. *Let $f(u, x, y)$ be an analytic function of u, x , and y which has continuous singularities $u = \alpha_r(x, y)$, and consider a contour C in the u -plane. If, for $\phi(x, y) = 0$, C is "nipped" between α_1 and α_2 , and the other singularities are such that $|\alpha_r - \alpha_s| > \delta > 0$, then*

$$F(x, y) = \int_C f(u, x, y) du$$

is regular for $\phi(x, y) = 0$ provided that

$$f(u, x, y) = f_1(u, x, y) + f_2(u, x, y),$$

where $f_1(u, x, y)$ is regular for $u = \alpha_1(x, y)$, and $f_2(u, x, y)$ is regular for $u = \alpha_2(x, y)$.

We may divide C into two portions γ_1 and γ_2 in such a way that

$$\int_{\gamma_1} f(u, x, y) du$$

is regular, and we shall have further

$$\int_{\gamma_2} f(u, x, y) du = \int_{\gamma_2} f_1(u, x, y) du + \int_{\gamma_2} f_2(u, x, y) du.$$

The contour is not "nipped" between two singularities for either of the integrals on the right, and so we may introduce two separate deformations of γ_2 which will show that the functions defined by these integrals remain regular even when α_1 and α_2 coincide. The lemma follows immediately.

The possibility of singularities for $x = 0$ or $y = 0$ is considered in the next section; in this section we shall assume that $|x| > \delta$, $|y| > \delta$. We consider the coincidence of a point α_1 given by $p(\alpha_1, 0) = 0$ with a point $\alpha_2(x, y)$ given by

$$p(\alpha_2, v) = q\left(\frac{x}{\alpha_2}, \frac{y}{v}\right) = 0,$$

* Hartogs, *Münchener Sitzungsberichte*, 36 (1906), 223.

and we show first that no singularity arises from such a coincidence unless $v \rightarrow 0$ when $a_2 \rightarrow a_1$. For suppose, on the contrary, that whenever $|v| < \epsilon$,

$$(8) \quad \left| q\left(\frac{x}{a_1}, \frac{y}{v}\right) \right| > \eta,$$

then we may write

$$\begin{aligned} \int_{C_u} f(u, v) g\left(\frac{x}{u}, \frac{y}{v}\right) \frac{dv}{v} \\ = \int_{C_{u_0}} f(u, v) g\left(\frac{x}{u}, \frac{y}{v}\right) \frac{dv}{v} + \frac{1}{2\pi i} f(u, 0) g\left(\frac{x}{u}, \infty\right), \end{aligned}$$

where C_{u_0} is a contour in the v -plane of the same type as C_u , indented to exclude the points (Γu) and also the origin. By virtue of the condition (8) the integral on the right is a function of u regular in the neighbourhood of $u = a_1$, though it is singular at $u = a_2$; on the other hand, $f(u, 0) g(x/u, \infty)$ is regular in the neighbourhood of $u = a_2$, though singular for $u = a_1$. It follows, from the lemma, that the integral over C is, in this case, a regular function of x and y in the neighbourhood of the point considered.

We conclude that the only possible singularities arising from coincidences of this type are given by

$$(9) \quad p(u, v) = q\left(\frac{x}{u}, \frac{y}{v}\right) = 0,$$

where $|v| \rightarrow 0$. We shall show that points (x, y) given in this way are necessarily isolated.

It follows, from the conditions imposed on $p(u, v)$, that $p(u, 0) = 0$ only for *isolated* points $a_1, a_2, \dots$, and likewise $q(x/u, \infty) = 0$ only for $u = x/\beta_s$; hence we see that singularities may arise for $x = a_r \beta_s$. But now, if we write $x = a_r \beta_s$ in (9), eliminate v , and let $u \rightarrow a_r$, we shall obtain certain isolated values of y ; that is to say, the only singularities which can arise are isolated, and, by Hartogs' theorem, we may ignore such points since singularities can occur only as points of a continuous locus. Similarly, we may show that points in the u -plane given by $q(x/u, 0) = 0$ can coincide with points given by (9), but that singularities of $F(x, y)$ will arise thus only for isolated values of x and y ; this may be proved by considering the integral

$$\int_{\bar{C}} \int_{\bar{C}_u} f\left(\frac{x}{u'}, \frac{y}{v'}\right) g(u', v') \frac{dv'}{v'} \frac{du'}{u'}.$$

Finally, by an interchange of u and v we may eliminate apparent singularities arising from the coincidence of points given by (9) with the points $u = 0$ or $u = \infty$.

7. In the preceding section we have assumed that x and y are not zero; we have still to consider the possibility of singularities in the planes $x = 0$ and $y = 0$. Now $F(0, y)$ is the function derived from $f(0, y)$ and $g(0, y)$ in the manner considered in the one-variable Hadamard theorem, and therefore can have none but isolated singularities unless the singular points of at least one of $f(0, y)$ and $g(0, y)$ have a limit-point. By virtue of our hypotheses regarding $p(u, v)$ and $q(u, v)$, this is not possible unless either $p(0, v)$ or $q(0, v)$ is identically zero; this alternative implies that $f(u, v)$ or $g(u, v)$ is not regular in the neighbourhood of $u = 0, v = 0$, contrary to our hypothesis. Hence we conclude that $F(0, y)$ can have singularities only for isolated values of y ; such singular points we can ignore, as we have shown above. A similar result is true of the singularities of $F(x, 0)$.

The only remaining possible singular loci are to be obtained from the coincidence of two different solutions of (9); that is to say, all the singularities of a branch of $F(x, y)$ are to be found among the points

$$x = uu', \quad y = vv',$$

where

$$p(u, v) = q(u', v') = 0$$

and

$$uv' \frac{\partial p}{\partial u} \frac{\partial q}{\partial v'} = u'v \frac{\partial p}{\partial v} \frac{\partial q}{\partial u'}.$$

8. The construction of series which illustrate the theorem is a matter of considerable difficulty; since, except in the very simplest cases, the task of verifying that the loci obtained by the rule are actually all the singular loci of the derived function is not easy.

(i) Hurwitz* mentions

$$f(x, y) = \frac{1}{1-x-y} = \sum \sum \frac{(m+n)!}{m! n!} x^m y^n,$$

$$g(x, y) = \frac{1}{1-xy} = \sum x^n y^n,$$

* Hurwitz, *Comptes rendus*, 128 (1899), 353.

as an example of the difficulty one might expect in establishing a general theorem. The rule which we have obtained gives, as possible singularities, $x = uu'$, $y = vv'$, where $u+v-1 = u'v'-1 = 0$ and $uv'u' = u'vv'$; whence we obtain $4xy = 1$. The actual derived function is

$$F(x, y) = \sum \frac{2m!}{(m!)^2} x^m y^n = \frac{1}{\sqrt{(1-4xy)}}.$$

(ii) If we write

$$f(x, y) = g(x, y) = \frac{1}{1-x-y},$$

the derived function is

$$F(x, y) = \sum \sum \frac{(m+n!)^2}{(m!n!)^2} x^m y^n = \frac{1}{\sqrt{\{(1-x-y)^2 - 4xy\}}}.$$

The rule gives $1 \pm \sqrt{x} \pm \sqrt{y} = 0$.

If we compound any two functions $\phi(x+y)$ and $\psi(x+y)$, then we can show that the derived function has all its singular loci of the form $\pm\sqrt{x} \pm \sqrt{y} = k$.

(iii) The function

$$\sum \sum \frac{1}{(m^2+n^2)^s} x^m y^n,$$

where s is not integral, has the singular locus

$$(x-1)(y-1)[(\log x)^2 + (\log y)^2] = 0.$$

If we combine two such functions we obtain a derived function of the same type. The rule gives as possible singularities: (i) $x = 1$, any y ; (ii) $y = 1$, any x ; (iii) $(\log x)^2 + (\log y)^2 = 0$. These are, in fact, the actual singularities.

9. The condition of behaviour as analytic functions imposed on $p(u, v)$ and $q(u', v')$ is, of course, not a necessary condition for the truth of the theorem. That some such condition is required can be seen by considering functions which have a singular line in the plane $u = 0$; it is well known* that the one-variable Hadamard theorem may break down in such circumstances, and, *a fortiori*, the two-variable theorem must fail also. Adapting Faber's example, consider $\alpha_{m,n}$ equal to 2 when

* Faber, *loc. cit.*, 288.

$m = 2^\mu$, $n = 2^\nu$, and equal to 1 otherwise; then both

$$\sum a_{m,n} x^m y^n \quad \text{and} \quad \sum \frac{1}{a_{m,n}} x^m y^n$$

have the region $|x| < 1$, $|y| < 1$ as their natural domain of existence, and the functions cannot be continued outside this domain. Now consider

$$f(x, y) = \sum \sum a_{m,n} x^m y^n,$$

$$g(x, y) = \sum \sum \left(\frac{1}{2^m} + \frac{1}{3^m} \right) \frac{1}{a_{m,n}} x^m y^n.$$

The derived function

$$F(x, y) = \sum \sum \left(\frac{1}{2^m} + \frac{1}{3^m} \right) x^m y^n$$

is singular for $x = 2$, for $x = 3$, and for $y = 1$; and the singularity $x = 3$ cannot be obtained by the rule unless we regard *all* points outside the domain of existence of $f(x, y)$ and $g(x, y)$ as singular points of these functions.

The theorem has been stated and proved under a further limitation similar to that imposed on the one-variable theorem, namely that we consider only the "principal branch" of the derived function. In the one-variable case, Faber* has shown that the theorem is true for all branches if we include the origin as a possible singularity of all branches other than the principal branch, and it would seem probable that a similar result is true also in this case. A trivial example is

$$f(x, y) = g(x, y) = \log(1 - xy),$$

whose derived function is

$$F(x, y) = \sum \frac{x^n y^n}{n^2}.$$

This function has the origin of either plane as a singularity for an infinity of branches.

I am much indebted to Mr. E. C. Titchmarsh for his criticism and advice: in particular, the use of Hartogs' theorem in § 6 is due to his suggestion.

* *Loc. cit.*, 289.

ON THE SOLVING NUCLEI OF CERTAIN INTEGRAL EQUATIONS WHOSE NUCLEI ARE HOMOGENEOUS AND OF DEGREE -1 , AND THE SOLUTION OF A CLASS OF LINEAR FUNCTIONAL EQUATIONS

By A. C. DIXON.

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A FORMER paper [*Proc. Lond. Math. Soc.* (2), 22 (1923), 201-222] treated of integral equations in which the nucleus was homogeneous and of degree -1 , the limits being 0 and ∞ . §§1-5A of the present paper deal with the equation in which the nucleus is $\theta/(z+t)$ and the upper limit is 1. The solving nucleus is derived from a special hypergeometric function. The method is extended in §§ 7-14 to other equations, of which (13.4) is the type. It involves the construction of functions which satisfy two different linear functional equations in the two halves of the plane. Similar functions for which the two equations are not different are discussed in §§ 6, 6A. In § 15 an equation is discussed whose nucleus is a gamma function: in one case the solving nucleus depends on Riemann's zeta function.

1. The solution of the integral equation

$$\phi(z) = g(z) + \theta \int_0^1 \frac{\phi(t)}{z+t} dt, \quad (1.1)$$

in which t is real, $\phi(t)$ is the unknown function, and $g(z)$ a given function, is presumed to be

$$\phi(z) = g(z) + \theta \int_0^1 g(t) \chi(z, t) dt, \quad (1.2)$$

where

$$\chi(z, a) \text{ or } \chi(z) = \frac{1}{z+a} + \theta \int_0^1 \frac{\chi(t, a)}{z+t} dt. \quad (1.3)$$

(1.3) is derived from (1.1) by giving the particular value $1/(z+a)$ to the known function $g(z)$. (1.3) is first an integral equation to be satisfied by χ for real values of z between 0 and 1; but, secondly, when χ is known for such values, it defines $\chi(z)$ as a function of z for complex values. This function must have the following properties.

If a cut is made in the plane from 0 to -1 along the real axis, then $\chi(z)$ is uniform within the resulting region, since $z+t$ never vanishes on the path of integration; $\chi(z)$ has a pole at $-a$ with residue 1, and has no other singularity away from the cut. Towards infinity $\chi(z)$ is a regular function of $1/z$, vanishing with $1/z$.

If a circle of small radius ϵ is described about a point b on the real axis between 0 and 1, and if the path of integration is deformed so as to include the upper semicircumference of this circle instead of its diameter, then the value of the right side of (1.3) is not affected so long as $-z$ is outside the semicircle contained between the new and old paths. Thus the function represented is generally unchanged, but by means of the new form it is continued across the cut into the lower half of the circle of radius ϵ about $be^{i\pi}$. Similarly, if the lower semicircumference of the circle about b were taken into the path instead of the diameter, the function would be continued into the upper half of the circle of radius ϵ about $be^{-i\pi}$.

Thus, so long as z is inside the circle about b ,

$$\chi(ze^{i\pi}) = \frac{1}{a-z} + \theta \int_0^1 \frac{\chi(t)}{t-z} dt, \quad (1.31)$$

the path including the upper semicircumference, and the expression for $\chi(ze^{-i\pi})$ is the same, but the path includes the lower semicircumference; the second path is equivalent to the first together with a circuit about z positively, and hence

$$\chi(ze^{-i\pi}) - \chi(ze^{i\pi}) = 2i\pi\theta\chi(z). \quad (1.4)$$

This equation may now be used to continue the function χ to all amplitudes so long as $|z| < 1$.

Hence if $\chi(z)$ is expanded in powers of z near the origin its form should be $\sum Az^\lambda$, where

$$e^{-\lambda i\pi} - e^{\lambda i\pi} = 2\theta i\pi$$

or

$$\sin \lambda\pi = -\theta\pi. \quad (1.41)$$

A circuit about the origin described positively may be treated as a circuit about -1 described negatively, since ∞ is an ordinary value of z for $\chi(z)$, and thus, if $1-z$ is in the plane as cut,

$$\chi(-1+z) - \chi(-1+ze^{-2i\pi}) = 2i\pi\theta\chi(1-z), \quad (1.5)$$

so that

$$\chi(-1+z) - \theta\chi(1-z) \log z$$

is uniform when z is near the origin. Except near $-a$, 0 , and -1 , $\chi(z)$ is regular in the cut plane.

If κ is that value of λ with numerically least real part which satisfies (1.41), so that $R(\kappa)$ is between $\pm\frac{1}{2}$, and if $\kappa' = 1-\kappa$, then a possible form of expansion of $\chi(z)$ near the origin seems to be

$$z^\kappa(A+Bz^2+Cz^4+\dots) + z^{\kappa'}(A'+B'z^2+C'z^4+\dots), \quad (1.6)$$

and if $|a| > 1$ this expansion should hold within the circle $|z|=1$. Substitute (1.6) for χ in (1.3).

If $m < Ra < m+1$, where $m = -1, 0, 1, 2, \dots$, and if $|z| < 1$, $|\text{amp } z| < \pi$,

$$\begin{aligned} \int_0^1 \frac{t^a dt}{z+t} &= \int_0^1 \{t^{a-1} - zt^{a-2} + \dots \text{ to } m+1 \text{ terms}\} dt + (-1)^{m+1} \int_0^1 \frac{t^{a-m-1} z^{m+1}}{z+t} dt \\ &= \frac{1}{a} - \frac{z}{a-1} + \dots + \frac{(-z)^m}{a-m} \\ &\quad + (-1)^m \int_1^\infty \frac{t^{a-m-1} z^{m+1}}{t+z} dt + (-z)^{m+1} \int_0^\infty \frac{t^{a-m-1}}{z+t} dt \\ &= \left(\frac{1}{a} - \frac{z}{a-1} + \frac{z^2}{a-2} \dots \text{ to } \infty \right) - \frac{\pi z^a}{\sin a\pi}. \end{aligned}$$

Thus $\int_0^1 \frac{\chi(t)}{z+t} dt$ becomes

$$\begin{aligned} & - \frac{\pi z^\kappa}{\sin \kappa\pi} (A+Bz^2+Cz^4+\dots) - \frac{\pi z^{\kappa'}}{\sin \kappa'\pi} (A'+B'z^2+\dots) \\ & + \left(\frac{A}{\kappa} + \frac{A'}{\kappa'} + \frac{B}{\kappa+2} + \frac{B'}{\kappa'+2} + \dots \right) \\ & - z \left(\frac{A}{\kappa-1} + \frac{A'}{\kappa'-1} + \frac{B}{\kappa+1} + \frac{B'}{\kappa'+1} + \dots \right) \\ & + \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \end{aligned}$$

When we substitute in (1.3) and equate coefficients we have again

$$\theta\pi = -\sin \kappa\pi = -\sin \kappa'\pi,$$

and further

$$\frac{A}{\kappa-n} + \frac{A'}{\kappa'-n} + \frac{B}{\kappa+2-n} + \frac{B'}{\kappa'+2-n} + \dots = -\frac{1}{\theta a^{n+1}} \quad (n=0, 1, 2, \dots). \quad (1.7)$$

Suppose $A/(\kappa-\mu) + A'/(\kappa'-\mu) + B/(\kappa+2-\mu) + \dots$ to represent a function $\psi(\mu)$ with poles at $\kappa, \kappa', \kappa+2, \kappa'+2, \dots$, then, in order to find an expression for $\psi(\mu)$, we note that, since $\{\Gamma(\mu)\}^{-1}$ is an integral function vanishing only at $0, -1, -2, \dots$, the function

$$\frac{\theta\psi(\mu)\Gamma(-\mu)c^{-\mu}}{\Gamma(\frac{1}{2}\kappa-\frac{1}{2}\mu)\Gamma(\frac{1}{2}\kappa'-\frac{1}{2}\mu)}$$

should have poles at $0, 1, 2, \dots$ only, the residue at n being

$$\frac{(-1)^n}{c^n a^{n+1} \Gamma(n+1) \Gamma(\frac{1}{2}\kappa-\frac{1}{2}n) \Gamma(\frac{1}{2}\kappa'-\frac{1}{2}n)}.$$

The integral
$$\frac{1}{2i\pi} \int_{-i\infty}^{i\infty} \frac{c^{-\lambda} a^{-\lambda-1} \Gamma(-\lambda) d\lambda}{(\mu-\lambda) \Gamma(\frac{1}{2}\kappa-\frac{1}{2}\lambda) \Gamma(\frac{1}{2}\kappa'-\frac{1}{2}\lambda)},$$

subject to conditions of convergence, represents a function with these singularities, if the path separates μ from $0, 1, 2, \dots$, and thus tentatively we put

$$\psi(\mu) = \frac{1}{2i\pi\theta} \int_{-i\infty}^{i\infty} \frac{c^{\mu-\lambda} a^{-\lambda-1} \Gamma(-\lambda) \Gamma(\frac{1}{2}\kappa-\frac{1}{2}\mu) \Gamma(\frac{1}{2}\kappa'-\frac{1}{2}\mu)}{(\mu-\lambda) \Gamma(-\mu) \Gamma(\frac{1}{2}\kappa-\frac{1}{2}\lambda) \Gamma(\frac{1}{2}\kappa'-\frac{1}{2}\lambda)} d\lambda,$$

where c is arbitrary, and

$$\begin{aligned} \chi(z) &= \frac{1}{2i\pi} \int_{-i\infty}^{i\infty} \psi(\mu) z^\mu d\mu \\ &= -\frac{1}{4\pi^2\theta} \int^c f_1(2\epsilon z) f_2(2\epsilon a) d\epsilon, \end{aligned} \quad (1.8)$$

where
$$f_1(u) = \int_{-i\infty}^{i\infty} \left(\frac{u}{2}\right)^\mu \frac{\Gamma(\frac{1}{2}\kappa-\frac{1}{2}\mu) \Gamma(\frac{1}{2}\kappa'-\frac{1}{2}\mu)}{\Gamma(-\mu)} d\mu, \quad (1.81)$$

$$f_2(u) = \int_{-i\infty}^{i\infty} \left(\frac{u}{2}\right)^{-\lambda-1} \frac{\Gamma(-\lambda)}{\Gamma(\frac{1}{2}\kappa-\frac{1}{2}\lambda) \Gamma(\frac{1}{2}\kappa'-\frac{1}{2}\lambda)} d\lambda. \quad (1.82)$$

The path of μ is on the left of κ, κ' , that of λ is on the left of $0, 1, 2, \dots$, and the lower limit in the integral (1.8) is not yet assigned.

The reasoning so far has not been at all rigorous, but now it is possible to recognize f_1, f_2 as being hypergeometric functions, and to verify that they furnish a solution of the proposed integral equation when used according to the above indications.

2. *Identification of f_1, f_2 with a certain hypergeometric function.*
Special properties of this function.

The following known transformations of the hypergeometric function $F(a, \beta, \gamma, x)$ are important in the discussion.

$$\begin{aligned} F(a, \beta, \gamma, x) &= \frac{\Gamma(\gamma) \Gamma(\gamma - a - \beta)}{\Gamma(\gamma - a) \Gamma(\gamma - \beta)} F(a, \beta, a + \beta - \gamma + 1, 1 - x) \\ &\quad + (1 - x)^{\gamma - a - \beta} \frac{\Gamma(\gamma) \Gamma(a + \beta - \gamma)}{\Gamma(a) \Gamma(\beta)} \\ &\quad \times F(\gamma - a, \gamma - \beta, \gamma - a - \beta + 1, 1 - x) \quad (2.1) \end{aligned}$$

$$\begin{aligned} &= \frac{\Gamma(\gamma) \Gamma(\beta - a)}{\Gamma(\beta) \Gamma(\gamma - a)} (1 - x)^{-a} F\left(a, \gamma - \beta, a - \beta + 1, \frac{1}{1 - x}\right) \\ &\quad + \frac{\Gamma(\gamma) \Gamma(a - \beta)}{\Gamma(a) \Gamma(\gamma - \beta)} (1 - x)^{-\beta} \\ &\quad \times F\left(\beta, \gamma - a, \beta - a + 1, \frac{1}{1 - x}\right). \quad (2.2) \end{aligned}$$

When $\gamma = a + \beta - 1 + h$, and h tends to zero as its limit, (2.1) gives

$$\begin{aligned} &F(a, \beta, a + \beta - 1, x) \\ &= \frac{\Gamma(a + \beta - 1)}{\Gamma(a) \Gamma(\beta) \Gamma(a - 1) \Gamma(\beta - 1)} \sum_{n=0}^{\infty} (1 - x)^{n-1} \frac{\Gamma(a + n - 1) \Gamma(\beta + n - 1)}{\Gamma(n + 1)} \\ &\quad \times \left\{ \frac{\Gamma'(a + n - 1)}{\Gamma(n) \Gamma(a + n - 1)} + \frac{\Gamma'(\beta + n - 1)}{\Gamma(n) \Gamma(\beta + n - 1)} \right. \\ &\quad \left. + \frac{d}{dn} \frac{1}{\Gamma(n)} - \frac{\Gamma'(n + 1)}{\Gamma(n) \Gamma(n + 1)} + \frac{\log(1 - x)}{\Gamma(n)} \right\}, \end{aligned}$$

which differs by a regular function from

$$\frac{\Gamma(a + \beta - 1)}{\Gamma(a - 1) \Gamma(\beta - 1)} F(a, \beta, 2, 1 - x) \log(1 - x) + \frac{\Gamma(a + \beta - 1)}{\Gamma(a) \Gamma(\beta)} \frac{1}{1 - x}. \quad (2.11)$$

The expansion suggested by (1.81) for $f_1(u)$ is the same as

$$4i\pi \left(\frac{u}{2}\right)^\kappa \frac{\Gamma(\frac{1}{2}-\kappa)}{\Gamma(-\kappa)} F(\tfrac{1}{2}\kappa+\tfrac{1}{2}, \tfrac{1}{2}\kappa+1, \kappa+\tfrac{1}{2}, u^2) \\ + 4i\pi \left(\frac{u}{2}\right)^{\kappa'} \frac{\Gamma(\frac{1}{2}-\kappa')}{\Gamma(-\kappa')} F(\tfrac{1}{2}\kappa'+\tfrac{1}{2}, \tfrac{1}{2}\kappa'+1, \kappa'+\tfrac{1}{2}, u^2), \quad (2.31)$$

which, by (2.1), may be reduced to

$$-2i\pi^{\frac{3}{2}} \kappa \kappa' u^\kappa F(\tfrac{1}{2}\kappa+\tfrac{1}{2}, \tfrac{1}{2}\kappa+1, 2, 1-u^2). \quad (2.32)$$

The expansion suggested by (1.82) for $f_2(u)$ is the same as

$$\frac{4i\pi}{u \Gamma(\tfrac{1}{2}\kappa) \Gamma(\tfrac{1}{2}\kappa')} F\left(1-\tfrac{1}{2}\kappa, 1-\tfrac{1}{2}\kappa', \tfrac{1}{2}, \frac{1}{u^2}\right) \\ - \frac{8i\pi}{u^2 \Gamma(\tfrac{1}{2}\kappa-\tfrac{1}{2}) \Gamma(\tfrac{1}{2}\kappa'-\tfrac{1}{2})} F\left(\tfrac{3}{2}-\tfrac{1}{2}\kappa, \tfrac{3}{2}-\tfrac{1}{2}\kappa', \tfrac{3}{2}, \frac{1}{u^2}\right), \quad (2.33)$$

which, by (2.2), may be reduced to

$$i\pi^{\frac{3}{2}} \kappa \kappa' u^\kappa F(\tfrac{1}{2}\kappa+\tfrac{1}{2}, \tfrac{1}{2}\kappa+1, 2, 1-u^2). \quad (2.34)$$

Finally, then, it is suggested that $f_1(u)$ and $f_2(u)$ are multiples of the same function (2.32), whose properties are known for all values of κ . We therefore write $f(u)$ for $-(\kappa\kappa'/2\theta) u^\kappa F(\tfrac{1}{2}\kappa+\tfrac{1}{2}, \tfrac{1}{2}\kappa+1, 2, 1-u^2)$ or its equivalent $-(\kappa\kappa'/2\theta) u^{\kappa'} F(\tfrac{1}{2}\kappa'+\tfrac{1}{2}, \tfrac{1}{2}\kappa'+1, 2, 1-u^2)$: u will be supposed primarily on the segment $(0, \sqrt{2})$, its logarithm being real; on this segment $f(u)$ has no singularity, except at 0.

On the segment $(0, 1)$, (2.31) is equivalent to $4i\theta\pi^{\frac{3}{2}} f(u)$, and is a valid expression for small values of u with any amplitude: thus $f(u)$ has the same property (1.4) as $\chi(u)$. On the segment $(1, \sqrt{2})$, (2.33) is equivalent to $-2i\theta\pi^{\frac{3}{2}} f(u)$, and is a valid expression for great values of u with any amplitude: thus $uf(u)$ is a regular function of $1/u$ when u is great. To find the singularity of $f(u)$ near -1 we have, by (2.31),

$$\theta\sqrt{\pi} f(ue^{i\pi}) = \frac{\Gamma(\frac{1}{2}-\kappa)}{\Gamma(-\kappa)} \left(\frac{u}{2}\right)^\kappa e^{\kappa i\pi} F(\tfrac{1}{2}\kappa+\tfrac{1}{2}, \tfrac{1}{2}\kappa+1, \kappa+\tfrac{1}{2}, u^2) \\ + \frac{\Gamma(\frac{1}{2}-\kappa')}{\Gamma(-\kappa')} \left(\frac{u}{2}\right)^{\kappa'} e^{\kappa' i\pi} F(\tfrac{1}{2}\kappa'+\tfrac{1}{2}, \tfrac{1}{2}\kappa'+1, \kappa'+\tfrac{1}{2}, u^2), \quad (2.4)$$

where the two hypergeometric functions on the right are both of the form $F(a, \beta, a+\beta-1, x)$, so that, by substitution from (2.11), the

singular part of $f(ue^{i\pi})$, when $\log u$ is near 0, is that of

$$\frac{e^{\kappa i\pi} \Gamma(\frac{1}{2}-\kappa) \Gamma(\frac{1}{2}+\kappa)}{2^{\kappa} \theta \sqrt{\pi} \Gamma(-\kappa) \Gamma(\frac{1}{2}\kappa+\frac{1}{2}) \Gamma(\frac{1}{2}\kappa+1)} \\ \times \left[\frac{u^{\kappa}}{1-u^2} - \frac{1}{4} \kappa \kappa' u^{\kappa} F(\frac{1}{2}\kappa+\frac{1}{2}, \frac{1}{2}\kappa+1, 2, 1-u^2) \log(1-u^2) \right]$$

plus a like expression with κ and κ' interchanged

The second expression in the bracket is

$$\frac{1}{2} \theta f(u) \log(1-u^2),$$

so that the singular part in the bracket is, in each case,

$$\frac{1}{2} (1-u)^{-1} + \frac{1}{2} \theta f(u) \log(1-u),$$

while the factors outside reduce to

$$-\frac{e^{\kappa i\pi} \tan \kappa \pi}{\theta \pi} \quad \text{and} \quad -\frac{e^{\kappa' i\pi} \tan \kappa' \pi}{\theta \pi},$$

whose sum is 2.

Hence the singular part of $f(u)$, when $\log u$ is near $i\pi$, is

$$(1+u)^{-1} + \theta f(ue^{-i\pi}) \log(1+u). \quad (2.5)$$

3. Connexion between the function f and the integral equation (1.1).

The value of $\int \frac{f(t) dt}{t-z}$ taken round the circle at infinity is zero, since $f(t)$ is $O(t^{-1})$ at infinity. Thus the value of the same integral round a contour C enclosing the cut and excluding z , and described positively, is $-2i\pi f(z)$.

But we have seen at (2.5) that $f(t)$ is of the form

$$(t+1)^{-1} + \log(t+1) h(t) + k(t), \quad (3.1)$$

near -1 , where $h(t)$, $k(t)$ are regular near -1 . The integral of

$$(t+1)^{-1} (t-z)^{-1}$$

round C is $-2i\pi/(z+1)$, and hence

$$-2i\pi f(z) = -\frac{2i\pi}{z+1} + \int_C \left\{ f(t) - \frac{1}{t+1} \right\} \frac{dt}{t-z}. \quad (3.20)$$

Here, when t approaches 0, the integrand is $O(t^{\kappa}) + O(t^{\kappa'}) + O(1)$, and it is $O\{\log(1+t)\}$ when t approaches -1 : thus, if $R\kappa$ and $R\kappa'$ exceed -1 , we may take C in its limiting form, consisting of the two edges of the cut;

on the one edge $t = xe^{i\pi}$ and x runs from 0 to 1, on the other $t = xe^{-i\pi}$ and x runs from 1 to 0. Thus

$$\begin{aligned} -2i\pi f(z) &= -\frac{2i\pi}{z+1} + \int_0^1 \left\{ f(xe^{i\pi}) - \frac{1}{1-x} \right\} \frac{dx}{x+z} \\ &\quad - \int_0^1 \left\{ f(xe^{-i\pi}) - \frac{1}{1-x} \right\} \frac{dx}{x+z} \\ &= -\frac{2i\pi}{z+1} + \int_0^1 \{ f(xe^{i\pi}) - f(xe^{-i\pi}) \} \frac{dx}{x+z}. \end{aligned} \quad (3.2)$$

But

$$\begin{aligned} f(xe^{i\pi}) - f(xe^{-i\pi}) &= 2i \sin \kappa\pi f(x) \\ &= -2i\theta\pi f(x), \end{aligned} \quad (3.3)$$

and thus

$$f(z) = \frac{1}{z+1} + \theta \int_0^1 \frac{f(x)}{x+z} dx, \quad (3.4)$$

so that $f(z)$ is a value for $\chi(z, a)$ satisfying (1.3) if $a = 1$. Here, for convergence, we must restrict κ to have a real part between -1 and 2 : $R(\kappa')$ is also between those limits.

4. *Choice of the constant of integration in (1.8). Verification that the proposed equation (1.3) is satisfied by the function thus determined.*

In following out the suggestion of (1.8) we have to decide on a constant of integration, and with this object we first discuss

$$\int_0^\infty f(at)f(au) da.$$

Now, if $|\text{amp } u| < \pi$,

$$4i\pi^3 f(u) = \int_{-\infty}^{i\infty} (\tfrac{1}{2}u)^\mu \Gamma(\tfrac{1}{2}\kappa - \tfrac{1}{2}\mu) \Gamma(\tfrac{1}{2}\kappa' - \tfrac{1}{2}\mu) \Gamma(\mu+1) d\mu, \quad (4.0)$$

the path of integration separating $-1, -2, \dots$ from $\kappa, \kappa', \kappa+2, \kappa'+2, \dots$ and tending to infinity along the lateral axis. Thus

$$\begin{aligned} &-16\pi^3 \int_0^\infty f(at)f(au) da \\ &= \int_0^\infty \int_{-\infty}^{i\infty} \int_{-\infty}^{i\infty} (\tfrac{1}{2}a)^{\lambda+\mu} t^\lambda u^\mu \Gamma(\tfrac{1}{2}\kappa - \tfrac{1}{2}\mu) \Gamma(\tfrac{1}{2}\kappa' - \tfrac{1}{2}\mu) \Gamma(\mu+1) \Gamma(\tfrac{1}{2}\kappa - \tfrac{1}{2}\lambda) \\ &\quad \times \Gamma(\tfrac{1}{2}\kappa' - \tfrac{1}{2}\lambda) \Gamma(\lambda+1) d\lambda d\mu da, \end{aligned} \quad (4.1)$$

if (a) $\frac{3}{2} > R\kappa > -\frac{1}{2}$ for the convergence of the integral on the left;

(b) the amplitudes of t, u are both between $\pm\pi$;

(c) the path of α is the real positive axis;

(d) the paths of λ, μ are straight lines parallel to the lateral axis, separating -1 from κ and κ' . To secure absolute convergency in the triple integral we take these paths so that $R(\lambda + \mu + 1)$ is positive when α runs from 0 to 1, but negative when α runs from 1 to ∞ . This is possible, since we have supposed in (a) that $R(\kappa)$ and $R(\kappa')$ are greater than $-\frac{1}{2}$, and $R\mu$ may in fact be taken to be $-\frac{1}{2}$.

Then the α -integration may be taken first, and after it we are to integrate

$$\frac{2^{-\lambda-\mu}}{\lambda+\mu+1} t^\lambda u^\mu \Gamma(\tfrac{1}{2}\kappa - \tfrac{1}{2}\mu) \Gamma(\tfrac{1}{2}\kappa' - \tfrac{1}{2}\mu) \Gamma(\mu+1) \Gamma(\tfrac{1}{2}\kappa - \tfrac{1}{2}\lambda) \Gamma(\tfrac{1}{2}\kappa' - \tfrac{1}{2}\lambda) \Gamma(\lambda+1)$$

with respect to λ , first from $-i\infty$ to $i\infty$ along a line on the right of $-\mu-1$, and then back along a line on the left of $-\mu-1$: on the whole λ describes a circuit about $-\mu-1$ positively, and thus (4.1) becomes

$$2i\pi \int_{-i\infty}^{i\infty} 2t^{-\mu-1} u^\mu \Gamma(\tfrac{1}{2}\kappa - \tfrac{1}{2}\mu) \Gamma(\tfrac{1}{2}\kappa' - \tfrac{1}{2}\mu) \Gamma(\mu+1) \\ \times \Gamma(\tfrac{1}{2}\kappa + \tfrac{1}{2}\mu + \tfrac{1}{2}) \Gamma(\tfrac{1}{2}\kappa' + \tfrac{1}{2}\mu + \tfrac{1}{2}) \Gamma(-\mu) d\mu, \quad (4.2)$$

$$\text{or} \quad \frac{4i\pi}{t} \int_{-i\infty}^{i\infty} \left(\frac{u}{t}\right)^\mu \frac{\pi}{\sin \tfrac{1}{2}(\kappa - \mu)\pi} \cdot \frac{\pi}{\sin \tfrac{1}{2}(\kappa' - \mu)\pi} \cdot \frac{\pi}{-\sin \mu\pi} d\mu, \quad (4.3)$$

$$\text{or} \quad \frac{8i\pi^4}{t \sin \kappa\pi} \int_{-i\infty}^{i\infty} \left(\frac{u}{t}\right)^\mu \left\{ \frac{1}{\sin \mu\pi - \sin \kappa\pi} - \frac{1}{\sin \mu\pi} \right\} d\mu. \quad (4.30)$$

Now let
$$\int_{-i\infty+c}^{i\infty+c} \frac{a^\mu d\mu}{\sin \mu\pi - \sin \kappa\pi} = J,$$

the path being the line $x=c$: the integral converges if $|\text{amp } a| < \pi$. The same integral along the line $x=c+2$ is

$$\int_{-i\infty+c}^{i\infty+c} \frac{a^{\mu+2} d\mu}{\sin (\mu+2)\pi - \sin \kappa\pi},$$

that is, Ja^2 . Thus $J(a^2-1)$ is the integral round the strip between these two lines. If c is $-\frac{1}{2}$, as here, the poles of the integrand in the strip are κ, κ' , where

$$-\frac{1}{2} < R(\kappa) \leq \frac{1}{2} \quad \text{and} \quad \frac{1}{2} \leq R(\kappa') < \frac{3}{2},$$

and the corresponding residues are

$$\frac{a^\kappa}{\pi \cos \kappa \pi} \quad \text{and} \quad \frac{a^{\kappa'}}{\pi \cos \kappa' \pi}.$$

Hence $J(a^2 - 1) = 2i \sec \kappa \pi (a^\kappa - a^{\kappa'}),$ (4.310)

and after reduction (4.30) becomes*

$$\frac{16\pi^3}{\theta} \left\{ \frac{u^\kappa t^{\kappa'} - u^{\kappa'} t^\kappa}{(u^2 - t^2) \cos \kappa \pi} + \frac{1}{t + u} \right\}. \quad (4.31)$$

This expression is symmetrical in κ and κ' and holds, therefore, when $-\frac{1}{2} < R(\kappa) < \frac{3}{2}$, except for the value $\frac{1}{2}$, where it is indeterminate. Since, however, J is a continuous function of κ in the domain of $\frac{1}{2}$, the value of (4.30) may be found from (4.31) by the usual method even when $\kappa = \frac{1}{2}$.

Hence we put, in agreement with (1.8), where c may be $\frac{1}{2}$,

$$\chi(t, u) = \theta \int_1^\infty f(at) f(au) da + \frac{1}{t + u} \quad (4.41)$$

$$= -\theta \int_0^1 f(at) f(au) da + \frac{u^\kappa t^{\kappa'} - u^{\kappa'} t^\kappa}{(t^2 - u^2) \cos \kappa \pi}, \quad (4.42)$$

and $\chi(t, u)$ is a symmetric function of t, u .

The form (4.41) shows that, if $|t| > 1$,

$$\chi(te^{i\pi}, u) - \chi(te^{-i\pi}, u) = 0, \quad (4.51)$$

but (4.42) shows that, if $|t| < 1$,

$$\chi(te^{i\pi}, u) - \chi(te^{-i\pi}, u) = 2i \sin \kappa \pi \chi(t, u). \quad (4.52)$$

When $|t|$ is great $\chi(t, u)$ is $O(t^{-1})$ and when $|t|$ is small it is of the form (1.6); as a function of t , it has a simple pole at $-u$ with residue 1: there is no other singularity for amplitudes between $\pm\pi$. At amplitude $\pm\pi$, the only singularity is at -1 and is that of $\theta \int_1^\gamma f(at) f(au) da$ where γ can have any value, except unity, through which the path may be made

* The proof holds if amp t and amp u differ by less than π , but the result may be proved so long as the difference is less than 2π .

by deformation to pass. Now, by (3.1),

$$\begin{aligned} \int_1^\gamma f(at) f(au) da &= \int_1^\gamma \left\{ \frac{1}{1+at} + \log(at+1) h(at) + k(at) \right\} f(au) da \quad (4.6) \\ &= -\frac{1}{t} f(u) \log \frac{1+t}{1+\gamma t} \\ &\quad + \int_1^\gamma \left[\frac{a-1}{at+1} \frac{f(au)-f(u)}{a-1} + \frac{\log(at+1)}{\log(a-1)} f(au) h(at) \log(a-1) \right. \\ &\quad \left. + f(au) k(at) \right] da, \quad (4.61) \end{aligned}$$

and when t approaches -1 we may suppose, by deforming the path if necessary, that a and $-1/t$ approach 1 in directions making a finite angle with each other so that

$$\frac{a-1}{at+1} \quad \text{and} \quad \frac{\log(at+1)}{\log(a-1)}$$

are both bounded. Hence every term in (4.61) is bounded when $t \rightarrow -1$ except the first, and $\chi(t, u)$ is infinite like

$$\theta f(u) \log(1+t). \quad (4.62)$$

It is now possible to verify that $\chi(z, a)$ satisfies the equation (1.3). For, since we are supposing in this section that $-\frac{1}{2} < R(\kappa) < \frac{3}{2}$, it follows that $\int \frac{\chi(t, u)}{t-z} dt$ tends to zero if taken round a circle

(1) of small radius about the origin, or

(2) of small radius about -1 , or

(3) of great radius about the origin.

The integrand has poles at $z, -u$ with residues $\chi(z, u)$ and $-(z+u)^{-1}$ respectively. Hence $\int \frac{\chi(t, u)}{t-z} dt$ round the great circle is equal to

$$2i\pi \chi(z, u) - \frac{2i\pi}{z+u} + \int \frac{\chi(t, u)}{t-z} dt$$

round a contour enclosing the cut. On this contour, as at (3.2), we may put $t = xe^{-i\pi}$ where x runs from 1 to 0, and then $t = xe^{i\pi}$ where x runs

from 0 to 1. We have then

$$\begin{aligned}\chi(z, u) - \frac{1}{z+u} &= -\frac{1}{2i\pi} \int_0^1 \{ \chi(xe^{i\pi}, u) - \chi(xe^{-i\pi}, u) \} \frac{dx}{x+z} \\ &= \theta \int_0^1 \frac{\chi(x, u)}{x+z} dx.\end{aligned}\quad (4.7)$$

Thus $\chi(z, a)$ is the desired function satisfying (1.3), and for it we have the two expressions

$$\begin{aligned}\frac{1}{z+a} + \frac{\kappa^2 \kappa'^2}{4\theta} \int_1^\infty (a^2 z a)^\kappa F(\tfrac{1}{2}\kappa + \tfrac{1}{2}, \tfrac{1}{2}\kappa + 1, 2, 1 - a^2 z^2) \\ \times F(\tfrac{1}{2}\kappa + \tfrac{1}{2}, \tfrac{1}{2}\kappa + 1, 2, 1 - a^2 a^2) da\end{aligned}$$

and

$$\begin{aligned}\frac{z^\kappa a^{\kappa'} - z^{\kappa'} a^\kappa}{(a^2 - z^2) \cos \kappa\pi} - \frac{\kappa^2 \kappa'^2}{4\theta} \int_0^1 (a^2 z a)^\kappa F(\tfrac{1}{2}\kappa + \tfrac{1}{2}, \tfrac{1}{2}\kappa + 1, 2, 1 - a^2 z^2) \\ \times F(\tfrac{1}{2}\kappa + \tfrac{1}{2}, \tfrac{1}{2}\kappa + 1, 2, 1 - a^2 a^2) da,\end{aligned}$$

and two others, in which κ and κ' are interchanged.

The ordinary reasoning of the theory of integral equations now shows that if (1.2) is not a solution of (1.1), then g is such that the order may not be changed in

$$\int_0^1 \int_0^1 \frac{g(t)}{u+z} \chi(u, t) du dt, \quad (4.81)$$

so that this integral cannot be absolutely convergent. Also, if (1.1) is satisfied by a function ϕ other than (1.2), it is such that the order of integration may not be changed in

$$\int_0^1 \int_0^1 \frac{\phi(t)}{u+t} \chi(z, u) du dt. \quad (4.82)$$

$\chi(z, t)$ is therefore a "solving" nucleus or function for (1.1).

5. Existence of a second solving function under certain conditions. Extension of the region within which κ may lie.

It must be noted that when $R(\kappa)$ is between 0 and $-\frac{1}{2}$ the solution (1.2) is not unique, for if, in (1.1), we put $z^{-\kappa-1}$ for $\phi(z)$ we have

$$g(z) = z^{-\kappa-1} - \theta \int_0^1 \frac{t^{-\kappa-1}}{t+z} dt = \theta \int_1^\infty \frac{t^{-\kappa-1}}{t+z} dt, \quad (5.01)$$

while, according to (4.7), this value of $g(z)$ gives

$$\phi(z) = \theta \int_1^\infty t^{-\kappa-1} \chi(z, t) dt, \tag{5.02}$$

and thus $g(z) = 0$ when

$$\phi(z) = z^{-\kappa-1} - \theta \int_1^\infty t^{-\kappa-1} \chi(z, t) dt. \tag{5.1}$$

This is not inconsistent with the statement at (4.82).

Similarly, by putting z^κ for $\phi(z)$ in (1.1) we find that $g(z)$ is zero when

$$\phi(z) = z^\kappa - \theta \int_1^\infty t^\kappa \chi(z, t) dt.$$

Here, however, the order of integration in (4.82) may be changed, and thus

$$z^\kappa = \theta \int_1^x t^\kappa \chi(z, t) dt, \tag{5.2}$$

as we verify if we multiply (4.7) by z^κ and integrate from 0 to ∞ . Hence, when $R(\kappa)$ is between 0 and $-\frac{1}{2}$, the solution of (1.1) contains one arbitrary constant, multiplied into the function on the right side of (5.1), so as to form a complementary function of the usual type.

Whatever value is given to the arbitrary constant the function ϕ will be infinite at the origin in general. If we had tried to avoid this from the beginning by taking $A = 0$ in (1.6) we should have been led to change κ to $\kappa+2$ in (1.81), (1.82), and to take $f_1(u)$, $f_2(u)$ to be constant multiples of $f_+(u)$, $f_-(u)$, where

$$2i\pi^{\frac{3}{2}}f_+(u) = \int_{-i\infty}^{i\infty} (\tfrac{1}{2}u)^\mu \Gamma(\tfrac{1}{2}\kappa - \tfrac{1}{2}\mu + 1) \Gamma(\tfrac{1}{2}\kappa' - \tfrac{1}{2}\mu) \Gamma(\mu + 1) d\mu, \tag{5.31}$$

$$-8i\pi^{\frac{3}{2}}f_-(u) = \int_{-i\infty}^{i\infty} (\tfrac{1}{2}u)^\mu \Gamma(\tfrac{1}{2}\kappa - \tfrac{1}{2}\mu) \Gamma(\tfrac{1}{2}\kappa' - \tfrac{1}{2}\mu - 1) \Gamma(\mu + 1) d\mu, \tag{5.32}$$

the path of integration being now on the left of $\kappa'-2$.

Thus
$$f_+(u) = \left(\kappa - u \frac{d}{du}\right)^* f(u), \tag{5.33}$$

$$\left(\kappa + 1 + u \frac{d}{du}\right) f_-(u) = f(u). \tag{5.34}$$

Hence

$$\int_0^1 f(at)f(au) da = \int_0^1 f_+(at)f_-(au) da + f(t)f_-(u) + \frac{t^\kappa u^{-\kappa-1}}{\theta \cos \kappa \pi}, \quad (5.4)$$

the last term being the limit of $-af(at)f_-(au)$ when $a \rightarrow 0$. Thus

$$\chi(t, u) = -\theta \int_0^1 f_+(at)f_-(au) da - \theta f(t)f_-(u) + \frac{u^\kappa t^{\kappa'} - u^{\kappa'-2} t^{\kappa+2}}{(t^2 - u^2) \cos \kappa \pi}. \quad (5.5)$$

We have in (5.5) a new expression for $\chi(t, u)$. The integral involved converges when $R\kappa$ is between $\frac{1}{2}$ and $-\frac{3}{2}$, since, when a is small,

$$f_+(at) = O(a^{\kappa+2}) + O(a^{\kappa'})$$

and

$$f_-(au) = O(a^\kappa) + O(a^{\kappa'-2}).$$

Now in (4.42) the integral converges when $R(\kappa)$ is between $-\frac{1}{2}$ and $\frac{3}{2}$, so that when $R(\kappa)$ is between $\pm\frac{1}{2}$ both (4.42) and (5.5) are valid, and they are equivalent: by (4.42) the range of $R(\kappa)$ is extended up to $\frac{3}{2}$, and by (5.5) down to $-\frac{3}{2}$. Similar extensions to other values of κ can be found, so that the function χ exists for all values of κ , as, in fact, is clear from the form (4.41) in which κ need not be restricted: integral values of κ are somewhat exceptional, but, when $\kappa = 0$ or 1 ,

$$\chi(t, u) = (t+u)^{-1},$$

and the integral equation (1.1) becomes trivial; for other integral values of κ the functions $f(u) \sin \kappa \pi$ and $\chi(t, u) \sin \kappa \pi$ exist, but do not specially concern us. When κ approaches the half of an odd integer there is no discontinuity if the form (4.0) is used, and the indeterminate forms which occur, as in (4.42) and (5.5), may be evaluated by the usual rules.

In the proof of (4.7) it is necessary to suppose $R(\kappa)$ between -1 and 2 in order that $\int \frac{\chi(t, u)}{t-z} dt$ may tend to zero on the small circle about the origin. With this proviso (4.7) holds and (5.2) follows from it if $R(\kappa)$ is between 0 and -1 . Some precautions must be taken in the proof. If we put $\chi(z, u) = Uz^\kappa + \psi(z, u)$, where $\psi(z, u)$ is $O(z^{\kappa'}) + O(z^{\kappa+2})$ when z is small, then, by (4.7),

$$\begin{aligned} \psi(z, u) - \frac{1}{z+u} - \theta \int_0^1 \frac{\psi(x, u)}{x+z} dx \\ = -Uz^\kappa + \theta U \int_0^1 \frac{x^\kappa}{x+z} dx = -\theta U \int_1^\infty \frac{x^\kappa}{x+z} dx. \end{aligned} \quad (5.60)$$

Here multiply by z^κ and integrate with respect to z from 0 to 1. Thus

$$\begin{aligned} \int_0^1 \psi(z, u) z^\kappa dz - \int_0^1 \frac{z^\kappa}{z+u} dz \\ = \theta \int_0^1 \left\{ \int_0^1 \frac{\psi(x, u)}{x+z} z^\kappa dz \right\} dx - \theta U \int_0^1 \int_1^\infty \frac{x^\kappa z^\kappa}{x+z} dx dz, \quad (5.601) \end{aligned}$$

the change of order in the first double integral being legitimate. Similarly from (4.7)

$$\int_1^\infty \chi(z, u) z^\kappa dz - \int_1^\infty \frac{z^\kappa}{z+u} dz = \theta \int_0^1 \left\{ \int_1^\infty \frac{\chi(x, u)}{x+z} z^\kappa dz \right\} dx. \quad (5.602)$$

The sum of the two second members in these equations is

$$\theta \int_0^1 \left\{ \int_0^\infty \frac{\psi(x, u)}{x+z} z^\kappa dz \right\} dx \quad \text{or} \quad \int_0^1 \psi(x, u) x^\kappa dx,$$

and thus

$$\int_1^\infty \chi(z, u) z^\kappa dz = \int_0^\infty \frac{z^\kappa}{z+u} dz = \frac{u^\kappa}{\theta},$$

which is (5.2).

Thus $-\kappa-1$ may be put for κ if $R(\kappa)$ is between 0 and -1 , and (4.7), (5.2) still hold with the new value of $\chi(t, u)$, which we may distinguish as $\chi(t, u, -\kappa-1)$. Also $f_+(t)$ and $f_-(t)$ are unaffected by the interchange of κ and $-\kappa-1$, but $f(t)$ is unaffected by that of κ and $1-\kappa$.

From (5.34) we have

$$f(u, \kappa) - f(u, -\kappa-1) = (2\kappa+1)f_-(u). \quad (5.61)$$

The right side of (5.5) is not affected by the change of κ to $-\kappa-1$ except in the second term (which, incidentally, is also the only part that becomes infinite when t is 0). We have then

$$\begin{aligned} \chi(t, u, \kappa) - \chi(t, u, -\kappa-1) &= -\theta f_-(u) \{f(t, \kappa) - f(t, -\kappa-1)\} \\ &= -\theta(2\kappa+1)f_-(t)f_-(u). \end{aligned} \quad (5.62)$$

Also the right side of (5.1) is a constant multiple of $f_-(z)$, for by means of (5.2) it may be written

$$\theta \int_1^\infty t^{-\kappa-1} \{ \chi(z, t, -\kappa-1) - \chi(z, t, \kappa) \} dt.$$

Thus when $R(\theta)$ is positive, so that $R(\kappa)$ is between 0 and -1 , there are two symmetrical forms of the solving nucleus, $\chi(z, t, \kappa)$ and

$\chi(z, t, -\kappa-1)$, and the solution includes an arbitrary multiple of $f_-(z)$ as a complementary function. All three functions tend to infinity at the origin. When κ crosses the line $R(\kappa) = 0$ one of the forms of the solving nucleus persists, and when κ crosses the line $R(\kappa) = -1$ the other form persists: in either case the complementary function disappears and the solving nucleus ceases to be infinite at the origin.

5A. Recapitulation and forecast.

In the foregoing work it appeared that a function χ satisfying the proposed integral equation (1.3) had to satisfy two functional equations in different parts of the plane, namely

$$\chi(ze^{i\pi}) - \chi(ze^{-i\pi}) = -2i\pi\theta\chi(z) \quad \text{near the origin,}$$

$$\chi(ze^{i\pi}) - \chi(ze^{-i\pi}) = 0 \quad \text{towards infinity;}$$

in other words, it was reduced to zero by the operators

$$e^{i\pi\mathfrak{S}} - e^{-i\pi\mathfrak{S}} + 2i\pi\theta \quad \text{near the origin,}$$

and

$$e^{i\pi\mathfrak{S}} - e^{-i\pi\mathfrak{S}} \quad \text{towards infinity}$$

where $\mathfrak{S}$ is $z(d/dz)$.

A function was then constructed having these properties and having an assigned pole, and this function was found to be a solving function for the integral equation, as was expected.

The two operators were found by following the function round two circles, one small and the other great. The small circle crossed one ray which was a path of integration; the great circle crossed no such ray. It may be presumed that more complicated operators would appear if there were more such rays, some starting from the origin and some going out to infinity. We shall verify this possibility by starting with two given operators of a general type, and shall thus find a class of integral equations soluble by the above method. We construct a function reduced to zero by the operators in different regions and having an assigned pole, and pass from it to the integral equation whose solution it affords.

The functions f, χ hitherto encountered have been uniform functions of $\log z$: it will be more convenient to deal with uniform functions of z , and we shall therefore work henceforward with D or d/dz instead of $\mathfrak{S}$ and with exponentials instead of powers of z .

6. *Solution of linear functional equations by definite integrals.*

If
$$S(\lambda) = Ae^{a\lambda} + Be^{b\lambda} + \dots,$$

where $a, b, \dots$ are constants, and $A, B, \dots$ may be constants or may be quantities $A(\lambda), B(\lambda), \dots$ in λ , then a function $\phi(z)$ such that

$$S(D)\phi(z) = 0^* \quad (6.0)$$

may often be represented by $\int \frac{e^{\lambda z}}{S(\lambda)} d\lambda$ where both the limits are infinite and z has such values as make the integral absolutely convergent. For examples of this see *Proc. London Math. Soc.* (2), 21, (1922), 277, 281, and 25 (1926), 425, 431-2; as another example, take the equation

$$\phi(z+1) = \phi'(z) \quad \text{where } S(D) \text{ is } e^D - D. \quad (6.01)$$

The integral $\int_{-\infty}^{\infty} \frac{e^{\lambda z}}{e^{\lambda} - \lambda} d\lambda$, where the path is the real axis, is absolutely convergent and may be differentiated freely under the integral sign, if $0 < R(z) < 1$. Let the function represented by it be called $\psi(z)$: this function is regular within the strip in which $0 < R(z) < 1$. In order to continue the function we note that, within the strip mentioned,

$$\int_0^{\infty} e^{\lambda z - \lambda} d\lambda = \frac{1}{1-z},$$

so that
$$\psi(z) = \int_{-\infty}^0 \frac{e^{\lambda z}}{e^{\lambda} - \lambda} d\lambda + \int_0^{\infty} \left(\frac{e^{\lambda z}}{e^{\lambda} - \lambda} - e^{\lambda z - \lambda} \right) d\lambda + \frac{1}{1-z}. \quad (6.02)$$

Here the integrals converge absolutely within the wider strip $0 < R(z) < 2$, and $\psi(z)$ is thus continued over this double strip and is regular in it, except for the pole at 1 with residue -1 .

In the added region, where $1 < R(z) < 2$,

$$\frac{1}{1-z} = - \int_{-\infty}^0 e^{\lambda z - \lambda} d\lambda,$$

and therefore

$$\psi(z) = \int_{-\infty}^{\infty} \left(\frac{e^{\lambda z}}{e^{\lambda} - \lambda} - e^{\lambda z - \lambda} \right) d\lambda = \int_{-\infty}^{\infty} \frac{\lambda e^{\lambda z - \lambda}}{e^{\lambda} - \lambda} d\lambda = \psi'(z-1). \quad (6.03)$$

Thus $-\psi(z-a+1)$ satisfies the functional equation (6.01) and has a pole at a with residue 1. It may be continued over the whole plane, and by means of it functions satisfying the functional equation (6.01) may

* Here D stands for d/dz , and $e^{aD}\phi(z)$, for instance, means $\phi(z+a)$.

be constructed with any assigned rational algebraic singularities that are consistent with (6.01).

There is another way of arriving at the result (6.03). The path of integration, here the real axis, tends to infinity in such a way as to separate two infinite sequences of zeros of $S(\lambda)$ whose amplitudes approach $\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$ respectively. Provided that this separation is maintained, the path may be altered without changing the function represented by the integral, but in general the region of validity of the representation is altered. The path must not be passed over any zero of $S(\lambda)$. It may be deformed into one running from infinity at amplitude σ in the second quadrant to infinity at amplitude ρ in the first quadrant, provided that the integrals along a circle about the origin between amplitudes 0, ρ , and also between σ , π , tend to zero as the radius increases. This is certainly the case in that part of the strip $0 < R(z) < 1$ which lies above the real axis, so that the new form and the old are both valid in this half of the strip. The new form holds so long as $e^{i\rho}(z-1)$ and $e^{i\sigma}z$ have negative real parts, that is, if z lies above two straight lines, one from 1 at amplitude $\frac{1}{2}\pi - \rho$ and the other from 0 at amplitude $\frac{3}{2}\pi - \sigma$. Thus z is confined to a wedge-shaped region, which becomes the strip $0 < R(z) < 1$ when $\rho = 0$, $\sigma = \pi$, and becomes the upper half-plane when $\rho = \sigma = \frac{1}{2}\pi$. Both z and $z+1$ lie within this region if $\text{amp } z$ is between $\frac{1}{2}\pi - \rho$ and $\frac{3}{2}\pi - \sigma$, so that

$$\psi(z+1) - \psi'(z) = \int \left(\frac{e^{\lambda z + \lambda}}{e^\lambda - \lambda} - \frac{\lambda e^{\lambda z}}{e^\lambda - \lambda} \right) d\lambda = \int e^{\lambda z} d\lambda = 0,$$

since $e^{\lambda z}$ now tends to zero at each limit.

Hence $\psi(z+1) = \psi'(z)$ so long as z is in the upper half-plane, for $\frac{1}{2}\pi - \rho$, $\sigma - \frac{1}{2}\pi$ may both be made arbitrarily small.

Similarly by bending the path into the lower half-plane we may prove that in the lower half-plane

$$\psi(z+1) = \psi'(z).$$

This method does not, however, give the nature of the singularities of $\psi(z)$.

6A. Solution of a general functional equation $S(D)\phi(z) = 0$.

In order to choose the path of integration for λ when

$$S(\lambda) = Ae^{a\lambda} + Ee^{b\lambda} + \dots,$$

we need to ascertain the distribution towards infinity of the zeros of

$S(z)$, and to do this we suppose the points $a, b, \dots$ plotted, and let K be the least convex polygon to which none of these points are external. If the outward normals to the sides of K are inclined to the real axis at angles $\alpha, \beta, \dots$, and if z tends to infinity at an amplitude θ that is not $-\alpha, -\beta, \dots$, then among the exponents $az, bz, \dots$ in $S(z)$ there is one whose real part is greatest. For the least convex polygon containing $az, bz, \dots$ is similar to K , but turned through an angle θ . If two exponents had equal real parts, each exceeding the real part of any other, the polygon would have on the right a side parallel to the lateral axis, which can only be when θ has one of the values $-\alpha, -\beta, \dots$. Hence the equation $S(z) = 0$ has no roots greater in absolute value than R , except with amplitudes between $-\alpha \pm \epsilon$ or $-\beta \pm \epsilon$ or $\dots$, where $\epsilon \rightarrow 0$ when $R \rightarrow \infty$, for if az has the greatest real part among the exponents when $\text{amp } z = \theta$, then as $|z|$ increases, the term Ae^{az} becomes the commanding term in $S(z)$ and cannot be cancelled by the other terms.

Let the half-plane in which $\text{amp } (z-a)$ lies between $\frac{1}{2}\pi - \rho$ and $\frac{3}{2}\pi - \rho$, that is, in which $(z-a)e^{i\rho}$ has a negative real part, be distinguished as (a, ρ) .

Among the constants $a, b, \dots$ involved in $S(z)$ suppose that b lies in the half-plane (a, ρ) and a in the half-plane (b, σ) , while c and the others lie in the region or wedge common to these half-planes. Then ρ, σ are not among $-\alpha, -\beta, \dots$. Let P denote a path avoiding the zeros of $S(z)$ and running from infinity at amplitude σ to infinity at amplitude ρ . Then $Ae^{a\lambda}$ is the commanding term at the upper limit of P and $Be^{b\lambda}$ at the lower: the convergence of $\int_{(P)} \frac{e^{\lambda z}}{S(\lambda)} d\lambda$ is absolute, and differentiation under the integral sign is legitimate, so long as z also lies in the wedge common to the half-planes (a, ρ) and (b, σ) . Let $\psi(z)$ be the function represented by this integral. The wedge becomes a strip when $\sigma - \rho = \pi$, but even then it may be changed into a wedge by a slight deformation of P , since ρ, σ are only restricted by inequalities. Then if z lies in the wedge common to the half-planes $(0, \rho), (0, \sigma)$, all the arguments $z+a, z+b, \dots$ lie in both the half-planes $(a, \rho), (b, \sigma)$, and

$$S(D)\psi(z) = S(D) \int_{(P)} \frac{e^{\lambda z}}{S(\lambda)} d\lambda = \int_{(P)} e^{\lambda z} d\lambda = 0, \quad (6.10)$$

since at each end of P the real part of λz is negative and great, so that $e^{\lambda z}$ tends to zero.

To discuss the singularity at a , let the path P be divided at some point κ into an upper part Q and a lower part R . Then in the region common

to the half-planes (a, ρ) , (b, σ)

$$\begin{aligned} A(D) \psi(z) &= \int_{(P)} \frac{A(\lambda) e^{\lambda z}}{S(\lambda)} d\lambda \\ &= \int_{(R)} \frac{A(\lambda) e^{\lambda z}}{S(\lambda)} d\lambda + \int_{(Q)} \left\{ \frac{A(\lambda) e^{\lambda z}}{S(\lambda)} - e^{\lambda(z-a)} \right\} d\lambda - \frac{e^{\kappa z - \kappa a}}{z-a}. \quad (6.11) \end{aligned}$$

The integral along R converges in the half-plane (b, σ) and that along Q in the half-plane common to $(2a-b, \rho)$, $(2a-c, \rho)$, ..., which contains a as an internal point. Hence the singularity of $A(D) \psi(z)$ at a is a simple pole with residue -1 , and the third expression (6.11) is valid for a greater region than the second.

Similarly, if $S(\lambda) = T(\lambda) + U(\lambda)$

where $T(\lambda)$ includes $Ae^{a\lambda}$, $U(\lambda)$ includes $Be^{b\lambda}$, and the other terms of S , if any, are distributed, some to $T(\lambda)$, say for instance $Ce^{c\lambda}$, and the rest to $U(\lambda)$, say for instance $Ge^{g\lambda}$, then, if z is in the wedge common to $(0, \rho)$, $(0, \sigma)$, we have by (6.10)

$$U(D) \psi(z) = -T(D) \psi(z), \quad (6.12)$$

and each of these may be expressed as

$$\begin{aligned} & - \int_{(P)} \frac{T(\lambda)}{S(\lambda)} e^{\lambda z} d\lambda \quad \text{or} \quad \int_{(P)} \frac{U(\lambda)}{S(\lambda)} e^{\lambda z} d\lambda \\ \text{or} \quad & - \int_{(R)} \frac{T(\lambda)}{S(\lambda)} e^{\lambda z} d\lambda + \int_{(Q)} \frac{U(\lambda)}{S(\lambda)} e^{\lambda z} d\lambda + \frac{e^{\kappa z}}{z}, \quad (6.13) \end{aligned}$$

the last term being the result of integrating $-e^{\lambda z}$ along Q . The form (6.13) shows that $U(D) \psi(z)$ has a pole at the origin with residue 1, and has no other singularity in the wedge common to the half-planes $(a-b, \rho)$, $(a-g, \rho)$, ..., $(b-a, \sigma)$, $(b-c, \sigma)$, ... Its other singularities are to be deduced by means of the functional equation (6.10).

The function $\psi(z)$ of this section satisfies the same functional equation (6.10) over the whole plane. Our next step is to construct functions, analogous to the hypergeometric function, satisfying different equations in different regions, as, for instance, in two half-planes. The clue to their formation is given by the form (4.0), where the factor $\Gamma(\mu+1)$ becomes infinite at some of the zeros of $\sin \mu\pi$, and $\Gamma_{\frac{1}{2}}(\kappa-\mu)$, $\Gamma_{\frac{1}{2}}(\kappa'-\mu)$ at some of the zeros of $\sin \mu\pi - \sin \kappa\pi$. We have then to consider the resolution

of $S(\lambda)$ into factors, and for simplicity we shall take $A, B, C, \dots$, the coefficients in $S(\lambda)$, to be constants.

7-8. *Resolution of $S(z)$ into factors when $A, B, C, \dots$ are constants.*

$$7. \text{ We have } \frac{d^2 \log S(z)}{dz^2} = \frac{\Sigma AB(a-b)^2 e^{(a+b)z}}{\{S(z)\}^2}, \quad (7.01)$$

which tends to zero exponentially when $|z|$ increases, unless $\text{amp } z$ tends to $-\alpha, -\beta, \dots$, as explained in § 6A. Let P have the same meaning as in § 6A, and let λ be a value of z not lying on P .

$$\text{Then} \quad \int_{(P)} \frac{d^2 \log S(z)}{dz^2} \frac{dz}{\lambda - z} \quad (7.02)$$

is absolutely convergent, and so long as λ does not cross the path P this integral represents a regular analytical function of λ , say $\varpi(\lambda)$; $\varpi(\lambda)$ is not changed by a deformation of P so long as it does not pass over λ or any zero of $S(z)$, and the initial and final amplitudes may be varied continuously so long as they do not approach $-\alpha, -\beta, \dots$, since the integral (7.02) along an arc at infinity may be neglected if the arc does not approach these directions.

P divides the circuit at infinity into two parts, say P' on the right and P'' on the left. To fix the ideas we shall suppose λ to lie between P and P' . Then in the region between P and P' , $\varpi(\lambda)$ has no singularities; but if between P and P'' , $S(z)$ has a zero of the n -th order at v , then $d^2 \log S(z)/dz^2$ has the singularity of $-n/(z-v)^2$ and the integral (7.02) round a contour enclosing v positively is

$$-\frac{2ni\pi}{(\lambda-v)^2}.$$

This quantity must be added to the integral if it is still to represent $\varpi(\lambda)$ when the path P is passed over v , and therefore this is the singular part of $\varpi(\lambda)$ in the domain of v . Thus $\varpi(\lambda)$ has poles of the second order at those zeros of $S(z)$ that lie on the left of P , and has no other singularities except at infinity.

In the special case when $S(z) = \sin \pi z$, and P is the line $Rz = \frac{1}{2}$, described upwards, $\varpi(\lambda)$ is

$$-2i\pi \frac{d^2 \log \Gamma(\lambda)}{d\lambda^2}.$$

An asymptotic expansion for $\varpi(\lambda)$, leading in the special case to Stirling's formula for $\Gamma(z)$, may be found as follows:

$$\varpi(\lambda) = \int_{(P)} \frac{d^2 \log S(z)}{dz^2} \left(\frac{1}{\lambda} + \frac{z}{\lambda^2} + \frac{z^2}{\lambda^2(\lambda-z)} \right) dz. \quad (7.1)$$

The first two terms in (7.1) may be integrated, and their contribution is

$$\frac{a-b}{\lambda} - \frac{1}{\lambda^2} \log \frac{A}{B},$$

since Ae^{az} , Be^{bz} are the commanding terms in $S(z)$ at the upper and lower limits respectively, so that we substitute for

$$\frac{S'(z)}{S(z)} \quad \text{and} \quad z \frac{S'(z)}{S(z)} - \log S(z)$$

at the upper limit a and $-\log A$, at the lower b and $-\log B$. The rest may be written

$$\frac{1}{\lambda^3} \int_{(P)} \frac{\lambda}{\lambda-z} \cdot z^2 \frac{d^2 \log S(z)}{dz^2} \cdot dz,$$

and since $\int_{(P)} z^2 \frac{d^2 \log S(z)}{dz^2} dz$ is absolutely convergent this is of the order of λ^{-3} if $\lambda/(\lambda-z)$ is bounded, which is the case when $\lambda \rightarrow \infty$ on P' , not approaching the limits of P .

If λ crosses to the left of P the right side of (7.1) is equal to

$$\varpi(\lambda) - 2i\pi \frac{d^2 \log S(\lambda)}{d\lambda^2}, \quad (7.2)$$

where the second term is insignificant at infinity except near the amplitudes $-a$, $-\beta$, ... [see (7.01)]. Thus, except near these amplitudes,

$$\varpi(\lambda) = \frac{a-b}{\lambda} - \frac{1}{\lambda^2} \log \frac{A}{B} + O\left(\frac{1}{\lambda^3}\right),$$

and the exception does not apply to P' . The limiting amplitudes on P need not be excepted, since they can be varied: in effect, the arcs P' , P'' overlap at each end.

Let $\mathfrak{S}(\lambda)$ be the integral of $\varpi(\lambda)$ with respect to λ ; it is a regular analytical function, except at those zeros of $S(\lambda)$ which lie on the left of

P : these are simple poles of $\mathfrak{S}(\lambda)$, the residue at an n -fold zero being $2ni\pi$.

The constant of integration may be so chosen that, when λ approaches P' ,

$$\mathfrak{S}(\lambda) = (a-b) \log \lambda + \frac{1}{\lambda} \log \frac{A}{B} + O\left(\frac{1}{\lambda^2}\right). \quad (7.31)$$

When λ approaches P'' ,

$$\mathfrak{S}(\lambda) - 2i\pi \frac{S'(\lambda)}{S(\lambda)} = (a-b) \log \lambda - \frac{1}{\lambda} \log \frac{A}{B} \quad (7.32)$$

also differs from a constant by $O(\lambda^{-2})$, and this constant is $-2i\pi b$ if P'' is entered from P' by the lower end of P , but $-2i\pi a$ if by the upper. The difference is due to the fact that $\log \lambda$ is not uniform, though $\mathfrak{S}(\lambda)$ is.

Let $\eta(\lambda)$ be $\exp \left\{ \int \mathfrak{S}(\lambda) d\lambda \div 2i\pi \right\}$: then $\eta(\lambda)$ is a regular analytic function all over the plane, vanishing nowhere on the right of P , but having an n -fold zero at any n -fold zero of $S(\lambda)$ to the left. The constant of integration may be so taken that when λ approaches P' ,

$$2i\pi \log \eta(\lambda) = (a-b) \lambda (\log \lambda - 1) + (\log A - \log B) \log \lambda + O(\lambda^{-1}), \quad (7.41)$$

and when λ approaches P'' , as entered from the lower end of P ,

$$2i\pi \{ \log \eta(\lambda) - \log S(\lambda) \} = (a-b) \lambda (\log \lambda - 1) - 2i\pi b\lambda - 2i\pi \log B \\ + (\log A - \log B) \log \lambda + O(\lambda^{-1}). \quad (7.42)$$

These asymptotic expansions could be carried to any further extent; they would be series in negative integral powers of λ , and the error in stopping at any term would be of the order of the first term omitted.

$S(\lambda)$ has now been resolved into two factors; one is $\eta(\lambda)$, an integral function having the zeros of $S(\lambda)$ that lie on the left of P ; while the other $S(\lambda)/\eta(\lambda)$ is an integral function having the zeros of $S(\lambda)$ that lie on the right of P . If the path is reversed, the arcs P' , P'' change places and so do the two factors of $S(\lambda)$, save for a factor such as $\exp(p\lambda + q)$. It is clear that such a factor is of no importance, for it does not essentially alter the functional equation, the effect of which depends only on the ratios of A , B , C , ... and the differences of a , b , c , ... Further, by a linear transformation of λ , without real loss of generality, we may reduce $b-a$ and B/A to any particular values, say $2i\pi$ and -1 , and may take

$$S(\lambda) = -e^{-i\pi\lambda} + e^{i\pi\lambda} + Ce^{c\lambda} + \dots$$

8. *Product of two factors such as $\eta(\lambda)$, belonging to different functions S .*

If there are two expressions, distinguished by suffixes, where

$$-a_1 = b_1 = -b_2 = a_2 = \log A_1 = -\log A_2 = i\pi, \quad \log B_1 = \log B_2 = 0, \quad (8.11)$$

so that $S_r(\lambda) = (-1)^r e^{-i\pi\lambda} - (-1)^r e^{i\pi\lambda} + C_r e^{c_r\lambda} + \dots$ ($r = 1, 2$),

and where $c_1, \dots, c_2, \dots$ lie in the region common to the half-planes

$$(-i\pi, \rho), \quad (i\pi, \sigma),$$

ρ being an angle in the first or second quadrant, σ one in the third or fourth, then for the resolution of S_1 we may use a path P_1 running from infinity at amplitude σ to infinity at amplitude ρ , and for that of S_2 the same path reversed, say P_2 . P_1 divides the line at infinity into two arcs, P'_1 or P''_2 on the right of P_1 , and P''_1 or P'_2 on its left.

The assumptions as to the logarithms of A_1, A_2, B_1, B_2 imply that the net change in $\log \{S_1(\lambda)/S_2(\lambda)\}$ on the path P_1 is zero. (8.12)

The method of § 7 gives for each suffix functions

$$\varpi_r(\lambda), \quad \mathfrak{S}_r(\lambda), \quad \eta_r(\lambda),$$

and, since $(a_1 - b_1) + (a_2 - b_2) = 0$ and $\log(A_1/B_1) + \log(A_2/B_2) = 0$, we have for great values of λ in general, but with some exceptions,

$$\varpi_1(\lambda) + \varpi_2(\lambda) = O(\lambda^{-3}).$$

The determination of the constants of integration in $\mathfrak{S}_1(\lambda), \eta_1(\lambda), \mathfrak{S}_2(\lambda), \eta_2(\lambda)$ depends on the particular branch of $\log \lambda$ chosen in (7.31), (7.41). If the choice is such that $\log \lambda$ has the same meaning in all the four functions towards the lower limit on P_1 , which is the upper limit on P_2 , then from (7.31) and (7.32) we find by addition that, towards P'_1 ,

$$\mathfrak{S}_1(\lambda) + \mathfrak{S}_2(\lambda) - 2i\pi S'_2(\lambda)/S_2(\lambda) = 2\pi^2 + O(\lambda^{-2}), \quad (8.41)$$

and towards P''_1

$$\mathfrak{S}_1(\lambda) + \mathfrak{S}_2(\lambda) - 2i\pi S'_1(\lambda)/S_1(\lambda) = 2\pi^2 + O(\lambda^{-2}). \quad (8.42)$$

Also from (7.41) and (7.42) towards P'_1

$$\log \frac{\eta_1(\lambda) \eta_2(\lambda)}{S_2(\lambda)} = i\pi(1 - \lambda) + O(\lambda^{-1}), \quad (8.51)$$

and towards P_1''

$$\log \frac{\eta_1(\lambda) \eta_2(\lambda)}{S_1(\lambda)} = -i\pi\lambda + O(\lambda^{-1}). \tag{8.52}$$

[If $S_1(\lambda) \equiv -S_2(\lambda)$ it follows that each of them is equal to

$$e^{i\pi\lambda} \eta_1(\lambda) \eta_2(\lambda). \tag{8.6}$$

since the ratio of this product to $S_1(\lambda)$ tends to 1 at infinity in any direction, and is nowhere zero or infinite. This completes the resolution into factors proposed in § 7.]

9, 10. Construction of a function analogous to $f(u)$ of § 2.

9. Since (8.51) and (8.52) hold towards both limits on the path P_1 , it follows that

$$\begin{aligned} e^{i\pi\lambda} \eta_1(\lambda) \eta_2(\lambda) \text{ is } O(e^{i\pi\lambda}) \text{ when amp } \lambda \text{ is } \sigma, \\ \text{and } O(e^{-i\pi\lambda}) \text{ when amp } \lambda \text{ is } \rho, \end{aligned}$$

and thus

$$\int_{(P_1)} \frac{e^{\lambda(t-i\pi)}}{\eta_1(\lambda) \eta_2(\lambda)} d\lambda \tag{9.1}$$

is absolutely convergent when t lies in the wedge common to the two half-planes $(-i\pi, \rho)$, $(i\pi, \sigma)$. In this region (9.1) represents a regular analytic function of t , say $f(t)$ or $f_{12}(t)$. If $\sigma - \rho = \pi$ the wedge becomes a strip; but if $\sigma - \rho > \pi$ it widens out to infinity on the left, and when the amplitude of t is between $\frac{5}{2}\pi - \sigma$ and $\frac{3}{2}\pi - \rho$ the integral

$$\int_{(P_1)} \frac{S_2(\lambda) e^{\lambda(t-i\pi)}}{\eta_1(\lambda) \eta_2(\lambda)} d\lambda \tag{9.2}$$

represents

$$A_2 f(t+a_2) + B_2 f(t+b_2) + \dots, \text{ or say } S_2\left(\frac{d}{dt}\right) f(t).$$

Now the integrand in (9.2) has no poles to the right of P_1 , and the path may therefore be moved away indefinitely towards P_1' on which, since $R(\lambda t)$ is negative, the logarithm of the integrand has a negative real part, of the order of λ , by (8.51). Hence (9.2) must vanish, that is

$$S_2\left(\frac{d}{dt}\right) f(t) = 0. \tag{9.3}$$

This holds so long as $\text{amp } t$ is between $\frac{5}{2}\pi - \sigma$ and $\frac{3}{2}\pi - \rho$, so that $t + a_2$, $t + b_2$, ... all lie in the wedge within which $f(t)$ is originally defined. By means of (9.3) the function may be continued across the bounding lines of the wedge under certain restrictions, but the continuation cannot be carried over the points $\pm i\pi$.

Again, ρ , σ are only restricted by inequalities, and there may be a wedge widening out to the right in which $f(t)$ is regular, and within which it will similarly have the property

$$S_1\left(\frac{d}{dt}\right)f(t) = 0. \quad (9.4)$$

The proof of this property when $\sigma - \rho$ cannot be less than π is less simple*. We cut the path P_1 at κ where it crosses the real axis, and denote the upper part of it by Q_1 , the lower by R_1 . Then if $f^-(t)$ and $f^+(t)$ denote the respective contributions of R_1 and Q_1 to (9.1), so that $f(t) = f^+(t) + f^-(t)$, the representation of $f^-(t)$ is valid over the half-plane $(i\pi, \sigma)$, and $f^-(t)$ is regular over that half-plane. Over the half-plane $(0, \sigma)$

$$\int_{(R_1)} \frac{S_1(\lambda) e^{\lambda(t-i\pi)}}{\eta_1(\lambda) \eta_2(\lambda)} d\lambda \quad (9.41)$$

is also absolutely convergent and represents a regular function, namely $S_1(d/dt)f^-(t)$. But in (9.41) the integrand has no singularities to the left of P_1 , and the path R_1 may be deformed into

(1) a part of P , from amplitude σ to π , followed by

(2) the real axis from $-\infty$ to κ .

The integral along (1) may be neglected on account of (8.52) if the amplitude of t is between $-\frac{1}{2}\pi$ and $\frac{3}{2}\pi - \sigma$, since λt has a negative real part of the order of λ , and for this range of values of $\text{amp } t$

$$S_1\left(\frac{d}{dt}\right)f^-(t) = \int_{-\infty}^{\kappa} \frac{S_1(\lambda) e^{\lambda(t-i\pi)}}{\eta_1(\lambda) \eta_2(\lambda)} d\lambda. \quad (9.42)$$

Now the function represented by (9.42) is regular over the half-plane in which Rt is positive, and is the continuation of $S_1(d/dt)f^-(t)$ over it. Hence if t is in this half-plane and all but one of $f^-(t+a_1)$, $f^-(t+b_1)$, ...

* It is not needed for the purposes of §§ 11-14.

are known, their arguments being in the half-plane $(i\pi, \sigma)$, that one, which will be $f^-(t+i\pi)$, is defined by (9.42). Thus the function $f^-(t)$ can be continued at once into a half-strip to the right of the lateral axis, bounded by the lines through $i\pi$ and $i\pi+i\tau$ at the slope $\frac{3}{2}\pi-\sigma$, where τ is a positive quantity.

Similarly $f^+(t)$ is defined as a regular function in the half-plane $(-i\pi, \rho)$, and when t is in both the half-planes $(0, \rho)$ and $(0, \pi)$,

$$S_1\left(\frac{d}{dt}\right)f^+(t)=-\int_{-\infty}^{\kappa}\frac{S_1(\lambda)e^{\lambda(t-i\pi)}}{\eta_1(\lambda)\eta_2(\lambda)}d\lambda. \tag{9.43}$$

Thus $f^+(t)$ may be continued into a half-strip to the right of the lateral axis, bounded by lines through $-i\pi$ and $-i\pi-i\nu$ at the slope $\frac{1}{2}\pi-\rho$, where ν is positive.

Then if t lies in the triangle bounded by the lateral axis, the line through $-i\nu$ at slope $\frac{1}{2}\pi-\rho$ and the line through $i\tau$ at slope $\frac{3}{2}\pi-\sigma$, the functions f^+, f^- are both defined for all the arguments $t+a_1, t+b_1, \dots$, and by adding (9.42), (9.43) we have

$$S_1\left(\frac{d}{dt}\right)f(t)=0. \tag{9.4}$$

A similar proof could be given for (9.3) if $\sigma-\rho$ could not be made greater than π ; it is clear that the function can be continued over the whole plane, except singular points, but it is not uniform if S_1 and $-S_2$ are different.

10. If the behaviour of $f(t)$ is known when t is in the domain of $i\pi$, the nature of any other of its singularities can be deduced by means of (9.3) and (9.4).

Now when t describes a small closed contour about $i\pi$, $f(t)$ is increased by a regular function*, say $\psi(t)$, and thus

$$f(t)=\frac{1}{2i\pi}\psi(t)\log(t-i\pi)+\omega(t), \tag{10.1}$$

where $\omega(t)$ is uniform.

Again, when t is near $i\pi$, $f^+(t)$ is regular, and by carrying (8.52) two steps further, we find that $f^-(t)$ is, in $(i\pi, \sigma)$, of the form

$$\int_{(R_1)}e^{\lambda(t-i\pi)}\left\{1+\frac{h_1}{\lambda}+\frac{h_2}{\lambda^2}+O\left(\frac{1}{\lambda^3}\right)\right\}d\lambda. \tag{10.2}$$

* Namely $-\left\{S_2\left(\frac{d}{dt}\right)+S_1\left(\frac{d}{dt}\right)\right\}f(t-i\pi)$.

Here the first term is $e^{\kappa(t-i\pi)}/(t-i\pi)$, which must form part of $\omega(t)$: this term has a simple pole at $i\pi$ with residue 1. In the second term h_1 is multiplied by

$$\int_{e^{i\sigma\infty}}^{\kappa} e^{\lambda(t-i\pi)} \frac{d\lambda}{\lambda}$$

or

$$\int_{e^{i\sigma\infty}}^0 \frac{e^{\lambda(t-i\pi)} - e^{\lambda g}}{\lambda} d\lambda + \int_0^{\kappa} \frac{e^{\lambda(t-i\pi)} - 1}{\lambda} d\lambda + \int_{\kappa}^0 \frac{e^{\lambda g} - 1}{\lambda} d\lambda + \int_{e^{i\sigma\infty}}^{\kappa} \frac{e^{\lambda g}}{\lambda} d\lambda, \quad (10.3)$$

where g is any fixed quantity such that $ge^{i\sigma}$ has a negative real part, to secure the convergence of the last term.

In (10.3) the first term is $\log\{t-i\pi/g\}$ and the rest is regular in the domain of $i\pi$. The rest of (10.2) is bounded when t approaches $i\pi$ in the half-plane $(i\pi, \sigma)$.

Hence, in (10.1), $\omega(t)$ has a simple pole at $i\pi$ with residue 1. At $-i\pi$ the singularity of $f(t)$ is of the same kind as at $i\pi$, by (9.3).

We have then constructed a function analogous to f of § 2, in that it satisfies two different functional equations, (9.3) and (9.4), in the two half-planes, and has similar singularities. Compare the results of this section with (2.5).

If $S_2(\lambda) = -S_1(\lambda)$, then, by (8.6), $f_{12}(t)$ becomes

$$\int_{(P_1)} \frac{e^{\lambda t}}{S_1(\lambda)} d\lambda,$$

the function discussed in § 6A, which we may now call

$$f_{11}(t). \quad (10.4)$$

Similarly $f_{22}(t)$ will denote

$$\int_{(P_2)} \frac{e^{\lambda t}}{S_2(\lambda)} d\lambda.$$

11, 12. Construction of a more general function satisfying the same functional equations in the case when the wedge becomes a strip.

11. In the case (§ 6A) of a function satisfying the same functional equation everywhere, no unreasonable complication was caused by supposing the coefficients $A, B, C, \dots$ in $S(\lambda)$ to be quantics in λ . When we discussed the resolution of $S(\lambda)$ into factors and constructed the

function $f(t)$ in § 9, it seemed best to simplify the work by taking $A, B, C, \dots$ as constants, with A, B , equal to ± 1 , and a, b equal to $\pm i\pi$; we shall now simplify it further by supposing $c, \dots$ to be in the strip between the lines $y = \pm \pi$, so that this strip may take the place of the wedges of § 9.

Let

$$\left. \begin{aligned} S_1(\lambda) &= e^{i\pi\lambda} - e^{-i\pi\lambda} + C_1 e^{c_1\lambda} + \dots \\ S_2(\lambda) &= e^{-i\pi\lambda} - e^{i\pi\lambda} + C_2 e^{c_2\lambda} + \dots \\ S_3(\lambda) &= S_1(-\lambda-1), \quad S_4(\lambda) = S_2(-\lambda-1) \end{aligned} \right\}, \tag{11.1}$$

so that

$$\begin{aligned} a_1 &= b_2 = a_3 = b_4 = -i\pi, \\ a_2 &= b_1 = a_4 = b_3 = i\pi, \\ \log A_1 &= \log A_3 = i\pi, \quad \log A_2 = \log A_4 = -i\pi, \\ \log B_r &= 0 \quad (r = 1, 2, 3, 4); \end{aligned}$$

$ic_1, ic_2 \dots$ are all between $\pm \pi$.

Let the paths P_1, P_3 be the same, namely, the line $x = -\frac{1}{2}$ from infinity in the third quadrant to infinity in the second quadrant, and let P_2 and P_4 coincide with P_1 or P_3 reversed. Suppose also that none of the zeros of the functions S_1, S_2 lie on this path; then the zeros must all lie outside a strip bounded by lines $x = -\frac{1}{2} \pm g$, where g is a certain positive quantity.

Let

$$f_{rs}(t) = \int_{(P_r)} \frac{e^{\lambda(t-i\pi)}}{\eta_r(\lambda)\eta_s(\lambda)} d\lambda \quad (r = 1, 3; s = 2, 4) \tag{11.2}$$

as at (9.1). This function is not affected by shifting the path through g to the right or left, and therefore, when $R(t)$ is positive,

$$f_{rs}(t) \text{ is } O(e^{-t(\frac{1}{2}+g)}), \tag{11.21}$$

and, when $R(t)$ is negative,

$$f_{rs}(t) \text{ is } O(e^{-t(\frac{1}{2}-g)}). \tag{11.22}$$

These hold also for the continuations of the function, given by such relations as (9.3).

Thus the integral*

$$\int_{-\infty}^{\infty} f_{12}(a+t)f_{34}(a+u)e^a da \tag{11.3}$$

* Suggested by the work of § 4.

converges absolutely if the path of integration is the real axis and if t, u are near that axis; for towards the lower limit the integrand is $O(e^{2ga})$ and towards the upper $O(e^{-2ga})$. If t or u approaches one of the lines $y = \pm\pi$ the path must be deformed to avoid the singularities of f_{12}, f_{34} at $\pm i\pi$, but this is not possible if $t-u = \pm 2i\pi$.

The integral (11.3) is

$$\int_{-\infty}^{\infty} \int_{(P_1)} \int_{(P_1)} \frac{e^{a(1+\lambda+\mu)+\lambda(t-i\pi)+\mu(u-i\pi)}}{\eta_1(\lambda)\eta_2(\lambda)\eta_3(\mu)\eta_4(\mu)} d\lambda d\mu da. \quad (11.31)$$

where the a -integration is taken last: in order to take it first we secure absolute convergence by breaking the a -path at 0, and shifting the λ -path slightly to the right when $a < 0$, and to the left when $a > 0$, so that $R(\lambda+\mu+1)$ is positive when $a \rightarrow -\infty$ and negative when $a \rightarrow +\infty$. Thus

$$\int_{-\infty}^0 e^{a(1+\lambda+\mu)} da = \frac{1}{\lambda+\mu+1} \quad (11.32)$$

and

$$\int_0^{\infty} e^{a(1+\lambda+\mu)} da = -\frac{1}{\lambda+\mu+1}, \quad (11.33)$$

and in the λ -integration the first of these is to be used on a path $x = -\frac{1}{2}+g$ (upwards) and the second on a path $x = -\frac{1}{2}-g$ (also upwards). The net effect is that the first is used on a simple circuit described positively about $-\mu-1$, so that (11.3) becomes

$$2i\pi \int_{(P_1)} \frac{e^{-(\mu+1)(t-i\pi)+\mu(u-i\pi)}}{\eta_1(-\mu-1)\eta_2(-\mu-1)\eta_3(\mu)\eta_4(\mu)} d\mu. \quad (11.34)$$

$$\text{But now} \quad S_3(z) = S_1(-z-1), \quad S_4(z) = S_2(-z-1). \quad (11.4)$$

Here, when λ is on the right of P_1 , $-\lambda-1$ is on the left, and by (7.2)

$$\begin{aligned} \varpi_1(-\lambda-1) - 2i\pi \frac{d^2 \log S_1(-\lambda-1)}{d\lambda^2} &= \int_{(P_1)} \frac{d^2 \log S_1(z)}{dz^2} \frac{dz}{-\lambda-1-z} \\ &= - \int_{(P_1)} \frac{d^2 \log S_3(z)}{dz^2} \frac{dz}{\lambda-z} \end{aligned}$$

(by the substitution of $-z-1$ for z)

$$= -\varpi_3(\lambda). \quad (11.51)$$

$$\text{Similarly} \quad \varpi_4(\lambda) + \varpi_2(-\lambda-1) = 2i\pi \frac{d^2 \log S_2(-\lambda-1)}{d\lambda^2}.$$

By addition,

$$\frac{d^2}{d\lambda^2} \log \frac{\eta_1(-\lambda-1) \eta_2(-\lambda-1) \eta_3(\lambda) \eta_4(\lambda)}{S_1(-\lambda-1) S_2(-\lambda-1)} = 0, \tag{11.52}$$

and thus, with the help of (8.51), (8.52),

$$\log \frac{\eta_1(-\lambda-1) \eta_2(-\lambda-1) \eta_3(\lambda) \eta_4(\lambda)}{S_3(\lambda) S_4(\lambda)} = 2i\pi; \tag{11.53}$$

so that

$$\begin{aligned} \int_{-\infty}^{\infty} f_{12}(a+t) f_{34}(a+u) e^a da &= -2i\pi \int_{(P_1)} \frac{e^{\mu u - \mu t - t}}{S_3(\mu) S_4(\mu)} d\mu \\ &= -2i\pi \int_{(P_1)} \frac{e^{\lambda t - \lambda u - u}}{S_1(\lambda) S_2(\lambda)} d\lambda \end{aligned} \tag{11.6}$$

by the substitution $\lambda = -\mu - 1$. The a -path has been taken along the real axis, but it may be shifted into a parallel position, provided that on the path $I(t+a)$ and $I(u+a)$ are both between $\pm\pi$, and thus (11.6) may be continued to any values of t, u such that $I(t), I(u)$ differ by less than 2π . The result (11.6) seems to correspond to (4.3), but it is not at once clear what is to correspond to (4.30). At this point the division of S into two parts T, U , as at (6.13), is useful.

12. Let θ be a real quantity between $\pm\pi$ and not equal to any one of $I(c_3), I(c_4), \dots$. Let

$$S_r(z) = T_r(z) + U_r(z) \quad (r = 3, 4),$$

where T_r includes the terms in which the coefficient of z in the exponent (such as c_3, c_4) falls above the line $y = \theta$. Denote

$$T_4(z) U_3(z) - T_3(z) U_4(z)$$

by $S_5(z)$ or by $A_5 e^{a_5 z} + B_5 e^{b_5 z} + \dots$. Then

$$\begin{aligned} \int_{-\infty}^{\infty} f_{12}(a+t) S_5\left(\frac{d}{du}\right) f_{34}(a+u) \cdot e^a da &= -2i\pi e^{-t} \int_{(P_1)} \frac{S_5(\mu) e^{\mu u - \mu t}}{S_3(\mu) S_4(\mu)} d\mu \\ &= -2i\pi e^{-u} \int_{(P_1)} \frac{S_5(-\lambda-1) e^{\lambda t - \lambda u}}{S_1(\lambda) S_2(\lambda)} d\lambda, \end{aligned} \tag{12.1}$$

provided that $u-t+a_5, u-t+b_5, \dots$ all have lateral parts between $\pm 2i\pi$. Now the lateral parts of $a_5, b_5, \dots$ are all between $i\theta \pm i(\pi-0)$, and therefore the expression (12.1) is valid if $I(t-u)$ is between $\theta \pm (\pi+0)$.

The separation of S_r into T_r and U_r ($r = 3, 4$) depends on θ , but is the same for values of θ within a certain range, say so long as $\theta_1 > \theta > \theta_2$.

Then we shall restrict u so that $I(u) + \theta_1$ is positive and $I(u) + \theta_2$ is negative, and (12.1) will be valid at least so long as $|I(t)| \leq \pi$, θ being equal to $-I(u)$.

Since $S_5 = -T_3 S_4 + T_4 S_3$, the third member of (12.1) may be written, according to (10.4),

$$2i\pi \left[T_3 \left(\frac{\partial}{\partial u} \right) \{ e^{-u} f_{11}(t-u) \} + T_4 \left(\frac{\partial}{\partial u} \right) \{ e^{-u} f_{22}(t-u) \} \right]. \quad (12.11)$$

When $|I(t)| \leq \pi$, let $F(t, u)$ be written for

$$\frac{1}{2i\pi} \int_0^\infty f_{12}(\alpha+t) e^\alpha S_5 \left(\frac{\partial}{\partial u} \right) f_{34}(\alpha+u) d\alpha - T_3 \left(\frac{\partial}{\partial u} \right) \{ e^{-u} f_{11}(t-u) \} \quad (12.21)$$

$$\text{or} \quad -\frac{1}{2i\pi} \int_{-\infty}^0 f_{12}(\alpha+t) e^\alpha S_5 \left(\frac{\partial}{\partial u} \right) f_{34}(\alpha+u) d\alpha + T_4 \left(\frac{\partial}{\partial u} \right) \{ e^{-u} f_{22}(t-u) \}, \quad (12.22)$$

which are equal by (12.11).

Consider the second part of (12.21). It is certain that $T_3(z)$ contains the term $e^{i\pi z}$ and thus

$$T_3 \left(\frac{\partial}{\partial u} \right) \{ e^{-u} f_{11}(t-u) \}$$

must contain

$$e^{-u-i\pi} f_{11}(t-u-i\pi),$$

which, as a function of t , has a pole at u with residue $-e^{-u}$ and has no other singularity when $I(t-u) \geq 0$ and $I(t-u) < 2\pi$ [see (6.11)].

If $T_3(z)$ contains also $C_3 e^{c_3 z}$ then

$$T_3 \left(\frac{\partial}{\partial u} \right) \{ e^{-u} f_{11}(t-u) \}$$

contains

$$C_3 e^{-u-c_3} f_{11}(t-u-c_3),$$

which has no singularity so long as $|I(t-u-c_3)| < \pi$. Now

$$I(u) = -\theta, \quad \pi > I(c_3) \geq \theta_1,$$

and hence this inequality is satisfied so long as $I(t) \geq I(u)$ and $I(t)$ is less than $\pi + \theta_1 - \theta$. So also for any other terms of $T_3(z)$.

Again, $f_{11}(t-u)$ is reduced to zero by the operation

$$S_1 \left(-\frac{\partial}{\partial u} \right) \quad \text{or} \quad S_3 \left(-1 + \frac{\partial}{\partial u} \right) \quad \text{or} \quad e^u \left\{ T_3 \left(\frac{\partial}{\partial u} \right) + U_3 \left(\frac{\partial}{\partial u} \right) \right\} e^{-u},$$

so that the second part of (12.21) may also be written

$$U_3 \left(\frac{\partial}{\partial u} \right) \{ e^{-u} f_{11}(t-u) \}, \quad (12.3)$$

and is seen to have no singularity when $I(u) > I(t) > -\pi + \theta_2 - \theta$. Hence the second part of (12.21) has the singularity of $e^{-u}/(t-u)$, and no other, so long as $\pi + \theta_1 - \theta > I(t) > -\pi + \theta_2 - \theta$. The same is true of the second part of (12.22).

When $R(t)$ is positive, $f_{12}(t)$ has no singularity in a half-strip in which $\pi + \tau > I(t) > -\pi - \nu$, where τ, ν are positive, and hence in this region the first part of (12.21) has no singularity. We may suppose that

$$\tau < \theta_1 - \theta, \quad \nu < \theta - \theta_2,$$

so that over the same half-strip the second part of (12.21) has no singularity but the pole at u , and the whole of (12.21) is a valid representation of $F(t, u)$. Then, if

$$R(t) > 0 \quad \text{and} \quad \tau > I(t) > -\nu,$$

$$\text{it follows, from (9.4), that } S_1 \left(\frac{\partial}{\partial t} \right) F(t, u) = 0, \quad (12.41)$$

and $R(t, u)$ may be continued over the half-plane $R(t) > 0$ by means of (12.41).

Similarly, if $R(t) < 0$, by (9.3),

$$S_2 \left(\frac{\partial}{\partial t} \right) F(t, u) = 0, \quad (12.42)$$

and by means of (12.42), $F(t, u)$ may be continued over the half-plane $R(t) < 0$.

When t is near $i\pi$ the form of $f_{12}(t)$ is given at (10.1), and by a method closely analogous to that of (4.6) we find that $F(t, u)$ is

$$O \{ \log(t - i\pi) \}.$$

The only singularities of $F(t, u)$ in the strip where $|I(t)| \leq \pi$ are the pole at u with residue e^{-u} and the logarithmic singularities at $\pm i\pi$.

When $R(t)$ is great and negative the form (12.22) shows that $F(t, u)$ is $O(e^{-t(\frac{1}{2}-\vartheta)})$, and when $R(t)$ is great and positive (12.21) shows that $F(t, u)$ is $O(e^{-t(\frac{1}{2}+\vartheta)})$.

13. *Discovery of the integral equation.*

The value of $\int \frac{F(z, u)}{e^z - e^v} e^z dz$, round a contour enclosing u and v , and no other singularity of the integrand, is

$$2i\pi F(v, u) + 2i\pi/(e^u - e^v). \quad (13.1)$$

Take $|I(v)|$ to be less than π , and suppose the contour to expand so as to contain the whole of the strip $-\pi < y < \pi$, except regions round the points $\pm i\pi$ and towards infinity, which regions are to be continually diminished. From the properties of $F(z, u)$, proved in § 12, it follows that these regions may be neglected in the limit, and thus

$$2i\pi F(v, u) + \frac{2i\pi}{e^u - e^v} = \int_{-\infty}^{\infty} \left\{ \frac{F(z - i\pi, u)}{e^z + e^v} - \frac{F(z + i\pi, u)}{e^z + e^v} \right\} e^z dz,$$

which, by (12.41) and (12.42), is equal to

$$\int_0^{\infty} \{C_1 F(z + c_1, u) + \dots\} \frac{e^z dz}{e^z + e^v} - \int_{-\infty}^0 \{C_2 F(z + c_2, u) + \dots\} \frac{e^z dz}{e^z + e^v}. \quad (13.2)$$

Thus, if

$$\chi(v, u) = F(\log v, \log u),$$

$$\begin{aligned} \chi(z, u) &= \frac{1}{z - u} - \frac{1}{2i\pi} \int_0^1 \{C_2 \chi(te^{c_2}, u) + \dots\} \frac{dt}{t + z} \\ &\quad + \frac{1}{2i\pi} \int_1^{\infty} \{C_1 \chi(te^{c_1}, u) + \dots\} \frac{dt}{t + z} \\ &= \frac{1}{z - u} - \frac{C_2}{2i\pi} \int_0^{\exp c_2} \frac{\chi(t, u) dt}{t + z \exp c_2} - \dots + \frac{C_1}{2i\pi} \int_{\exp c_1}^{\infty} \frac{\chi(t, u) dt}{t + z \exp c_1} + \dots \end{aligned} \quad (13.3)$$

Hence $\chi(z, u)$ is a value of $\phi(z)$ satisfying the integral equation

$$\phi(z) = g(z) - \frac{C_2}{2i\pi} \int_0^{\exp c_2} \frac{\phi(t) dt}{t + z \exp c_2} - \dots + \frac{C_1}{2i\pi} \int_{\exp c_1}^{\infty} \frac{\phi(t) dt}{t + z \exp c_1} + \dots, \quad (13.4)$$

if the known function $g(z)$ has the special value $(z - u)^{-1}$. Moreover, our knowledge of $\chi(z, u)$ enables us to write down a solution of (13.4) in general, namely

$$\begin{aligned} \phi(z) &= g(z) - \frac{C_2 e^{-c_2}}{2i\pi} \int_0^{\exp c_2} g(t) \chi(z, -te^{-c_2}) dt - \dots \\ &\quad + \frac{C_1 e^{-c_1}}{2i\pi} \int_{\exp c_1}^{\infty} g(t) \chi(z, -te^{-c_1}) dt + \dots, \end{aligned} \quad (13.5)$$

provided that, when the right-hand side of (13.5) is put for $\phi(z)$ in (13.4), the order of the integrations may be changed in the resulting double integrals.

The number of integrals of either kind on the right in (13.4) is not restricted, and the terms in (13.5) correspond. The equation (1.1) is the special case of (13.4), in which there is only one integral, that which depends on C_2 , c_2 , and in which $c_2 = 0$.

There is an apparent ambiguity in (13.5), since the construction of F or χ (12.21) involves S_5 , and depends on the amplitude of the second argument of χ , which here may be taken in each integral as $+\pi$ or $-\pi$. However, when $\text{amp } u$ is $+\pi-0$, $S_5(z)$ is $-e^{-i\pi z}\{S_3(z)+S_4(z)\}$, and when $\text{amp } u$ is $-\pi+0$, $S_5(z)$ is $-e^{i\pi z}\{S_3(z)+S_4(z)\}$, so that $\chi(z, u)$ is the same function of z whichever amplitude is taken, and there is no ambiguity.

Thus, finally, to solve such an equation as (13.4) where $C_1, \dots, C_2, \dots, \exp c_1, \dots, \exp c_2, \dots$ are constants, we first form $S_1(z), S_2(z)$ as at (11.1), and then, using for the path P the line $x = -\frac{1}{2}$, resolve each of them into factors as at (8.6). Having ascertained that (8.12) holds, we then form the function f_{12} , as at (9.1), and the similar function f_{34} from $S_1(-z-1)$ and $S_2(-z-1)$. From these we form $F(t, u)$, as at (12.21), and the function $\chi(z, u)$ in (13.5) is $F(\log z, \log u)$.

14. Effect of renouncing the assumption (8.12).

The discussion has been simplified by the important assumption (8.12), namely, that the change in $\log \{S_2(z)/S_1(z)\}$, from one end of P_1 to the other, is 0; this is equivalent to assuming that the sum of the coefficients of λ^{-2} in the expansions of $\varpi_1(\lambda), \varpi_2(\lambda)$, as at (7.1), is zero, or that when k changes gradually from 0 to 1, as many zeros of $kS_1-(1-k)S_2$ cross the path P_1 from left to right as from right to left.

The limit of S_2/S_1 towards either end of P_1 is -1 , and thus the change in its logarithm can only be a multiple of $2i\pi$, which multiple may conceivably be altered to any other by passing P_1 over zeros of S_1 or S_2 . If the logarithm at the upper end exceeds that at the lower by $2ni\pi$ we may put

$\log B_1 = \log B_2 = 0, \quad \log A_2 = -i\pi, \quad \log A_1 = -(2n-1)i\pi, \quad (14.1)$

taking λ , as before, to be primarily at the lower end. Then

$\varpi_1(\lambda) + \varpi_2(\lambda) = \frac{2ni\pi}{\lambda^2} + O\left(\frac{1}{\lambda^3}\right) \quad \text{generally;} \quad (14.2)$

a term $-2n\pi/\lambda$ must be added on the right in (8.41), (8.42), and $-n \log \lambda$ in (8.51), (8.52).

Section 9 is unaffected, but the series in (10.2) is

$$\lambda^n \left(1 + \frac{h_1}{\lambda} + \frac{h_2}{\lambda^2} + \dots \right), \quad (14.3)$$

and the pole in $\omega(t)$ is of the order $n+1$ if n is positive, absent if n is negative. Sections 11, 12 are unaffected, except in the matter of the singularity of $F(t, u)$ when $t \rightarrow \pm i\pi$.

When n is negative, (13.2) and (13.3) still hold. It may also be proved, by discussing $\int \frac{f_{12}^{(r)}(z)}{e^z - e^v} e^z dz$, that

$$2i\pi f_{12}^{(r)}(v) = \int_0^\infty \{ C_1 f_{12}^{(r)}(z + c_1) + \dots \} \frac{e^z dz}{e^z + e^v} - \int_{-\infty}^0 \{ C_2 f_{12}^{(r)}(z + c_2) \dots \} \frac{e^z dz}{e^z + e^v} \\ (r = 0, 1, \dots, -n-1), \quad (14.4)$$

so that $f_{12}^{(r)}(\log z)$, the r -th derivative of $f_{12}(\log z)$, satisfies the equation (13.4) when 0 is put for $g(z)$; the solution of this equation therefore contains linearly $-n$ arbitrary constants, in addition to the expression given in (13.5).

When n is positive, $F(t \pm i\pi, u)$ take the forms $\Phi(t) \log t + \Omega(t) \pm \Theta(t)$ respectively where $\Phi(t)$ and $\Theta(t)$, as also $t^n \Omega(t)$, are regular near the origin. It is possible, and will prove convenient, to identify the singular part of $\Omega(t)$ with that of

$$\sum_{r=1}^n Q_r (1 - e^t)^{-r},$$

where, for each value of r , Q_r is a certain linear combination of $S_5(d/du) f_{34}(u)$ and its first $n-1$ derivatives, with constant coefficients: we shall also denote Q_r by $Q_r(e^n)$. The corresponding part of $F(t, u)$ when t approaches $i\pi$ or $-i\pi$ is

$$\sum Q_r (1 + e^t)^{-r} \quad \text{or say} \quad Q(e^t, e^u).$$

Then (13.2) must be increased on account of the regions round $\pm i\pi$ by

$$\int \frac{Q(-e^z, e^u)}{e^z + e^v} e^z dz,$$

taken negatively round a small circle about the origin, that is, by $2i\pi Q(e^v, e^u)$.

Hence, in (13.3), we change the first member to

$$\chi(z, u) - Q(z, u). \quad (14.5)$$

Then, if we write for the second members of (13.4) and (13.5) respectively $g(z) + M\phi(z)$ and $g(z) + Ng(z)$, it is a consequence of (13.5) that

$$M\{\phi(z) - g(z)\} = MN g(z)$$

$$\begin{aligned} &= (N - M)g(z) + \frac{C_2 e^{-c_2}}{2i\pi} \int_0^{\exp c_2} g(t) Q(z, -te^{-c_2}) dt \\ &\quad + \dots - \frac{C_1 e^{-c_1}}{2i\pi} \int_{\exp c_1}^{\infty} g(t) Q(z, -te^{-c_1}) dt. \end{aligned} \quad (14.6)$$

Thus $\phi(z) - g(z) - M\phi(z)$ or $Ng(z) - M\phi(z)$ is equal to

$$\frac{K_1}{z+1} + \dots + \frac{K_n}{(z+1)^n}$$

where

$$\begin{aligned} K_r &= -\frac{C_2 e^{-c_2}}{2i\pi} \int_0^{\exp c_2} g(t) Q_r(-te^{-c_2}) dt \dots \\ &\quad + \frac{C_1 e^{-c_1}}{2i\pi} \int_{\exp c_1}^{\infty} g(t) Q_r(-te^{-c_1}) dt + \dots \end{aligned}$$

and (13.4) is only satisfied if $K_1, \dots, K_n$ all vanish, which is the case, from the construction of Q_r , if the function g satisfies the n conditions

$$\begin{aligned} &C_2 e^{-c_2} \int_0^{\exp c_2} g(t) \left(t \frac{d}{dt}\right)^r S_5\left(t \frac{d}{dt}\right) f_{34}(\log t - c_2 - i\pi) dt + \dots \\ &= C_1 e^{-c_1} \int_{\exp c_1}^{\infty} g(t) \left(t \frac{d}{dt}\right)^r S_5\left(t \frac{d}{dt}\right) f_{34}(\log t - c_1 - i\pi) dt + \dots, \end{aligned} \quad (14.7)$$

where $S_5(z) = -e^{i\pi z} \{S_3(z) + S_4(z)\}$, so that for

$$\left(t \frac{d}{dt}\right)^r S_5\left(t \frac{d}{dt}\right) f_{34}(\log t - c - i\pi)$$

we may write

$$-\left\{S_3\left(\frac{d}{d \log t}\right) + S_4\left(\frac{d}{d \log t}\right)\right\} f_{34}^{(r)}(\log t - c).$$

Some further investigation is needed to complete the theory of the equation (13.4) and of its adjoint, which is not quite of the same form, and also of the similar results when $C_1, C_2, \dots$ are not constants, but are quantities as in § 6A.

15. Another integral equation of interest is

$$\phi(z) = g(z) + \theta \int_{-i\infty}^{i\infty} \Gamma(z-t) \phi(t) dt, \quad (15.1)$$

which can, in fact, be transformed into one of the class discussed in my former paper [*Proc. Lond. Math. Soc.* (2), 22 (1923), 216-9]. For definiteness, take z to be on the right of the path of integration.

Put as a trial $\phi(z) = a^z$, where $|\text{amp } a| < \frac{1}{2}\pi$; then

$$g(z) = a^z - \theta \int_{-i\infty}^{i\infty} \Gamma(z-t) a^t dt = (1 - 2i\pi\theta e^{-a}) a^z. \quad (15.2)$$

Thus, if

$$g(z) = \int_0^\infty f(a) a^{z-1} da \quad (15.31)$$

where

$$f(a) = \frac{1}{2i\pi} \int_{-i\infty}^{i\infty} g(z) a^{-z} dz, \quad (15.32)$$

we are led to put

$$\begin{aligned} \phi(z) &= \int_0^\infty \frac{f(a) a^{z-1}}{1 - 2i\pi\theta e^{-a}} da = g(z) + 2i\pi\theta \int_0^\infty \frac{f(a) e^{-a} a^{z-1}}{1 - 2i\pi\theta e^{-a}} da \\ &= g(z) + \theta \int_0^\infty \int_{-i\infty}^{i\infty} \frac{e^{-a} a^{z-t-1} g(t)}{1 - 2i\pi\theta e^{-a}} dt da \\ &= g(z) + \theta \int_{-i\infty}^{i\infty} g(t) \chi(z-t) dt, \end{aligned} \quad (15.4)$$

by a change in the order of the integrations, if

$$\chi(z) = \int_0^\infty \frac{a^{z-1} da}{e^a - 2i\pi\theta}. \quad (15.5)$$

This integral defines $\chi(z)$ as a regular function so long as $R(z)$ is positive. It may be written

$$\int_1^\infty \frac{a^{z-1}}{e^a - 2i\pi\theta} da + \int_0^1 a^{z-1} \left\{ \frac{1}{e^a - 2i\pi\theta} - \frac{1}{1 - 2i\pi\theta} \right\} da + \frac{1}{z(1 - 2i\pi\theta)}. \quad (15.51)$$

which defines $\chi(z)$ so long as $R(z) > -1$, and which, when

$$0 > R(z) > -1,$$

may be written

$$\int_0^\infty \left\{ \frac{a^{z-1}}{e^a - 2i\pi\theta} - \frac{a^{z-1}}{1 - 2i\pi\theta} \right\} da.$$

Similarly, the function may be continued into the strip $-1 > R(z) > -2$, and so on. It is regular over the whole plane except for simple poles at $0, -1, -2, \dots$, and for an essential singularity at infinity. When $2i\pi\theta = 1$, there is an extra pole at $+1$, and $\chi(z)/\Gamma(z)$ is Riemann's function $\zeta(z)$.

To verify its connexion with the integral equation (15 . 1) we discuss

$$\int_{-i\infty}^{i\infty} \chi(z-t) \Gamma(t-u) dt, \tag{15 . 6}$$

where the path of t is supposed to be parallel to the axis of y and to separate z on the right from u on the left. The integral may be written

$$\int_{-i\infty}^{i\infty} \int_0^\infty \int_0^\infty \frac{a^{z-t-1}}{e^u - 2i\pi\theta} e^{-b} b^{t-u-1} da db dt. \tag{15 . 61}$$

Here the amplitude of b/a need not be the same for all values of t , and to secure absolute convergence we shall suppose it to have the same sign as $I(t)$; then the t -integration may be taken first.

Thus we break the t -path where it crosses the real axis, say, at k , and have to integrate

$$\frac{a^{z-k-1} b^{k-u-1}}{e^b(e^a - 2i\pi\theta)} \div \log \frac{b}{a} \tag{15 . 62}$$

with respect to a , first from 0 to ∞ above b , and then from ∞ to 0 below b ; this is equivalent to a circuit about b negatively, and thus (15 . 61) reduces to

$$\int_0^\infty \frac{b^{z-u-2}}{e^b(e^b - 2i\pi\theta)} 2i\pi b db \tag{15 . 63}$$

or
$$\theta^{-1} \chi(z-u) - \theta^{-1} \Gamma(z-u). \tag{15 . 64}$$

Thus
$$\chi(z-u) = \Gamma(z-u) + \theta \int_{-i\infty}^{i\infty} \chi(z-t) \Gamma(t-u) dt, \tag{15 . 71}$$

so long as $R(z-u)$ is positive, and by continuation for other positions. Thus $\chi(z-t)$ is the solving nucleus for (15 . 1). By putting $-u, -t, -z$ for z, t, u in (15 . 71) we also have

$$\chi(z-u) = \Gamma(z-u) + \theta \int_{-i\infty}^{i\infty} \Gamma(z-t) \chi(t-u) dt. \tag{15 . 72}$$

If $|2\pi\theta| > 1$, the logarithms of $2i\pi\theta$, say $b_1, b_2, \dots$, lie to the right of the lateral axis, and a change of path in (15 . 5) might increase $\chi(z)$

by an expression

$$B_1 b_1^z + B_2 b_2^z + \dots, \quad (15.81)$$

but would not affect the work. In fact, if such an expression as (15.81) with arbitrary constant coefficients $B_1, B_2, \dots$ were added to $\phi(z)$ in (15.1), the value of $g(z)$ would be unaffected. Thus the solution of (15.1) is not unique, but contains a complementary function of the form (15.81). Similarly the solution of the adjoint equation contains a complementary function

$$B'_1 b_1^{-z} + B'_2 b_2^{-z} + \dots \quad (15.82)$$

The restriction that $|2\pi\theta| > 1$ is not necessary for this in the case of (15.1) itself, if the path of integration is bent into the first and fourth quadrants; but in the case of the adjoint the bending must be the other way.

MEAN-VALUE THEOREMS IN THE THEORY OF THE RIEMANN ZETA-FUNCTION

By A. E. INGHAM.

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Introduction.

The present paper falls into two parts. The first part (§§ 1-5) deals with the asymptotic behaviour of the integral

$$(1) \quad \int_1^T \xi^{(\mu)}(\alpha+ti) \xi^{(\nu)}(\beta-ti) dt$$

as $T \rightarrow \infty$; here μ and ν are positive integers or zero, $\xi^{(\mu)}(s)$ is the μ -th derivative of Riemann's Zeta-function, and α and β are supposed real. It is known that, if α and β are fixed and $\alpha > -\frac{1}{2}$, $\beta > -\frac{1}{2}$, $\alpha+\beta > 1$, then

$$(2) \quad \int_1^T \xi^{(\mu)}(\alpha+ti) \xi^{(\nu)}(\beta-ti) dt \sim \xi^{(\mu+\nu)}(\alpha+\beta) T;$$

this formula was proved in successive stages by Hadamard, Landau, and Schnee, and is now classical*.

For the special case $\mu = \nu = 0$, $\alpha = \beta (= \sigma)$, that is for the integral

$$J_2(T, \sigma) = \int_1^T |\xi(\sigma+ti)|^2 dt,$$

* See Schnee **1**, Landau **H**, 2, 806-819, 905-906, and the references there given. (The references **H**, **1**, **2**, ... are to the bibliography at the end of this paper.) Actually these authors take the integral over the range $(-T, T)$; this necessitates a slight modification of the formula (2) when $\alpha = 1$ or $\beta = 1$, to allow for poles of the integrand. It is easy to see, however, that our equation (2) is equivalent to the formulae given in Landau's *Handbuch*.

much more is known. In the first place, asymptotic formulae have been found for *every* fixed value of σ , namely

$$(3) \quad J_2(T, \sigma) \sim \zeta(2\sigma) T \quad (\sigma > \tfrac{1}{2}).$$

$$(4) \quad J_2(T, \sigma) \sim (2\pi)^{2\sigma-1} \frac{\zeta(2-2\sigma)}{2-2\sigma} T^{2-2\sigma} \quad (\sigma < \tfrac{1}{2}),$$

$$(5) \quad J_2(T, \tfrac{1}{2}) \sim T \log T.$$

Of these equations (3) is a particular case of (2), (4) is easily deduced from it by means of the functional equation satisfied by $\zeta(s)^*$, and (5) is due to Hardy and Littlewood†. Further, the formulae (3) and (5) have been considerably refined, and combined into a single formula holding uniformly over a certain range of σ , by Littlewood‡. He proves first a formula of the type

$$(6) \quad \int_T^{T+H} |\zeta(\sigma+it)|^2 dt = \left[L_\sigma(t) \right]_T^{T+H} + O(T^{1-\sigma+\epsilon}) + O(T^\epsilon) + O(HT^{-\frac{1}{2}\sigma+\epsilon})$$

uniformly for $0 \leq H \leq T$, $\frac{1}{2} \leq \sigma \leq 2$; from which may be deduced

$$(7) \quad J_2(T, \sigma) = L_\sigma(T) + O(T^{1-\sigma+\epsilon}) + O(T^\epsilon) + O(T^{1-\frac{1}{2}\sigma+\epsilon})$$

uniformly in $\frac{1}{2} \leq \sigma \leq 2$. Here $L_\sigma(t)$ is an explicit function whose precise form need not be specified at the moment, as we are mainly interested in the order of the error term; $L_\sigma(t)$ is for any fixed σ an elementary function of t .

In this paper we reduce the order of the error term in (7), and obtain corresponding formulae for the general integral (1). So far as $J_2(T, \sigma)$ is concerned, the principal refinement consists in the elimination of the last term on the right of (7). Precisely what we prove is:

$$J_2(T, \sigma) = L_\sigma(T) + O[(T^{1-2\sigma} + T^{1-\sigma} + 1) \log^2 T]$$

uniformly in any bounded interval of σ , a formula which evidently

* This was pointed out by Hardy and Littlewood 1, 123.

† 1, 151-156 (Theorem 2.41); 2, 59-61 (Theorem E).

‡ Littlewood 1, §1. No proofs are given in this note. I am indebted to Mr. Littlewood for placing at my disposal his manuscript containing the proof of (6); the proof depends on the approximate functional equation (Hardy and Littlewood 2). Mr. Littlewood conjectured the possibility of a further refinement such as that effected in the present paper.

includes both (6) and (7)*. It appears, further, that, if σ is confined to a closed (bounded) interval not containing either of the points $\sigma = 0$, or $\sigma = 1$, then the factor $\log^2 T$ in the error term may be replaced by $\log T$. In particular, in the formula for $J_2(T, \frac{1}{2})$, we replace Littlewood's $O(T^{\frac{3}{2}+\epsilon})$ by $O(T^{\frac{1}{2}} \log T)$.

Actually we generalise the integral (1) slightly by replacing a and β by complex numbers a and b . This enables us to deduce the formulae for the general integral (1) by mere differentiation from the special case $\mu = \nu = 0$, and so to effect considerable formal simplifications in the proof. The final conclusion of this part of the paper may now be stated as follows.

Theorem A. *Suppose that h, h', k are fixed,*

$$(8) \quad h > 0, \quad 0 < h' < k,$$

and let

$$(9) \quad a = \alpha + \alpha'i, \quad b = \beta + \beta'i,$$

$$\lambda(\sigma) = \text{Max} \left(0, \frac{1}{2}(1-\sigma), \frac{1}{2}-\sigma \right) = \begin{cases} 0 & (\sigma \geq 1), \\ \frac{1}{2}(1-\sigma) & (0 \leq \sigma \leq 1), \\ \frac{1}{2}-\sigma & (\sigma \leq 0), \end{cases}$$

$$(10) \quad F(T, c) = \int_1^T \left(\xi(c) + \tau^{1-c} \bar{\xi}(2-c) \right) d\tau = \int_1^T f(\tau, c) d\tau,$$

$f(\tau, 1)$ being defined as $\lim_{c \rightarrow 1} f(\tau, c)$.

Let μ and ν be any fixed positive integers or zero. Then (i)

$$(11) \quad \int_k^T \bar{\xi}^{(\mu)}(a+ti) \bar{\xi}^{(\nu)}(b-ti) dt = 2\pi F_{\mu+\nu} \left(\frac{T}{2\pi}, a+b \right) + R_{\mu, \nu}(T, a, b),$$

where

$$(12) \quad F_r(T, c) = \frac{\partial^r F}{\partial c^r} = \int_1^T \frac{\partial^r f}{\partial c^r} d\tau \quad (r = 0, 1, 2, \dots),$$

and

$$R_{\mu, \nu}(T, a, b) = O[T^{\lambda(a)+\lambda(b)}(\log T)^{\mu+\nu+2}]$$

* The extension of the range of σ represents no real generalisation; for (1) the extension to the right of $\sigma = 2$ is trivial, and (2) the behaviour of $J_2(T, \sigma)$ for $\sigma < \frac{1}{2}$ can be inferred, by the functional equation, from its behaviour for $\sigma > \frac{1}{2}$. The new error term $O(T^{1-2\sigma} \log^2 T)$, which dominates when $\sigma < 0$, arises merely from this extension of the range of σ .

uniformly in

$$(13) \quad -h \leq \alpha \leq h, \quad -h' \leq \alpha' \leq h', \quad -h \leq \beta \leq h, \quad -h' \leq \beta' \leq h',$$

as $T \rightarrow \infty$;

(ii) if a (for example) is further restricted so that a lies outside each of the intervals $(-\delta, \delta)$ and $(1-\delta, 1+\delta)$, where δ is a fixed positive number, then the factor $(\log T)^{\mu+\nu+2}$ may be replaced by $(\log T)^{\mu+\nu+1}$.

The existence of the limit defining $f(\tau, 1)$ is most readily seen from the formula

$$(14) \quad f(\tau, c) = Z(c) + \tau^{1-c} Z(2-c) + \int_1^\tau u^{-c} du,$$

where $Z(s)$ is the integral function $\zeta(s) - (s-1)^{-1}$. The same formula shows that $f(\tau, c)$ is continuous in τ and c for $\tau \geq 1$ and all complex c , and is (for fixed $\tau \geq 1$) an integral function of c . This justifies the second of the equations (12), and shows that $F(\mathbf{T}, c)$ is (for fixed $\mathbf{T} \geq 1$) an integral function of c . The function $2\pi F(\mathbf{T}, c)$ is substantially Littlewood's $L_{\frac{1}{2}c}(2\pi\mathbf{T})$, and Theorem A includes all the theorems asserted above about $J_2(T, \sigma)$. To see that it also includes (2), we have only to observe that (as is easily verified) the conditions $\alpha > -\frac{1}{2}$, $\beta > -\frac{1}{2}$, $\alpha + \beta > 1$, imply $\lambda(\alpha) + \lambda(\beta) < 1$.

The second part of the paper (§§ 6, 7), which is entirely independent of the first part, is concerned with the integral

$$J_4(T, \sigma) = \int_1^T |\zeta(\sigma + ti)|^4 dt.$$

Hardy and Littlewood* have proved

$$(15) \quad J_4(T, \sigma) \sim T \sum_1^\infty \frac{d^2(n)}{n^{2\sigma}} = \frac{\zeta^4(2\sigma)}{\zeta^4(4\sigma)} T \quad (\sigma > \tfrac{1}{2}),$$

$$(16) \quad J_4(T, \sigma) = O(T \log^4 T) \text{ uniformly in } \tfrac{1}{2} \leq \sigma \leq 2,$$

where $d(n)$ denotes the number of positive divisors of n . From (15) we deduce, again by the functional equation,

$$(17) \quad J_4(T, \sigma) \sim (2\pi)^{4\sigma-2} \frac{\zeta^4(2-2\sigma)}{\zeta^4(4-4\sigma)} \frac{T^{3-4\sigma}}{3-4\sigma} \quad (\sigma < \tfrac{1}{2}).$$

* 2, 59-67 (Theorem D).

In this paper we discuss $J_4(T, \sigma)$ only for fixed σ , and our object is to fill in the missing value $\sigma = \frac{1}{2}$ by proving

Theorem B. *We have, as $T \rightarrow \infty$,*

$$\int_0^T |\zeta(\tfrac{1}{2} + ti)|^4 dt = \frac{1}{2\pi^2} T \log^4 T + O(T \log^3 T).$$

Hardy and Littlewood prove the formulae (15) and (16) by means of their approximate functional equation for $\zeta(s)$. A new and interesting proof of (15) has recently been found by Carlson*, who deduces the result from a general theorem on Dirichlet's series without appealing to the deeper special properties of the function $\zeta(s)$. Our proof of Theorem B follows the general lines of the Hardy-Littlewood proof. We find, however, that the ordinary approximate functional equation does not suffice for our purpose, and have to appeal to an analogous approximate functional equation for the square of $\zeta(s)$ †. The formula in question is also due to Hardy and Littlewood, but no proof has yet been published. The particular form which we require is‡

$$(18) \quad \zeta^2(\tfrac{1}{2} + ti) = \sum_1^{t/2\pi} d(n) n^{-\frac{1}{2}-ti} + i \left(\frac{t}{2\pi e} \right)^{-2ti} \sum_1^{t/2\pi} d(n) n^{-\frac{1}{2}+ti} + O(\log t),$$

as $t \rightarrow \infty$. This formula will be assumed in the relevant parts of this paper.

A. The mean value of $\zeta^{(\mu)}(a+ti) \zeta^{(\nu)}(b-ti)$.

1. Our first object is to prove Theorem A in the special case $\mu = \nu = 0$. We start from the formula

$$(19) \quad \hat{\zeta}(s) = \frac{1}{2i \cos \frac{1}{2}s\pi} \int_{C(0)} \frac{z^{-s} dz}{e^{-2\pi z} - 1},$$

* 1, 11-12.

† The formula $\int_0^T |\zeta(\tfrac{1}{2} + it)|^4 dt \sim \frac{1}{2\pi^2} T \log^4 T$

was first stated by Ingham (1, Theorem 2). A new proof not depending on approximate functional equations has now been found by E. C. Titchmarsh (1). Titchmarsh applies his method, which is based on the theory of Fourier transforms, to the proof of a number of analogous formulae.

‡ I am indebted to Prof. Hardy and Mr. Littlewood for communicating this result to me; also for suggesting the problem of $J_4(T, \frac{1}{2})$.

where $C(0)$ is a contour of the Hankel type enclosing the negative real axis, encircling the point $z = 0$ once in the positive sense, but excluding the points $\pm ni$ ($n = 1, 2, \dots$), and z^{-s} has its principal value*. We suppose, for simplicity, that the two branches of $C(0)$ go to infinity in two definite directions $\arg z = \pm\Theta$, where $\frac{1}{2}\pi < \Theta < \pi$; it is clear from Cauchy's theorem that the value of the integral is independent of the particular value of Θ . We denote by $C(x)$, where $x \geq 0$, a contour of the same general form as $C(0)$, but including those, and only those, of the points $\pm ni$ ($n = 1, 2, \dots$) for which $n \leq x$; and by C a contour of the same type, but with no restriction relating to the points $\pm ni$.

Now let x and y be any two positive numbers. Transforming (19) by Cauchy's theorem, we obtain

$$\begin{aligned}\zeta(s) &= \sum_1^x m^{-s} + \frac{1}{2i \cos \frac{1}{2}s\pi} \int_{C(x)} \frac{z^{-s} dz}{e^{-2\pi z} - 1} \\ &= \sum_1^x m^{-s} + \frac{1}{2i \cos \frac{1}{2}s\pi} \left(\sum_1^y \int_{C(x)} z^{-s} e^{2n\pi z} dz + \int_{C(x)} \frac{z^{-s} e^{2\pi[y]z}}{e^{-2\pi z} - 1} dz \right).\end{aligned}$$

Changing z to z/n in each term of the second sum, and observing that these integrals may be taken along any contour of the type C , we deduce

$$(20) \quad \zeta(s) = \sum_1^x m^{-s} + \chi(s) \sum_1^y n^{s-1} + Q(x, y, s),$$

where

$$(21) \quad Q(x, y, s) = \frac{1}{2i \cos \frac{1}{2}s\pi} \int_{C(x)} \frac{z^{-s} e^{2\pi[y]z}}{e^{-2\pi z} - 1} dz,$$

$$\chi(s) = \frac{1}{2i \cos \frac{1}{2}s\pi} \int_C z^{-s} e^{2\pi z} dz = 2(2\pi)^{s-1} \sin \frac{1}{2}s\pi \Gamma(1-s),$$

by Hankel's formula for the Gamma function†. We write now

$$\tau = \tau(t) = \frac{t}{2\pi}, \quad Q_a(\pm ti) = Q(\sqrt{\tau}, \sqrt{\tau}, a \pm ti) \quad (t > 0),$$

* The formula (19) is a slight variant of Riemann's well-known contour integral formula for $\zeta(s)$. It is most easily verified by observing that the integral on the right is, for $\Re s > 1$, equal to $-2\pi i$ times the sum of the residues of the integrand at the points $z = \pm ni$ ($n = 1, 2, \dots$).

† See, for example, Whittaker and Watson, *Modern analysis*, § 12.22.

and obtain as particular cases of (20)

$$\zeta(a+ti) = \sum_1^{\sqrt{\tau}} m^{-a-ti} + \chi(a+ti) \sum_1^{\sqrt{\tau}} n^{a-1+ti} + Q_a(ti) = U_1 + U_2 + U_3,$$

$$\zeta(b-ti) = \sum_1^{\sqrt{\tau}} m^{-b+ti} + \chi(b-ti) \sum_1^{\sqrt{\tau}} n^{b-1-ti} + Q_b(-ti) = V_1 + V_2 + V_3;$$

whence
$$I \equiv \int_k^T \zeta(a+ti) \zeta(b-ti) dt = \sum_{i,j=1}^3 \int_k^T U_i V_j dt = \sum_{i,j=1}^3 I_{ij}.$$

The use of formula (20) with this particular choice of x and y is suggested by the approximate functional equation*, and our formulae so far correspond exactly to those used by Littlewood in his proof of (6). Our subsequent argument differs essentially from Littlewood's only in the treatment of those of the integrals I_{ij} (other than I_{33}) which involve Q . It is precisely these integrals which give rise to the error term which it is our object to eliminate. We effect this by appealing directly to the formula (21) instead of merely using the inequality for Q provided by the approximate functional equation.

It will be useful to observe that, as deductions from the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$ combined with (20), we have

$$\chi(s)\chi(1-s) = 1,$$

$$Q(x, y, s) = \chi(s)Q(y, x, 1-s);$$

in particular

$$(22) \quad Q_a(\pm ti) = \chi(a \pm ti)Q_{1-a}(\mp ti).$$

2. Preliminary lemmas. In Lemmas A1-A9 below we suppose always that the numbers h, h', k, a, b , so far as they are actually relevant, satisfy the conditions (8), (9), (13) of Theorem A. We denote by A an absolute positive constant, by K a positive constant depending only on h, h', k , and by $K(p, q, \dots)$ a positive constant depending in addition on the parameters $p, q, \dots$. An A or a K , unless distinguished by a suffix, does not necessarily denote the same constant at different occurrences.

* Hardy and Littlewood 2. It is possible, by using (20) as a starting point, to obtain a proof of the approximate functional equation formally simpler than that given by the authors, though naturally identical in principle.

Lemma A 1. *We have*

$$\left| \chi(a+ti) \chi(b-ti) - \left(\frac{t}{2\pi} \right)^{1-a-b} \right| < K t^{-a-\beta} \quad (t \geq k).$$

This follows easily from the asymptotic formula for the Gamma function. We have, in fact, uniformly in a, b , as $t \rightarrow \infty$,

$$\begin{aligned} \chi(a+ti) \chi(b-ti) &= \frac{\chi(b-ti)}{\chi(1-a-ti)} \\ &= (2\pi)^{a+b-1} \frac{\sin \frac{1}{2}(b-ti)\pi}{\sin \frac{1}{2}(1-a-ti)\pi} \frac{\Gamma(1-b+ti)}{\Gamma(a+ti)} \\ &= (2\pi)^{a+b-1} e^{\frac{1}{2}\pi i(b-1+a)} (1+O(e^{-\pi t})) (ti)^{1-b-a} (1+O(t^{-1})) \\ &= \left(\frac{t}{2\pi} \right)^{1-a-b} (1+O(t^{-1})), \end{aligned}$$

from which the result follows, since $\chi(a+ti) \chi(b-ti)$ is continuous for $t \geq k$ and the stated ranges of a and b .

Lemma A 2. *If $T \geq T_0 > 0$, $\lambda > 0$, $c = \gamma + \gamma' i$, $-2h' \leq \gamma' \leq 2h'$, then*

$$\left| \int_{T_0}^T \tau^{1-c} e^{\pm i\lambda\tau} d\tau \right| < \frac{K}{\lambda} \text{Max} (T^{1-\gamma}, T_0^{1-\gamma}).$$

It is enough, by the second mean-value theorem, to prove this for $\gamma = 1$. Now

$$(23) \quad \left| \lambda \int_{T_0}^T \tau^{-\gamma' i} e^{\pm i\lambda\tau} d\tau \right| = \left| \int_{\lambda T_0}^{\lambda T} u^{-\gamma' i} e^{\pm iu} du \right| = |I_1 + I_2|,$$

where I_1 and I_2 are the contributions of those parts of the interval $(\lambda T_0, \lambda T)$ which lie respectively inside and outside the interval $(0, 4h')$.

Clearly $|I_1| < 4h'$. Also

$$\begin{aligned} |I_2| &= \left| \int_{U_1}^{U_2} \frac{d(u^{-\gamma' i} e^{\pm iu})}{1 \mp (\gamma'/u)} \right| \quad (4h' \leq U_1 \leq U_2), \\ &\leq \text{Max}_{u=U_1, U_2} \frac{1}{1 \mp (\gamma'/u)} \cdot 2 \leq 4, \end{aligned}$$

by the second mean-value theorem, since $\{1 \mp (\gamma'/u)\}^{-1}$ is positive and monotonic for $u \geq 4h'$. Hence

$$|I_1 + I_2| < 4h' + 4,$$

and the lemma follows from (23).

Lemma A3. Suppose that $\mathfrak{S}$, q are positive constants,

$$\xi > 0, \quad \xi \neq 1; \quad T \geq k; \quad T' \geq \mathfrak{S}T, \quad |T' - T| \leq q\sqrt{T};$$

and let

$$J = J(a, T, T') = \int_{L(1)} \left| (z/i)^{-a-T'} e^{Tz} dz \right|,$$

$$J_\xi = J_\xi(a, T, T') = \int_{L(1)} \left| \frac{(z/i)^{-a-T'}}{\log(z/\xi i)} e^{Tz} dz \right|,$$

where $L(x)$ denotes generally the contour consisting of the two straight lines starting from the point x and going to infinity in the directions $\arg z = \pm \frac{3}{4}\pi$. Then

$$J < \frac{K(\mathfrak{S}, q)}{\sqrt{T}}, \quad J_\xi < \frac{K(\mathfrak{S}, q)}{\sqrt{T} |\log \xi|}.$$

Here and in all that follows $\log(z/\xi i)$ is defined, for any positive value of ξ , to be $\log z - \log \xi - \frac{1}{2}\pi i$, where $\log z$ and $\log \xi$ have their principal values; and then

$$\left(\frac{z}{\xi i} \right)^{-s} = \exp \left(-s \log \frac{z}{\xi i} \right).$$

Writing

$$z = -u + vi = re^{i\theta} \quad (r > 0, \quad -\pi < \theta < \pi),$$

we have

$$\left| \left(\frac{z}{i} \right)^{-T'} e^{Tz} \right| = e^{T(\phi+u)-Tu} = e^{T\phi+(T-T')u},$$

where

$$\phi = \phi(z) = \Im \log \frac{z}{i} - u = \theta - \frac{1}{2}\pi - u.$$

On the upper half of L , $u \geq -1$ and $v = 1+u$, so that

$$\phi = \arctan \frac{u}{1+u} - u \leq \begin{cases} -\frac{u^2}{1+u} & (u \geq 0), \\ -u^2(1+\frac{2}{3}u) & (-1 \leq u \leq 0); \end{cases}$$

the first of the inequalities follows from $\arctan x \leq x$ ($x \geq 0$), and the second from

$$\frac{d\phi}{du} = \frac{1}{(1+u)^2 + u^2} - 1 = \frac{-2u(1+u)}{1+2u(1+u)} \geq -2u(1+u) \quad (-1 \leq u \leq 0).$$

These inequalities hold *a fortiori* when z is on the lower half of L , for then $\theta \leq -\theta$, and so $\phi(z) \leq \phi(\bar{z})$.

Now, assuming (as we clearly may) $\mathfrak{S} \leq 1$, we divide J into two parts J_1 and J_2 corresponding to $-1 \leq u \leq U$ and $u \geq U$ respectively, where $1+U = 3/\mathfrak{S}$. In J_1 we have $\phi \leq -\frac{1}{3}\mathfrak{S}u^2$ ($u \geq 0$) and $\phi \leq -\frac{1}{3}u^2$ ($u \leq 0$), and so in either case $\phi \leq -\frac{1}{3}\mathfrak{S}u^2$; also $|(z/i)^{-\alpha}| < K(\mathfrak{S})$ and $|T-T'| \leq q\sqrt{T}$; hence

$$\begin{aligned} J_1 &< K(\mathfrak{S}) \int_{-1}^U e^{-\frac{1}{3}\mathfrak{S}Tu^2 + q\sqrt{T}|u|} du \\ &< \frac{K(\mathfrak{S})}{\sqrt{T}} \int_{-\infty}^{\infty} e^{-\frac{1}{3}\mathfrak{S}w^2 + q|w|} dw = \frac{K(\mathfrak{S}, q)}{\sqrt{T}}. \end{aligned}$$

In J_2 we have $\phi+u \leq u/(1+u) \leq u/(1+U) = \frac{1}{3}\mathfrak{S}u$, $T' \geq \mathfrak{S}T$, and $|(z/i)^{-\alpha}| < K(\mathfrak{S})u^{-\alpha} < K(\mathfrak{S})u^h$; hence

$$\begin{aligned} J_2 &< K(\mathfrak{S}) \int_U^{\infty} u^h e^{-\frac{1}{3}\mathfrak{S}Tu} du < K(\mathfrak{S}) e^{-\frac{1}{3}\mathfrak{S}UT} \int_U^{\infty} u^h e^{-\frac{1}{3}\mathfrak{S}ku} du \\ &< K(\mathfrak{S}) e^{-\frac{1}{3}\mathfrak{S}UT} < \frac{K(\mathfrak{S})}{\sqrt{T}}. \end{aligned}$$

The first inequality $J < K(\mathfrak{S}, q)/\sqrt{T}$ is thus proved. The second is easily deduced. We have, in fact, writing $\log(z/i) = X + Yi$, $\log \xi = W$,

$$\begin{aligned} |\log(z/i) \log(z/\xi i)|^2 &= (X^2 + Y^2)(X^2 - 2XW + W^2 + Y^2) \\ &= (X^2 + Y^2 - XW)^2 + Y^2 W^2 \geq |YW|^2; \end{aligned}$$

hence

$$\left| \frac{\log \xi}{\log(z/\xi i)} \right| \leq \left| \frac{\log(z/i)}{\mathfrak{S} \log(z/i)} \right|,$$

provided that the denominator of the expression on the right is different from zero. Now this expression, when suitably defined at $z = i$, is a continuous function of z on L , and is $O(\log|z|) = O(|z|)$ as $z \rightarrow \infty$ along L ; it is therefore less than $A|z|$ for all z on L . From this it follows that

$$(\log \xi) J_{\xi}(a, T, T') < AJ(a-1, T, T') < \frac{K(\mathfrak{S}, q)}{\sqrt{T}},$$

by the first part of the lemma (with $h+1$ in place of h).

Lemma A 4. If $T \geq T_0 \geq \xi > 0$, $2\pi T_0 \geq k$, then*

$$(24) \quad \left| \int_{2\pi T_0}^{2\pi T} \xi^{ti} \chi(a+ti) dt \right| < K \text{Max} (T^{1-\alpha}, T_0^{1-\alpha}) / \log \frac{T_0 + \frac{1}{2}\sqrt{T_0}}{\xi}.$$

We consider first the corresponding integral in which $\chi(s)$ is replaced by

$$g(s) = (1 + e^{\pi si}) \chi(s) = -i \int_C \left(\frac{z}{i} \right)^{-s} e^{2\pi z} dz.$$

We choose the contour C in such a way that it encloses the point ξi . The last integral being uniformly convergent over any bounded range of s , we have, writing $T = 2\pi T$, $T_0 = 2\pi T_0$, etc.,

$$(25) \quad \begin{aligned} \int_{T_0}^T \xi^{ti} g(a+ti) dt &= -i \int_C \left(\frac{z}{i} \right)^{-a} e^{2\pi z} dz \int_{T_0}^T \left(\frac{z}{\xi i} \right)^{-ti} dt \\ &= G(T) - G(T_0), \end{aligned}$$

where

$$G(T) = \xi^{Ti} \int_C \frac{(z/i)^{-a-Ti} e^{2\pi z}}{\log(z/\xi i)} dz.$$

Now, if $T \geq \xi$, we can take C in the last formula to be the contour $L(T) = L(T + \frac{1}{2}\sqrt{T})$ defined in Lemma A 3. Doing this and changing the variable from z to $T'z$, we obtain

$$|G(T)| \leq T'^{1-\alpha} \int_{L(1)} \left| \frac{(z/i)^{-a-T'i} e^{T'z}}{\log(zT'/\xi i)} dz \right|.$$

Here $0 < T' - T = \pi\sqrt{T} = \sqrt{(\frac{1}{2}\pi T)}$; hence, by Lemma A 3 [with $\mathfrak{S} = 1$, $q = \sqrt{(\frac{1}{2}\pi)}$],

$$(26) \quad |G(T)| < \frac{KT'^{1-\alpha}}{\sqrt{T} \log(T'/\xi)} < KT^{\frac{1}{2}-\alpha} / \log \frac{T + \frac{1}{2}\sqrt{T}}{\xi},$$

if $2\pi T \geq k$, since then $1 < T'/T \leq 1 + \frac{1}{2}\sqrt{(2\pi/k)}$.

* We could also prove Lemma A 4 (in a slightly less precise form) by first approximating to χ by Stirling's theorem and then using known results concerning the integral

$$\int_{T_0}^T t^s \left(\frac{t}{e\xi} \right)^{ti} dt.$$

(Cf. Lemma B 4 below.) This is the method adopted by Littlewood at the corresponding point in his proof of (6). We prefer, however, to use a method more in keeping with those which we have to employ in other parts of the paper.

The formula (24) now follows easily from (25) and (26) by partial integration. We have, in fact,

$$\chi(a+ti) = \omega(t) g(a+ti),$$

where

$$|\omega(t)| < K, \quad |\omega'(t)| < Ke^{-\pi t} \quad (t \geq k);$$

hence

$$\begin{aligned} \left| \int_{T_0}^T \xi^{ti} \chi(a+ti) dt \right| &= \left| \int_{T_0}^T \omega(t) G'(t) dt \right| \\ &\leq \text{Max}_{T_0 \leq t \leq T} |G(t)| \left\{ |\omega(T)| + |\omega(T_0)| + \int_{T_0}^T |\omega'(t)| dt \right\} \\ &\leq K \text{Max}_{T_0 \leq t \leq T} |G(t)| < K \text{Max}(T^{\frac{1}{2}-a}, T_0^{\frac{1}{2}-a}) / \log \frac{T_0 + \frac{1}{2}\sqrt{T_0}}{\xi}, \end{aligned}$$

Lemma A5*.

$$|Q_a(\pm ti)| < Kt^{-\frac{1}{2}a} \quad (t \geq k).$$

It is enough to consider the upper sign, since $Q_a(-ti)$ is the conjugate of $Q_a(ti)$. In (21) we put $s = a+ti$, $x = y = \sqrt{\tau} = \sqrt{(t/2\pi)}$, and take $C(x)$ to be $L([\sqrt{\tau}] + \frac{1}{2})$; we obtain

$$Q_a(ti) = \frac{-i}{1 + e^{\pi i(a+ti)}} \int_{L([\sqrt{\tau}] + \frac{1}{2})} \left(\frac{z}{i}\right)^{-a-ti} e^{2\pi\sqrt{\tau}z} \frac{e^{-2\pi(\sqrt{\tau} - [\sqrt{\tau}])z}}{e^{-2\pi z} - 1} dz.$$

Now $|1 + e^{\pi i(a+ti)}|^{-1} < K$ ($t \geq k$), and on $L([\sqrt{\tau}] + \frac{1}{2})$

$$\left| \frac{e^{-2\pi(\sqrt{\tau} - [\sqrt{\tau}])z}}{e^{-2\pi z} - 1} \right| < A,$$

since $0 \leq \sqrt{\tau} - [\sqrt{\tau}] < 1$ and $|z - mi| \geq \frac{1}{4}\sqrt{2}$ for every integer m .

$$\begin{aligned} \text{Hence} \quad |Q_a(ti)| &< K \int_{L([\sqrt{\tau}] + \frac{1}{2})} \left| \left(\frac{z}{i}\right)^{-a-ti} e^{2\pi\sqrt{\tau}z} dz \right| \\ &= K([\sqrt{\tau}] + \frac{1}{2})^{1-a} \int_{L(1)} \left| \left(\frac{z}{i}\right)^{-a-ti} e^{t'z} dz \right|, \end{aligned}$$

where

$$t' = 2\pi\sqrt{\tau}([\sqrt{\tau}] + \frac{1}{2}).$$

Now

$$t' \geq 2\pi\sqrt{\tau} \text{Max}(\sqrt{\tau} - \frac{1}{2}, \frac{1}{2}) \geq \pi\tau = \frac{1}{2}t,$$

$$|t' - t| = 2\pi\sqrt{\tau} |[\sqrt{\tau}] + \frac{1}{2} - \sqrt{\tau}| \leq \pi\sqrt{\tau} = \sqrt{(\frac{1}{2}\pi t)}.$$

* Lemma A5 is of course immediately deducible from the approximate functional equation. We insert a proof, however, for the sake of completeness.

Hence, by Lemma A 3,

$$|Q_a(ti)| < K([\sqrt{\tau}] + \frac{1}{2})^{1-a} t^{-\frac{1}{2}} < Kt^{-\frac{1}{2}}.$$

Lemma A 6. *If m and n are positive integers, and*

$$n^2 \leq T_1 \leq T_2 \leq (n+1)^2, \quad 2\pi T_1 \geq k,$$

then
$$\left| \int_{2\pi T_1}^{2\pi T_2} m^{ti} Q_a(ti) dt \right| < K n^{-a} \left| \log \frac{n+\frac{1}{2}}{m} \right|^{-1}.$$

As in Lemma A 4, we consider first the corresponding integral with $Q_a(ti)$ replaced by

$$q_a(ti) = (1 + e^{\pi i(a+ti)}) Q_a(ti).$$

For $2\pi n^2 \leq t < 2\pi(n+1)^2$ we have $[\sqrt{\tau}] = n$, and we can take the integral defining $Q_a(ti)$ along the path $L(n+\frac{1}{2})$; therefore

$$q_a(ti) = -i \int_{L(n+\frac{1}{2})} \frac{(z/i)^{-a-ti} e^{2\pi n z}}{e^{-2\pi z} - 1} dz,$$

whence, if $n^2 \leq T_1 \leq T < (n+1)^2$, $T_1 = 2\pi T_1$, $T = 2\pi T$,

$$\int_{T_1}^T m^{ti} q_a(ti) dt = P(T) - P(T_1),$$

where
$$P(T) = m^{Ti} \int_{L(n+\frac{1}{2})} \frac{(z/i)^{-a-Ti} e^{2\pi n z}}{(e^{-2\pi z} - 1) \log(z/mi)} dz.$$

On $L(n+\frac{1}{2})$, $|e^{-2\pi z} - 1|^{-1} < A$; hence

$$\begin{aligned} |P(T)| &< A \int_{L(n+\frac{1}{2})} \left| \frac{(z/i)^{-a-Ti} e^{2\pi n z}}{\log(z/mi)} dz \right| \\ &= A(n+\frac{1}{2})^{1-a} \int_{L(1)} \left| \frac{(z/i)^{-a-Ti} e^{T'z}}{\log(z/\xi i)} dz \right|, \end{aligned}$$

where $T' = 2\pi n(n+\frac{1}{2})$, $\xi = m/(n+\frac{1}{2})$. Now

$$T' > 2\pi n^2 \geq \frac{1}{2}\pi(n+1)^2 > \frac{1}{4}T,$$

$$|T' - T| \leq 2\pi(\frac{3}{2}n+1) \leq 5\pi n \leq 5\pi \sqrt{\left(\frac{T}{2\pi}\right)};$$

hence, by Lemma A 3,

$$|P(T)| < \frac{K(n+\frac{1}{2})^{1-a}}{\sqrt{T} |\log \xi|} < K n^{-a} \left| \log \frac{n+\frac{1}{2}}{m} \right|^{-1}.$$

From this inequality the desired result follows by partial integration, as in Lemma A 4.

Lemma A 7. *Under the same conditions as in Lemma A 6, we have*

$$(27) \quad \left| \int_{2\pi T_1}^{2\pi T_2} m^{ti} \chi(a+ti) Q_b(-ti) dt \right| < K n^{1-2a-\beta} \left(\left| \log \frac{n+\frac{1}{2}}{m} \right|^{-1} + n^{-1} \right).$$

We deduce this from Lemma A 6 by means of the relation (22). This, combined with Lemma A 1, gives, for the integral on the left of (27),

$$\int_{T_1}^{T_2} m^{ti} \left(\frac{t}{2\pi} \right)^{1-c} Q_{1-b}(ti) dt + \rho = J + \rho,$$

say, where $c = a+b = \gamma + \gamma'i$, $T_1 = 2\pi T_1$, $T_2 = 2\pi T_2$, and

$$|\rho| < K \int_{T_1}^{T_2} t^{-\gamma} |Q_{1-b}(ti)| dt < K \int_{T_1}^{T_2} t^{-\gamma-\frac{1}{2}(1-\beta)} dt < K n^{-2a-\beta},$$

by Lemma A 5. Writing now

$$j(T) = \int_{T_1}^T m^{ti} Q_{1-b}(ti) dt,$$

we have, by partial integration and Lemma A 6 (with $h+1$ in place of h),

$$\begin{aligned} |J| &= \left| \left(\frac{T_2}{2\pi} \right)^{1-c} j(T_2) - (1-c)(2\pi)^{c-1} \int_{T_1}^{T_2} t^{-c} j(t) dt \right| \\ &< K(n^{2(1-\gamma)} + n \cdot n^{-2\gamma}) n^{\beta-1} \left| \log \frac{n+\frac{1}{2}}{m} \right|^{-1} < K n^{1-2a-\beta} \left| \log \frac{n+\frac{1}{2}}{m} \right|^{-1}, \end{aligned}$$

which completes the proof of Lemma A 7.

Lemma A 8. *Suppose that $\omega \geq 1$, $\xi > 0$, $\epsilon > 0$, σ real. Then**

$$\begin{aligned} \sum_{\substack{n=1 \\ |n-\xi| \geq \epsilon}}^{\omega} \frac{n+\xi}{|n-\xi|} n^{-\sigma} &< \omega^{\sigma_1-\sigma} \left(\frac{4}{\epsilon} + 5 \sum_1^{\omega} n^{-\sigma_1} \right) \\ &< 5 \text{Max}(\omega^{1-\sigma}, 1) \left(\log \omega + 1 + \frac{1}{\epsilon} \right), \end{aligned}$$

where

$$\sigma_1 = \text{Max}(\sigma, 1).$$

It is enough to prove the first inequality, the second being a trivial consequence.

* This result is of a familiar type. Cf. Landau, **H** 2, 806 Hilfssatz 1.

Suppose first that $\sigma \geq 1$. Denoting the sum on the left by $S_\sigma(\omega)$, we have, since $n + \xi \leq |n - \xi| + 2n$,

$$S_\sigma(\omega) \leq \sum_1^\omega n^{-\sigma} + 2 \sum_{(\epsilon)}^\omega \frac{n^{1-\sigma}}{|n - \xi|},$$

the second sum being taken over the range $1 \leq n \leq \omega$, $|n - \xi| \geq \epsilon$. This sum contains at most two terms for which $|n - \xi| < 1$, and each of these is less than or equal to $1/\epsilon$. Hence

$$S_\sigma(\omega) \leq \sum_1^\omega n^{-\sigma} + 2 \sum_{(1)}^\omega \frac{n^{1-\sigma}}{|n - \xi|} + \frac{4}{\epsilon} = U_\sigma(\omega) + 2\xi_\sigma(\omega) + \frac{4}{\epsilon},$$

say. It is therefore enough, for this case, to prove

$$\xi_\sigma(\omega) < 2U_\sigma(\omega) \quad (\sigma \geq 1).$$

This is immediate for $\sigma = 1$; for, if $\xi_1(\omega)$ contains ω' terms for which $n < \xi$ and ω'' for which $n > \xi$, it is clear that $\xi_1(\omega) \leq U_1(\omega') + U_1(\omega'')$, and this is less than $2U_1(\omega)$ since $\omega' + \omega'' \leq \omega$. The general case follows by partial summation; in fact, if $\sigma \geq 1$, we have

$$\begin{aligned} \xi_\sigma(\omega) &= \sum_1^\omega \{ \xi_1(n) - \xi_1(n-1) \} n^{1-\sigma} \\ &= \sum_1^{\omega-1} \xi_1(n) \{ n^{1-\sigma} - (n+1)^{1-\sigma} \} + \xi_1(\omega) [\omega]^{1-\sigma} \\ &< 2 \sum_1^{\omega-1} U_1(n) \{ n^{1-\sigma} - (n+1)^{1-\sigma} \} + 2U_1(\omega) [\omega]^{1-\sigma} \\ &= 2 \sum_1^\omega \{ U_1(n) - U_1(n-1) \} n^{1-\sigma} = 2U_\sigma(\omega). \end{aligned}$$

The lemma is thus proved for $\sigma \geq 1$. The case $\sigma < 1$ follows immediately; for

$$S_\sigma(\omega) = \sum_{(\epsilon)}^\omega \frac{n + \xi}{|n - \xi|} n^{-1} \cdot n^{1-\sigma} \leq \omega^{1-\sigma} S_1(\omega) \quad (\sigma < 1).$$

Lemma A 9. If $\omega \geq 1$, then*

$$\left| \sum_1^\omega n^{-a} - \zeta(a) - \frac{\omega^{1-a}}{1-a} \right| < K\omega^{-a}.$$

* The expression in $|\dots|$ is supposed defined, when $a = 1$, as $\lim_{a \rightarrow 1}$; this expression may be written as

$$\sum_1^\omega n^{-a} - Z(a) - \int_1^\omega u^{-a} du,$$

where

$$Z(s) = \zeta(s) - (s-1)^{-1}.$$

Suppose first that $a > 1$. Then, by the Euler-Maclaurin sum formula, applied to the series $\sum_{n > \omega} n^{-a}$,

$$\zeta(a) - \sum_1^{\omega} n^{-a} - \frac{\omega^{1-a}}{a-1} = \sum_1^M \frac{a(a+1) \dots (a+m-2)}{m!} \frac{\bar{\phi}_m(\omega)}{\omega^{a+m-1}} - \frac{a(a+1) \dots (a+M-1)}{M!} \int_{\omega}^{\infty} \frac{\bar{\phi}_M(u)}{u^{a+M}} du,$$

where $\bar{\phi}_m(u)$ is for each m a periodic function of u with period 1, bounded in the interval $0 \leq u < 1$, and therefore bounded for all real u . This formula holds for $a > 1-M$, since each side is a regular function of a in this half-plane. In particular, if $M > h+1$, it holds for $-h \leq a \leq h$, $-h' \leq a' \leq h'$. Taking $M = [h] + 3$, we have

$$\left| \sum_1^{\omega} n^{-a} - \zeta(a) - \frac{\omega^{1-a}}{1-a} \right| < K \left(\omega^{-a} + \int_{\omega}^{\infty} \frac{du}{u^{a+M}} \right) \\ = K \omega^{-a} \left(1 + \omega^{1-M} \int_1^{\infty} \frac{dv}{v^{a+M}} \right) < K \omega^{-a},$$

since $\omega \geq 1$, $M > 1$, and $a+M > -h+(h+2) = 2$.

3. *Proof of Theorem A for $\mu = \nu = 0$.* Throughout the proof we write

$$k = 2\pi\kappa, \quad T = 2\pi\mathbf{T}, \quad \text{etc.}, \quad \mathbf{T}_{m,n} = \text{Max}(m^2, n^2, \kappa),$$

$$\tau = \tau(t) = \frac{t}{2\pi}, \quad c = a+b = \gamma + \gamma' i, \quad \Sigma' = \sum_{m \neq n},$$

and suppose always that $\mathbf{T} \geq \text{Max}(\kappa, 1)$. All O 's are uniform in the range (13) of a, b .

We revert to the formulae of § 1 and discuss the nine integrals I_{ij} in turn. We have first

$$(28) \quad I_{11} = \int_k^T \sum_{m, n=1}^{\sqrt{\tau}} m^{-a-ti} n^{-b+ti} dt = I'_{11} + I''_{11},$$

where I'_{11} is the contribution of those terms for which $m = n$. Now

$$(28.1) \quad I'_{11} = \int_k^T \sum_1^{\sqrt{\tau}} n^{-c} dt = 2\pi \int_{\kappa}^{\mathbf{T}} \sum_1^{\sqrt{\tau}} n^{-c} d\tau,$$

$$(28.2) \quad I''_{11} = \sum'_{m, n=1}^{\sqrt{\mathbf{T}}} m^{-a} n^{-b} \int_{2\pi\mathbf{T}_{m,n}}^{2\pi\mathbf{T}} \left(\frac{n}{m}\right)^{ti} dt = O\left(\sum'_{m, n=1}^{\sqrt{\mathbf{T}}} \frac{m^{-a} n^{-b}}{|\log(m/n)|}\right).$$

We have next, by Lemma A 1,

$$\begin{aligned} I_{22} &= \int_k^T \chi(a+ti) \chi(b-ti) \sum_{m, n=1}^{\sqrt{\tau}} m^{a-1+ti} n^{b-1-ti} dt \\ &= 2\pi \int_{\kappa}^T \tau^{1-c} \sum_{m, n=1}^{\sqrt{\tau}} m^{a-1+2\pi\tau i} n^{b-1-2\pi\tau i} d\tau + O\left(\int_{\kappa}^T \tau^{-\gamma} \sum_{m, n=1}^{\sqrt{\tau}} m^{a-1} n^{b-1} d\tau\right) \\ (29) \quad &= I'_{22} + I''_{22} + O(I'''_{22}), \end{aligned}$$

say, where I'_{22} represents the contribution to the first integral of those terms for which $m = n$. Now

$$\begin{aligned} (29.1) \quad I'_{22} &= 2\pi \int_{\kappa}^T \tau^{1-c} \sum_1^{\sqrt{\tau}} n^{c-2} d\tau, \\ I''_{22} &= 2\pi \sum'_{m, n=1}^{\sqrt{T}} m^{a-1} n^{b-1} \int_{T_{m,n}}^T \tau^{1-c} \left(\frac{m}{n}\right)^{2\pi\tau i} d\tau; \end{aligned}$$

hence, by Lemma A 2 [with $\lambda = 2\pi |\log(m/n)|$],

$$\begin{aligned} I''_{22} &= O\left(\sum'_{m, n=1}^{\sqrt{T}} \frac{m^{a-1} n^{b-1}}{|\log(m/n)|} \text{Max}(T^{1-\gamma}, T_{m,n}^{1-\gamma})\right) \\ (29.2) \quad &= O\left(\sum'_{m, n=1}^{\sqrt{T}} \frac{T^{1-\gamma} m^{a-1} n^{b-1} + m^{-\beta} n^{-\alpha}}{|\log(m/n)|}\right), \end{aligned}$$

since $T \geq T_{m,n} \geq mn$.

$$\begin{aligned} \text{Next} \quad I_{21} &= \int_k^T \chi(a+ti) \sum_{m, n=1}^{\sqrt{\tau}} m^{-b+ti} n^{a-1+ti} dt \\ &= \sum_{m, n=1}^{\sqrt{T}} m^{-b} n^{a-1} \int_{2\pi T_{m,n}}^{2\pi T} (mn)^{ti} \chi(a+ti) dt. \end{aligned}$$

To the integrals on the right we can apply Lemma A 4, since

$$T \geq T_{m,n} \geq mn > 0, \quad 2\pi T_{m,n} \geq k;$$

we thus obtain, observing, further, that

$$T_{m,n} \geq \text{Max}(m^2, n^2) \geq n^2,$$

$$\begin{aligned} I_{21} &= O\left(\sum_{m, n=1}^{\sqrt{T}} m^{-\beta} n^{a-1} \text{Max}\{T^{\frac{1}{2}-\alpha}, (n^2)^{\frac{1}{2}-\alpha}\} \lambda_{m,n}\right) \\ (30) \quad &= O\left(\sum_{m, n=1}^{\sqrt{T}} (T^{\frac{1}{2}-\alpha} m^{-\beta} n^{a-1} + m^{-\beta} n^{-\alpha}) \lambda_{m,n}\right), \end{aligned}$$

where

$$\frac{1}{\lambda_{m,n}} = \log \frac{\text{Max}(m^2, n^2) + \frac{1}{2} \sqrt{\text{Max}(m^2, n^2)}}{mn} = \log \frac{\text{Max}(m, n) + \frac{1}{2}}{\text{Min}(m, n)}.$$

Now I_{12} is obtained from I_{21} by changing a, b to $\bar{b}, \bar{a}$, and taking conjugates; hence

$$(31) \quad I_{12} = O\left(\sum_{m, n=1}^{\sqrt{T}} (T^{\frac{1}{2}-\beta} m^{-a} n^{\beta-1} + m^{-a} n^{-\beta}) \lambda_{m, n}\right).$$

We come now to

$$I_{31} = \int_k^T Q_a(t) \sum_1^{\sqrt{\tau}} m^{-b+ti} dt.$$

The part of the interval (k, T) which lies inside the interval $(0, 2\pi)$ may be ignored, since the integrand is there zero. The remainder of the interval (k, T) we divide up by the points $2\pi n^2$ ($n = 1, 2, 3, \dots$). We thus obtain a formula of the type

$$I_{31} = \sum_{n=n_0}^{\sqrt{T}} \int_{2\pi T_n}^{2\pi T'_n} Q_a(t) \sum_{m=1}^{\sqrt{\tau}} m^{-b+ti} dt,$$

where $n_0 = \text{Max}(1, [\sqrt{\kappa}])$, $n^2 \leq T_n \leq T'_n \leq (n+1)^2$, $2\pi T_n \geq k$. For $2\pi T_n \leq t < 2\pi T'_n$ we have $[\sqrt{\tau}] = n$; hence

$$I_{31} = \sum_{n=n_0}^{\sqrt{T}} \sum_{m=1}^n m^{-b} \int_{2\pi T_n}^{2\pi T'_n} m^{ti} Q_a(t) dt.$$

To these integrals we apply Lemma A 6, and obtain

$$I_{31} = O\left(\sum_{n=n_0}^{\sqrt{T}} \sum_{m=1}^n m^{-\beta} n^{-a} / \log \frac{n+\frac{1}{2}}{m}\right) = O\left(\sum_{m, n=1}^{\sqrt{T}} m^{-\beta} n^{-a} \lambda_{m, n}\right).$$

I_{13} is obtained from I_{31} by changing a, b to $\bar{b}, \bar{a}$, and taking conjugates; hence

$$(32) \quad I_{31} + I_{13} = O\left(\sum_{m, n=1}^{\sqrt{T}} m^{-\beta} n^{-a} \lambda_{m, n}\right).$$

In a similar way we deduce from Lemma A 7,

$$\begin{aligned} I_{23} &= \int_k^T \chi(a+ti) Q_b(-ti) \sum_1^{\sqrt{\tau}} m^{a-1+ti} dt \\ &= O\left\{\sum_{n=n_0}^{\sqrt{T}} \sum_{m=1}^n m^{a-1} n^{1-2a-\beta} \left(\frac{1}{\log\{(n+\frac{1}{2})/m\}} + 1\right)\right\} \end{aligned}$$

$$= O\left(\sum_{1 \leq m \leq n \leq \sqrt{T}} m^{a-1} n^{-\beta} \text{Max}(m^{1-2a}, T^{\frac{1}{2}-a}) (\lambda_{m, n} + 1)\right)$$

$$(33) \quad = O\left(\sum_{m, n=1}^{\sqrt{T}} (m^{-a} n^{-\beta} + T^{\frac{1}{2}-a} m^{a-1} n^{-\beta}) (\lambda_{m, n} + 1)\right),$$

$$(34) \quad I_{32} = O\left(\sum_{m, n=1}^{\sqrt{T}} (m^{-\beta} n^{-a} + T^{\frac{1}{2}-\beta} m^{\beta-1} n^{-a}) (\lambda_{m, n} + 1)\right).$$

Finally by Lemma A 5,

$$(35) \quad I_{33} = O\left(\int_k^T t^{-\frac{1}{2}\alpha} t^{-\frac{1}{2}\beta} dt\right) = O[(T^{1-\frac{1}{2}\gamma} + 1) \log T] = O(T^{\mathbf{A}+\mathbf{B}} \log T),$$

where $\mathbf{A} = \lambda(\alpha)$, $\mathbf{B} = \lambda(\beta)$.

Combining equations (28)–(35), we obtain

$$(36) \quad I = \sum_{i,j=1}^3 I_{ij} = 2\pi \int_{\kappa}^{\sqrt{\tau}} \left(\sum_1^{\sqrt{\tau}} n^{-\alpha} + \tau^{1-\alpha} \sum_1^{\sqrt{\tau}} n^{\alpha-2} \right) d\tau + O(S) + O(I_{22}''') \\ + O(T^{\mathbf{A}+\mathbf{B}} \log T),$$

where

$$S = \sum_{m,n=1}^{\sqrt{T}} (m^{-\alpha} + T^{\frac{1}{2}-\alpha} m^{\alpha-1}) (n^{-\beta} + T^{\frac{1}{2}-\beta} n^{\beta-1}) (\lambda_{m,n} + \lambda'_{m,n} + 1),$$

$$\lambda'_{m,n} = \begin{cases} |\log(m/n)|^{-1} & (m \neq n), \\ 0 & (m = n), \end{cases}$$

$$I_{22}''' = \int_{\kappa}^T \tau^{-\gamma} \sum_{m,n=1}^{\sqrt{\tau}} m^{\alpha-1} n^{\beta-1} d\tau.$$

We have now to estimate S and I_{22}''' . If $m \geq n$,

$$\lambda_{m,n}^{-1} = -\log\left(1 - \frac{m-n+\frac{1}{2}}{m+\frac{1}{2}}\right) > \frac{m-n+\frac{1}{2}}{m+\frac{1}{2}} > \frac{|n-m|+\frac{1}{2}}{n+m+\frac{1}{2}},$$

$$\lambda_{m,n} < \frac{n+m+\frac{1}{2}}{|n-m|+\frac{1}{2}};$$

the last inequality holds also, by symmetry, for $m \leq n$. Similarly, if $m > n$,

$$\lambda'_{m,n} = -\log\left(1 - \frac{m-n}{m}\right) > \frac{m-n}{m} > \frac{m-n+\frac{1}{2}}{2m} > \frac{1}{2} \frac{|n-m|+\frac{1}{2}}{n+m+\frac{1}{2}};$$

$$\lambda'_{m,n} < 2 \frac{n+m+\frac{1}{2}}{|n-m|+\frac{1}{2}};$$

this holds by symmetry for $m < n$, and trivially for $m = n$. Hence

$$\lambda_{m,n} + \lambda'_{m,n} + 1 < 4 \frac{n+m+\frac{1}{2}}{|n-m|+\frac{1}{2}} \leq 4 \frac{n+m+\frac{1}{2}}{|n-m-\frac{1}{2}|},$$

$$S < 4 \sum_{m=1}^{\sqrt{T}} (m^{-\alpha} + T^{\frac{1}{2}-\alpha} m^{\alpha-1}) \sum_{n=1}^{\sqrt{T}} (n^{-\beta} + T^{\frac{1}{2}-\beta} n^{\beta-1}) \frac{n+m+\frac{1}{2}}{|n-m-\frac{1}{2}|}.$$

U 2

By Lemma A 8 (with $\omega = \sqrt{T}$, $\xi = m + \frac{1}{2}$, $\epsilon = \frac{1}{2}$), the inner sum is, for each m ,

$$< 5 [\text{Max} (T^{\frac{1}{2}(1-\beta)}, 1) + T^{\frac{1}{2}-\beta} \text{Max} (T^{\frac{1}{2}\beta}, 1)] (\frac{1}{2} \log T + 3) \leq 5 T^{\beta} (\log T + 6).$$

Therefore

$$S < 20 T^{\beta} (\log T + 6) \sum_{m=1}^{\sqrt{T}} (m^{-\alpha} + T^{\frac{1}{2}-\alpha} m^{\alpha-1}).$$

Now the sum on the right is

$$\begin{aligned} &\leq \text{Max} (T^{\frac{1}{2}(1-\alpha)}, 1) l_{1-\alpha}(\sqrt{T}) + T^{\frac{1}{2}-\alpha} \text{Max} (T^{\frac{1}{2}\alpha}, 1) l_{\alpha}(\sqrt{T}) \\ &\leq T^{\frac{1}{2}} \{l_{\alpha}(T) + l_{1-\alpha}(T)\}, \end{aligned}$$

where

$$l_{\sigma}(\omega) = \text{Min} \left(1 + \log \omega, 1 + \frac{1}{|\sigma|} \right). *$$

Writing, for brevity, $l(\omega) = l_0(\omega) = 1 + \log \omega$, we deduce

$$(37) \quad S < 20 T^{A+B} \{l(T) + 5\} \{l_{\alpha}(T) + l_{1-\alpha}(T)\}.$$

We have next

$$\begin{aligned} I_{22}''' &\leq \sum_{m,n=1}^{\sqrt{T}} m^{\alpha-1} n^{\beta-1} \int_{mn}^T \tau^{-\gamma} d\tau < \sum_{m,n=1}^{\sqrt{T}} m^{\alpha-1} n^{\beta-1} \text{Max} \{ (mn)^{1-\gamma}, T^{1-\gamma} \} l_{1-\gamma}(T) \\ &< \left(\sum_{m,n=1}^{\sqrt{T}} m^{-\beta} n^{-\alpha} + T^{1-\gamma} \sum_{m,n=1}^{\sqrt{T}} m^{\alpha-1} n^{\beta-1} \right) l_{1-\gamma}(T) \\ &\leq T^{A+B} [l_{1-\alpha}(T) l_{1-\beta}(T) + l_{\alpha}(T) l_{\beta}(T)] l_{1-\gamma}(T). \end{aligned}$$

Now, since $(1-\beta)-(1-\gamma) = \alpha$, one at least of the numbers $|1-\beta|$ and $|1-\gamma|$ must be greater than or equal to $\frac{1}{2}|\alpha|$, so that

$$l_{1-\beta}(T) l_{1-\gamma}(T) \leq l(T) \text{Min} \left(l(T), 1 + \frac{2}{|\alpha|} \right) < 2l(T) l_{\alpha}(T).$$

* This expression is to be interpreted as $1 + \log \omega$ when $\sigma = 0$. The inequality

$$\sum_1^{\omega} n^{\sigma-1} \leq \text{Max} (\omega^{\sigma}, 1) l_{\sigma}(\omega) \quad (\omega \geq 1, \sigma \geq 0)$$

which we use here is easily proved; for, writing $\Omega = \text{Max} (\omega^{\sigma}, 1)$, we have

$$\sum_1^{\omega} n^{\sigma-1} \leq \text{Max} (\omega^{\sigma-1}, 1) + \int_1^{\omega} u^{\sigma-1} du \leq \Omega + \frac{\omega^{\sigma}-1}{\sigma} < \Omega \left(1 + \frac{1}{|\sigma|} \right) \quad (\sigma \neq 0),$$

and

$$\sum_1^{\omega} n^{\sigma-1} \leq \Omega \sum_1^{\omega} n^{-1} \leq \Omega (1 + \log \omega).$$

Similarly $l_\beta(\mathbf{T}) l_{1-\gamma}(\mathbf{T}) < 2l(\mathbf{T}) l_{1-a}(\mathbf{T})$.

Hence

$$(38) \quad I_{22}''' < 4\mathbf{T}^{A+B} l(\mathbf{T}) l_a(\mathbf{T}) l_{1-a}(\mathbf{T}) \leq 12\mathbf{T}^{A+B} l(\mathbf{T}) \{l_a(\mathbf{T}) + l_{1-a}(\mathbf{T})\},$$

since either $|a|$ or $|1-a|$ must be greater than or equal to $\frac{1}{2}$, and therefore $l_a(\mathbf{T})$ or $l_{1-a}(\mathbf{T}) \leq 3$.

From (36), (37), (38) it is clear that both parts of Theorem A, in the case $\mu = \nu = 0$, will be proved if we show that

$$(39) \quad \int_0^{\mathbf{T}} \left(\sum_1^{\sqrt{\tau}} n^{-c} + \tau^{1-c} \sum_1^{\sqrt{\tau}} n^{c-2} \right) d\tau = F(\mathbf{T}, c) + O(\mathbf{T}^{A+B} \log \mathbf{T}).$$

Now the left-hand side is, by Lemma A 9 (with h, h' replaced by $2h+2, 2h'$),

$$\begin{aligned} &= \int_1^{\mathbf{T}} \left\{ \left(\zeta(c) + \frac{\tau^{\frac{1}{2}(1-c)}}{1-c} \right) + \tau^{1-c} \left(\zeta(2-c) + \frac{\tau^{\frac{1}{2}(c-1)}}{c-1} \right) \right\} d\tau \\ &\quad + O\left(\int_1^{\mathbf{T}} (\tau^{-\frac{1}{2}\gamma} + \tau^{1-\gamma} \tau^{\frac{1}{2}(\gamma-2)}) d\tau \right) + O(1) \\ &= F(\mathbf{T}, c) + O[(\mathbf{T}^{1-\frac{1}{2}\gamma} + 1) \log \mathbf{T}] = F(\mathbf{T}, c) + O(\mathbf{T}^{A+B} \log \mathbf{T}). \end{aligned}$$

This proves (39), and so Theorem A for $\mu = \nu = 0$.

4. *Proof of Theorem A in the general case.* We consider the function $R(T, a, b) = R_{0,0}(T, a, b)$, defined by (11) and (12), as a function of the two complex variables a and b , T being supposed fixed and greater than $\text{Max}(k, 2)$. This function is clearly regular in the domain $|a'| < k, |\beta'| < k$, its partial derivatives being given by

$$\frac{\partial^{\mu+\nu} R(T, a, b)}{\partial a^\mu \partial b^\nu} = R_{\mu,\nu}(T, a, b),$$

where $R_{\mu,\nu}(T, a, b)$ is defined by (11) and (12). Now, by the particular case $\mu = \nu = 0$ of Theorem A, already proved, we have

$$|R(T, a, b)| < K_1 T^{\lambda(a)+\lambda(\beta)} \log^2 T,$$

if $|a|, |\beta| \leq h+d$, and $|a'|, |\beta'| \leq h'+d$, where $d = \frac{1}{2}(k-h')$. It follows from Cauchy's inequalities that, if $|a|, |\beta| \leq h$, and $|a'|, |\beta'| \leq h'$,

and ρ is any number satisfying $0 < \rho \leq d$, then

$$\begin{aligned} |R_{\mu, \nu}(T, a, b)| &< \mu! \nu! \max_{|z-a|=|w-b|=\rho} |R(T, z, w)| \rho^{-\mu-\nu} \\ &< K_1 \mu! \nu! T^{\lambda(a-\rho)+\lambda(\beta-\rho)} \log^2 T \cdot \rho^{-\mu-\nu} \\ &\leq K_1 \mu! \nu! T^{2\rho} \cdot T^{\lambda(a)+\lambda(\beta)} \log^2 T \cdot \rho^{-\mu-\nu}, \end{aligned}$$

since $\lambda(a-\rho) \leq \lambda(a)+\rho$. We now take $\rho = \min(d, 1/\log T)$, and deduce

$$\begin{aligned} |R_{\mu, \nu}(T, a, b)| &< K_1 \mu! \nu! e^2 T^{\lambda(a)+\lambda(\beta)} \log^2 T \left(\log T + \frac{1}{d} \right)^{\mu+\nu} \\ &< K(\mu, \nu) T^{\lambda(a)+\lambda(\beta)} (\log T)^{\mu+\nu+2}. \end{aligned}$$

This proves Theorem A(i). Part (ii) is proved in the same way; in this case we take $\rho = \min(\frac{1}{2}\delta, d, 1/\log T)$ and obtain

$$|R_{\mu, \nu}(T, a, b)| < K(\delta, \mu, \nu) T^{\lambda(a)+\lambda(\beta)} (\log T)^{\mu+\nu+1}.$$

5. *Particular cases.* We state explicitly some interesting particular cases of Theorem A.

Theorem A'. *We have*

$$\int_0^T |\xi(\tfrac{1}{2}+ti)|^2 dt = T \log T - (1 + \log 2\pi - 2C)T + O(T^{\frac{1}{2}} \log T),$$

where C is Euler's constant.

To prove this we have only to take $a = b = \frac{1}{2}$, $\mu = \nu = 0$ in Theorem A, and to observe that [as is at once obvious from equation (14)],

$$F(T, 1) = \int_1^T (2C + \log \tau) d\tau = T \log T - (1 - 2C)(T - 1).$$

More generally we have

Theorem A''.

$$\begin{aligned} \int_0^T |\xi^{(\nu)}(\tfrac{1}{2}+ti)|^2 dt &= \frac{T}{2\nu+1} \sum_{n=0}^{2\nu} c_n \binom{2\nu+1}{n} \left(\log \frac{T}{2\pi} \right)^{2\nu+1-n} + \left(c_{2\nu} + \frac{2c_{2\nu+1}}{2\nu+1} \right) T \\ &\quad + O[T^{\frac{1}{2}} (\log T)^{2\nu+1}], \end{aligned}$$

* Owing to a slip in calculation, Littlewood (1, § 1) gives the coefficient of T as $-(1 + \log 2\pi)$.

where the coefficients c_n are defined by

$$(40) \quad \frac{x\zeta(1+x)}{1+x} = \sum_0^{\infty} \frac{c_n}{n!} x^n \quad (|x| < 1)$$

(so that, in particular, $c_0 = 1$, $c_1 = C-1$).

Theorem A, with $a = b = \frac{1}{2}$, $\mu = \nu$, gives, in fact,

$$\int_1^T |\zeta^{(\nu)}(\tfrac{1}{2} + ti)|^2 dt = 2\pi \left[\frac{\partial^{2\nu} F(\mathbf{T}, c)}{\partial c^{2\nu}} \right]_{c=1} + O[T^{\frac{1}{2}}(\log T)^{2\nu+1}] \quad \left(\mathbf{T} = \frac{T}{2\pi} \right),$$

and it remains only to calculate the first term on the right. Now

$$\frac{1}{r!} \left[\frac{\partial^r F(\mathbf{T}, c)}{\partial c^r} \right]_{c=1} = \text{coefficient of } x^r \text{ in } F(\mathbf{T}, 1+x).$$

If $x \neq 0$ or 1 , then

$$F(\mathbf{T}, 1+x) = (\mathbf{T}-1)\zeta(1+x) + \frac{\mathbf{T}^{1-x}-1}{1-x} \zeta(1-x)$$

$$= \mathbf{T}G(\mathbf{T}, x) - G(1, x),$$

where

$$G(\mathbf{T}, x) = \zeta(1+x) + \frac{\mathbf{T}^{-x}\zeta(1-x)}{1-x} = \frac{(1+x)\phi(x) - \mathbf{T}^{-x}\phi(-x)}{x},$$

$\phi(x)$ being the function (40). The coefficient of x^r ($r \geq 0$) in the expansion of $G(\mathbf{T}, x)$ in ascending powers of x is

$$\begin{aligned} & \frac{c_{r+1}}{(r+1)!} + \frac{c_r}{r!} - \sum_{n=0}^{r+1} \frac{(-1)^n c_n}{n!} \frac{(-\log \mathbf{T})^{r+1-n}}{(r+1-n)!} \\ &= \frac{1}{(r+1)!} \left\{ (-1)^r \sum_{n=0}^r \binom{r+1}{n} c_n (\log \mathbf{T})^{r+1-n} + \{1 + (-1)^r\} c_{r+1} + (r+1)c_r \right\} \\ &= \frac{\Lambda_r(\mathbf{T})}{(r+1)!}, \end{aligned}$$

say. Hence

$$\int_1^T |\zeta^{(\nu)}(\tfrac{1}{2} + ti)|^2 dt = \frac{2\pi \mathbf{T} \Lambda_{2\nu}(\mathbf{T})}{2\nu+1} - \frac{2\pi \Lambda_{2\nu}(1)}{2\nu+1} + O[T^{\frac{1}{2}}(\log T)^{2\nu+1}],$$

which is equivalent to the result stated.

B. The mean value of $|\zeta(\frac{1}{2} + it)|^4$.

6. **Lemma B 1.** We have, as $x \rightarrow \infty$,

$$\sum_1^x \frac{d^2(n)}{n} = \frac{1}{4\pi^2} \log^4 x + O(\log^3 x).$$

This result follows by partial summation from Ramanujan's formula*

$$D(x) \equiv \sum_1^x d^2(n) = \frac{1}{\pi^2} x \log^3 x + O(x \log^2 x).$$

We have, in fact, for $x \geq 1$,

$$\begin{aligned} \sum_1^x \frac{d^2(n)}{n} &= \sum_1^{[x]} \frac{D(n) - D(n-1)}{n} = \sum_1^{[x]-1} D(n) \left(\frac{1}{n} - \frac{1}{n+1} \right) + \frac{D(x)}{[x]} \\ &= \int_1^x \frac{D(u) du}{u^2} + \frac{D(x)}{x} \\ &= \frac{1}{\pi^2} \int_1^x \frac{\log^3 u}{u} du + O \left(\int_1^x \frac{\log^2 u}{u} du \right) + O(\log^3 x), \end{aligned}$$

from which the lemma follows.

Lemma B 2. If k is a positive integer and $x \geq k$, then

$$\sum_{r=1}^{x-k} d(r) d(r+k) < 4x(\log x + 1)^2 \sum_{d|k} \frac{1}{d}.$$

The sum on the left is clearly equal to

$$\sum_{\substack{\mu m + \nu n = k \\ \mu m \leq x}} 1 = \Sigma$$

say, the lower limit of summation for each of the variables μ, ν, m, n being 1. We denote by S the sum of those terms for which $mn \leq x$, and by S' the sum of those for which $\mu\nu \leq x$. Each term of Σ belongs to one at least of these sums, since $\mu\nu \cdot mn = \mu m \cdot \nu n < (\mu m)^2 \leq x^2$. Also $S = S'$ by symmetry. Hence

$$\Sigma \leq S + S' = 2S = 2 \sum_{mn \leq x} N(m, n),$$

* Ramanujan, 1. Wilson, 1, 235-240.

where $N(m, n) = N(m, n, x, k)$ is the number of pairs of integers μ, ν satisfying

$$\mu m - \nu n = k, \quad 0 < \mu \leq x/m, \quad \nu > 0.$$

Let $(m, n) = d$, $m = dm'$, $n = dn'$. If $d \nmid k$, clearly $N(m, n) = 0$. If $d \mid k$, the equation $\mu m - \nu n = k$ has integral solutions, and these are all contained in the formula

$$\mu = \mu_0 + rn/d, \quad \nu = \nu_0 + rm/d,$$

where μ_0, ν_0 is a particular solution and r is an arbitrary integer. Hence $N(m, n)$ is at most equal to the number of integers r satisfying

$$0 < \mu_0 + rn/d \leq x/m;$$

that is

$$N(m, n) \leq \left[\frac{xd}{mn} - \frac{\mu_0 d}{n} \right] - \left[-\frac{\mu_0 d}{n} \right] < \frac{xd}{mn} + 1 \leq \frac{2xd}{mn} = \frac{2x}{dm'n'}.$$

if $mn \leq x$. Hence

$$\begin{aligned} \Sigma &< 4x \sum_{d \mid k} \frac{1}{d} \sum_{\substack{m'n' \leq x/d^2 \\ (m', n')=1}} \frac{1}{m'n'} \\ &\leq 4x \sum_{d \mid k} \frac{1}{d} \left(\sum_1^x \frac{1}{m'} \right)^2 \leq 4x \sum_{d \mid k} \frac{1}{d} (\log x + 1)^2. \end{aligned}$$

Lemma B3. As $x \rightarrow \infty$,

$$\sum'_{m, n=1}^x \frac{d(m)d(n)}{\sqrt{(mn)} |\log(m/n)|} = O(x \log^2 x).$$

Here Σ' indicates, as usual, that the terms for which $m = n$ are omitted.

If $m < n$ we have

$$\begin{aligned} |\log(m/n)|^{-1} &= \left\{ -\log \left(1 - \frac{n-m}{n} \right) \right\}^{-1} < \frac{n}{n-m} \\ &= 1 + \frac{m}{n-m} < 1 + \frac{\sqrt{(mn)}}{n-m}. \end{aligned}$$

Hence the sum in question is

$$\begin{aligned} \sum'_{m, n=1}^x &= 2 \sum_{1 \leq m < n \leq x} < 2 \sum_{1 \leq m < n \leq x} \frac{d(m)d(n)}{\sqrt{(mn)}} + 2 \sum_{1 \leq m < n \leq x} \frac{d(m)d(n)}{n-m} \\ &= 2\Sigma_1 + 2\Sigma_2, \end{aligned}$$

say. We have first

$$\Sigma_1 < \left(\sum_1^x \frac{d(n)}{\sqrt{n}} \right)^2 = O(\sqrt{x \log x})^2 = O(x \log^2 x).$$

For Σ_2 we have, writing $n-m=k$ and using Lemma B 2,

$$\begin{aligned} \Sigma_2 &= \sum_{k=1}^{x-1} \frac{1}{k} \sum_{m=1}^{x-k} d(m) d(m+k) = O(x \log^2 x) \sum_{k=1}^x \frac{1}{k} \sum_{d|k} \frac{1}{d} \\ &= O(x \log^2 x) \sum_{d' \leq x} \frac{1}{d'} \cdot \frac{1}{d} = O(x \log^2 x) \sum_{d=1}^{\infty} \frac{1}{d^2} \sum_{d'=1}^x \frac{1}{d'} \\ &= O(x \log^3 x). \end{aligned}$$

Lemma B 4. If $T \geq T_0 \geq \xi > 0$, then

$$\left| \int_{T_0}^T \left(\frac{t}{e\xi} \right)^{2ti} dt \right| < \begin{cases} \frac{A}{\log(T_0/\xi)} & (T_0 > \xi), \\ A\sqrt{T} & \text{always,} \end{cases}$$

where A is an absolute constant.

These are known results*.

7. *Proof of Theorem B.* We can write the approximate functional equation (18) in the form

$$\xi^2(\tfrac{1}{2} + ti) = S + \bar{\psi} \bar{S} + O(\log t) = S + \bar{\psi} \bar{S} + \rho = \Sigma + \rho,$$

where $\psi = -i \left(\frac{t}{2\pi e} \right)^{2ti}$, $S = \sum_1^{t/2\pi} d(n) n^{-\frac{1}{2}-ti}$.

Now $|\Sigma|^2 = (S + \bar{\psi} \bar{S})(\bar{S} + \psi S) = 2|S|^2 + 2\Re(\psi S^2) \leq 4|S|^2$.

Hence

$$(41) \quad \int_0^T |\xi(\tfrac{1}{2} + ti)|^2 dt = 2 \int_0^T |S|^2 dt + 2\Re \int_0^T \psi S^2 dt + R = 2J_1 + 2\Re J_2 + R,$$

say, where

$$(42) \quad |R| \leq 2 \int_0^T |\Sigma \rho| dt + \int_0^T |\rho|^2 dt \leq 4 \int_0^T |S \rho| dt + \int_0^T |\rho|^2 dt = 4R_1 + R_2.$$

* Landau, 1, 707 (Hilfssatz 10); Hardy and Littlewood, 1, 168-169 (Lemma 3.51).

We have first

$$J_1 = \int_0^T S \bar{S} dt = \int_0^T \sum_{m, n=1}^{t/2\pi} \frac{d(m) d(n)}{\sqrt{mn}} \left(\frac{n}{m}\right)^{ti} dt = J_{11} + J_{12},$$

where J_{11} is the contribution of those terms for which $m = n$. Now, by Lemma B 1,

$$\begin{aligned} J_{11} &= \int_0^T \sum_{n=1}^{t/2\pi} \frac{d^2(n)}{n} dt = \int_{2\pi}^T \left(\frac{\log^4(t/2\pi)}{4\pi^2} + O(\log^3 t) \right) dt \\ &= \frac{1}{4\pi^2} T \log^4 T + O(T \log^3 T), \end{aligned}$$

Also, writing $T' = T'_{m, n} = 2\pi \text{Max}(m, n)$, we have

$$\begin{aligned} J_{12} &= \sum'_{m, n=1}^{T/2\pi} \frac{d(m) d(n)}{\sqrt{mn}} \int_{T'}^T \left(\frac{n}{m}\right)^{ti} dt = O\left(\sum'_{m, n=1}^{T/2\pi} \frac{d(m) d(n)}{\sqrt{mn} |\log(m/n)|}\right) \\ &= O(T \log^3 T) \end{aligned}$$

by Lemma B 3. Hence

$$(49) \qquad J_1 = \frac{1}{4\pi^2} T \log^4 T + O(T \log^3 T).$$

Next

$$J_2 = -i \int_0^T \left(\frac{t}{2\pi e}\right)^{2ti} \sum_{m, n=1}^{t/2\pi} \frac{d(m) d(n)}{(mn)^{\frac{1}{2}+ti}} dt = -i \sum_{m, n=1}^{T/2\pi} \frac{d(m) d(n)}{\sqrt{mn}} \int_{T'}^T \left(\frac{t}{e\xi}\right)^{2ti} dt,$$

where $\xi = \xi_{m, n} = 2\pi\sqrt{mn}$. It is clear that $\xi \leqslant T'$, and indeed $\xi < T'$ unless $m = n$. Further $\log(T'/\xi) = |\frac{1}{2} \log(m/n)|$. Hence, by Lemma B 4,

$$\begin{aligned} |J_2| &< A \sum'_{m, n=1}^{T/2\pi} \frac{d(m) d(n)}{\sqrt{mn} |\frac{1}{2} \log(m/n)|} + A \sum_1^{T/2\pi} \frac{d^2(n)}{n} \sqrt{T} \\ (44) \qquad &= O(T \log^3 T) + O(\sqrt{T} \log^4 T) = O(T \log^3 T), \end{aligned}$$

by Lemmas B 3 and B 1.

It remains to estimate R . We have obviously

$$R_2 = O(T \log^2 T),$$

which, combined with (43), gives, with the aid of Schwarz's inequality,

$$R_1 \leqslant \sqrt{(J_1 R_2)} = O(T \log^3 T).$$

Hence

$$(45) \qquad R = O(T \log^3 T).$$

From (41), (43), (44), and (45), Theorem B follows at once.

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A POSTSCRIPT TO PONCELET'S THEOREM

By R. VAIDYANATHASWAMY.

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1. *The two issues in Poncelet's theorem.*

Let E be a fundamental conic, and F_1, F_2 two other conics. If x, y, z are points on E , such that zx is tangent to F_1 and zy to F_2 , then the envelope of xy is, in general, a curve ϕ of class eight. For the case in which F_1, F_2 have the same four points p_1, p_2, p_3, p_4 in common with E , Poncelet's theorem asserts that the envelope of xy is composed of two conics ϕ_1, ϕ_2 also passing through p_1, p_2, p_3, p_4 ; that is, the envelope ϕ splits up into the two conics ϕ_1, ϕ_2 , each counted twice.

In studying Poncelet's theorem algebraically we may let x, y, z denote also the parameters of the points x, y, z on E ; then $(x, z), (y, z)$ are connected by two irreducible symmetric doubly-quadratic relations

$$F_1(x, z) = 0, \quad F_2(y, z) = 0.$$

The two symmetric double-quadratics F_1, F_2 are so related that they have the same discriminant points p_1, p_2, p_3, p_4 . The algebraic statement of Poncelet's theorem therefore shows that the z -resultant of two irreducible symmetric double-quadratics $F_1(x, z), F_2(y, z)$ with identical discriminant points is the product of two other symmetric double-quadratics $\phi_1(x, y), \phi_2(x, y)$, with the same discriminant points as F_1, F_2 .

As thus stated, it appears that the theorem is concerned with the character of the resultant of two irreducible double-quadratics $F_1(x, z), F_2(y, z)$. Accordingly, there are two issues involved in Poncelet's

theorem, namely :

- (1) the resolubility of the resultant of $F_1(x, z)$, $F_2(y, z)$;
- (2) the simultaneous symmetry of F_1 , F_2 and their resultant.

It will be useful to keep these issues apart in the discussion. It appears to be a fact that the symmetry of the resultant of two symmetric irreducible double-quadratics implies also its resolubility*. In any case, the present paper will show that (2) is a much more stringent requirement than (1). We accordingly consider the resolubility of the resultant to be the more fundamental of the two issues involved.

2. The resolubility of the resultant.

To study the circumstances in which the resultant $\phi(x, y)$ resolves, we need assume only the irreducibility of $F_1(x, z)$, $F_2(y, z)$, not their symmetry. The equations

$$F_1(x, z) = 0, \quad F_2(y, z) = 0$$

define y as a four-valued function of x , mediately through z . If, then, the resultant ϕ is to resolve, there must be two branches of y which do not pass over continuously into one another for any circuit of x , the complex variables x, z, y being represented in the usual manner by points in three Argand planes.

Let the value x of x correspond in F_1 to the values z_1, z_2 of z ; and let z_1, z_2 correspond in F_2 to the values $(y_1, y'_1), (y_2, y'_2)$ of y . Since F_1 is irreducible, there exist x -circuits which interchange z_1 and z_2 ; these x -circuits must interchange (y_1, y'_1) with (y_2, y'_2) in some order—say y_1 with y_2 and y'_1 with y'_2 . Now, if there also exists an x -circuit which changes y_1 into y'_1 , then it is clear that we can find x -circuits which change y_1 into any of the remaining branches, so that the resultant ϕ does not resolve. Thus, for the resolubility of ϕ it is necessary that no circuit

* This is not easy to prove. The general question involved in the second issue is not discussed in this paper; there is, however, good ground for believing that the following theorem is true:

If the resultant of two irreducible symmetric double-quadratics F_1, F_2 is also symmetric, then either the discriminants of F_1, F_2 are identical or at least one of the (three) quadratic factors of their sextic covariants is the same for both.

An example of the latter case occurs when F_1, F_2 reduce to two permutable involutions between the point-pairs of the same involution.

of x changes y_1 into y'_1 . This condition is also sufficient, for, since y'_1 can interchange with y'_2 , it follows that y_1 can change neither into y'_1 nor into y'_2 , so that ϕ must split up into two factors corresponding to the two pairs of branches $(y_1, y_2), (y'_1, y'_2)$.

Now let ζ be a branch point of y qua function of z . The passing over of y_1 into y'_1 cannot happen except through a circuit of z_1 round one of the points ζ . Since, in our case, there should not exist x -circuits which change y_1 into y'_1 , it follows that no single circuit of x should produce a circuit of z_1 about any of the points ζ . Or—reversing this statement by considering x as a function of z —a circuit of z_1 about any of the points ζ should *not* produce a complete circuit of x . That is to say, the points ζ should also be branch points of x qua function of z . Nor can x have branch points other than the points ζ , for, if it had, it would follow, by reversing our argument, that $\phi(x, y) = 0$ defined x as an irreducible function of y , but y as a reducible function of x —which is impossible if, according to our hypothesis, F_1 and F_2 are irreducible. We have thus

THEOREM I. *The z -resultant of two irreducible double-quadratics $F_1(x, z), F_2(y, z)$ resolves, if at all, into the product of two other double-quadratics $\phi_1(x, y), \phi_2(x, y)$; the necessary and sufficient condition for this to happen is the identity of the branch points of x and y , defined as functions of z by $F_1(x, z) = 0, F_2(y, z) = 0$.*

We easily obtain further information about the factors $\phi_1(x, y), \phi_2(x, y)$ of the resultant. From the previous proof it follows that the same x -circuits interchange y_1 with y_2 and y'_1 with y'_2 —namely, the circuits about the branch points of z qua function of x . Thus x and y have the same branch points, when defined as functions of each other by equating *either* of the factors of the resultant to zero. More precisely, the fundamental property of the factors of the resultant is expressed by

THEOREM II. *The functions t of x defined by each of the equations*

$$\phi_1(x, t) = 0, \quad \phi_2(x, t) = 0, \quad F_1(x, t) = 0,$$

have the same branch points. Similarly the functions t of y defined by the equations

$$\phi_1(t, y) = 0, \quad \phi_2(t, y) = 0, \quad F_2(y, t) = 0,$$

have the same branch points.

It is clear that these two theorems contain the algebraic basis of Poncelet's theorem; but the important thing to notice is that they

are more extensive than this theorem. For the condition imposed by Theorem I relates only to the *branch points*, and not to the discriminant points of the functions defined by $F_1 = 0$, $F_2 = 0$. Now, for an irreducible function x of z defined by $F_1(x, z) = 0$, there are two possibilities to be considered, namely :

(1) x has four branch points; in this case the x -discriminant of F_1 has four simple roots which give the four branch points. Also $F_1 = 0$ is of genus unity, and therefore requires elliptic functions for its uniformization.

(2) x has only two branch points. This may happen in two ways : either the x -discriminant has a double root (which is not a branch point) and two simple roots which give the branch points, or it has a triple root and a simple root, both of which give branch points. The second way is a special case of the first, and need not be treated separately. For this second case, the doubly-quadratic relation $F_1 = 0$ is of zero genus, and can be uniformized by rational quadratic functions.

Also, from the known theorem of the linear transformability of the two discriminants of a double-quadratic $F_1(x, z)$ into one another*, it is clear that x and z defined as functions of one another by $F_1(x, z) = 0$ will either both possess four or both two branch points. Let us say that the double-quadratic F_1 is of the *first* or *elliptic type* in the former case, and of the *second* or *rational type* in the latter. Thus the type determines the number of branch points, and double-quadratics which define functions with identical branch points must necessarily be of the same type. We may therefore paraphrase Theorems I and II as follows :

THEOREM III. *The necessary and sufficient conditions for the resolvability of the resultant of two irreducible double-quadratics $F_1(x, z)$, $F_2(y, z)$ are that*

- (1) F_1, F_2 should be of the same type ;
- (2) if they are of the elliptic type, their x - and y -discriminants should be identical to a constant factor ;
- (3) if they are of the second or rational type, the simple roots of the same discriminants should be identical ; the double roots need not be identical, since they do not give branch points.

* See the author's "Algebraic (2, 2) correspondence", *Journal of the Indian Math. Soc.*, 16 (1926), 164-199, where further references are given.

In any case, the factors ϕ_1, ϕ_2 of the resultant are of the same type as F_1, F_2 .

It is with the case in which F_1, F_2 are of the elliptic type that Poncelet's theorem is concerned. The other case, in which F_1, F_2 are of the rational type, also yields a closure theorem, which is studied in this paper under the name of *Companion theorem of Poncelet*.

When F_1, F_2 are both symmetric, and of the rational type, the geometrical equivalent of Theorems I, II, and III is

THEOREM IV. *Let F_1, F_2 be two conics which cut a fundamental conic E in the same two points i, j , and touch it elsewhere at p_1, p_2 respectively. If x, y, z are points on E such that zx is tangent to F_1 , and zy to F_2 , then the envelope ϕ of xy splits up into two curves ϕ_1, ϕ_2 of class 4, with double points at i, j .*

It will be shown below that, for given F_1, i, j, p_2 , it is possible to choose F_2 uniquely, so that one of the resulting envelopes becomes a repeated conic ϕ_1 . The conic ϕ_1 will then pass through i and j , and touch E elsewhere at p_3 , and the three conics F_1, F_2, ϕ_1 will have closure properties analogous to those derived from Poncelet's theorem. If i, j are the circular points, it will become a question of closure properties associated with triads (and n -ads) of contact-circles F_1, F_2, ϕ_1 of a fundamental circle E .

It is not to Theorem IV, but to the case in which one of the resulting envelopes is a repeated conic, that we reserve the name "*Companion theorem*".

3. The canonical uniformization of the double-quadratic of the second type.

Consider a symmetric double-quadratic $F(x, z)$ of the second type with the branch points i, j . The double discriminant point λ of F is also its double self-corresponding point, and may be conveniently referred to as the *double point* of F . The two other self-corresponding points of F , namely the two roots μ_1, μ_2 other than λ of $F(x, x) = 0$, will be called the *latent points* of F . In the general case, when F is of the elliptic type, its self-corresponding and discriminant tetrads are members of a syzygetic pencil; when F is of the second type, this leads to

THEOREM V. *If λ be the double point, i, j the branch points, and μ_1, μ_2 the latent points of F , $(i, j; \mu_1, \mu_2; \lambda, \lambda)$ belong to an involution.*

To effect the uniformization of $F(x, z)$ we choose, once for all, a triad Ω with the Hessian points i, j , and denote by P the polar (2, 1)-correspondence defined by it. Thus, if Ω is given by the roots of the equation $(ix)^3 + k(jx)^3 = 0$, then $x' = P(x)$ is given by

$$(ix)^2(ix') + k(jx)^2(jx') = 0.$$

Suppose now that I is an arbitrary involution with the fixed points p, q , and consider the correspondence PIP^{-1} . This is obviously a symmetric (2, 2)-correspondence. Further

$$PIP^{-1}(i) = PI(j, j) = PI(j), PI(j).$$

Thus i and j are discriminant points of PIP^{-1} . Also, if r, s form the common harmonic pair of p, q and i, j , it is clear that the only other discriminant point of PIP^{-1} is the point λ , whose first polar $P^{-1}(\lambda)$ is rs . Hence λ must be a double discriminant point, and so PIP^{-1} is a symmetric (2, 2)-correspondence of the second type with the branch points i, j and the double point λ .

Conversely, the given symmetric correspondence $F(x, z)$ with the branch points i, j and the double point λ can be put in the form PIP^{-1} and only in two ways. For, let μ_1, μ_2 be the latent pair of F , and let the first polar $P^{-1}(\mu_r)$ be μ_{r1}, μ_{r2} ($r = 1, 2$). Since the point-pairs $\mu_1, \mu_2; i, j; \lambda, \lambda$ belong to an involution (Theorem V), their first polars with respect to the triad Ω should belong to a pencil of tetrads. Hence, if r, s be the first polar of λ , the three tetrads $(i, j, i, j), (r, s, r, s), (\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22})$ belong to a pencil. Since rs is harmonic to ij , the pencil must be a syzygetic pencil. Hence the tetrad $(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22})$ can be divided into two pairs— $(\mu_{11}, \mu_{21}), (\mu_{12}, \mu_{22})$, say—each harmonic to r, s . Denote by I, I' the involutions defined by $(\mu_{11}, \mu_{21}), (\mu_{12}, \mu_{22})$ respectively. Since the involution J whose fixed points are i, j carries the pair μ_{11}, μ_{21} into the pair μ_{12}, μ_{22} , it follows that $I' = JIJ$. The correspondence PIP^{-1} is, as before, a symmetric (2, 2)-correspondence with branch points i, j and double point λ . Remembering that $PJ = P^*$, we easily show that μ_1, μ_2 are the latent points of PIP^{-1} , so that $PIP^{-1} = F$. For

$$\begin{aligned} PIP^{-1}(\mu_1) &= PI(\mu_{11}, \mu_{12}) = P(\mu_{11}) \text{ and some other point} \\ &= \mu_1 \text{ and some other point.} \end{aligned}$$

Hence

$$F = PIP^{-1} = P'P^{-1} = PJIP^{-1}.$$

* See "The (2, 1)-Correspondence", *Proc. Camb. Phil. Soc.*, 23 (1926), 242.

That is to say, whenever x, z correspond in $F(x, z) = 0$, $P^{-1}(x)$, $P^{-1}(z)$ correspond in the involution I (as well as in the involution $I' = JIJ$). This means that $F(x, z) = 0$ is identically satisfied if we put $x = PI(t)$, $z = P(t)$.

4. The characteristic involution.

Either of the involutions I, I' which effect the above canonical uniformization of $F(x, z) = 0$ will be called the *characteristic involution* of F . It was seen that I, I' are connected by the relation $I' = JIJ$, where J is the involution defined by the branch points i, j of F ; we shall say that two involutions I, I' connected by this relation are *converses* of one another, so that *converse values of the characteristic involution give rise to the same correspondence F* . It should not be forgotten that both the polar correspondence P and the characteristic involution I depend—the former directly and the latter indirectly—upon the triad Ω with Hessian points i, j , which has been chosen once for all.

The two fundamental relations obtained above, between F and its characteristic involution, may be restated here for purposes of reference.

THEOREM VI. *If λ be the double point of F , and J the involution defined by its branch points, the common pair of J with the characteristic involution is the first polar $P^{-1}(\lambda)$, and the transforms by P of the fixed points of the characteristic involution are the latent points of F . The three degenerate types of symmetric double-quadratics $F(x, z)$ included in the family with the branch points i, j are:*

(1) $F(x, z) = (xz)^2$; the correspondence F is the identical correspondence.

(2) $F(x, z) = (\lambda x)^2 (\lambda z)^2$; the correspondence F is singular and carries every point into (λ, λ) .

(3) $F(x, z) = \{J'(x, z)\}^2$, where J' is an involution harmonic to J ; in this case the correspondence F is the involution J' counted doubly.

In (1) the characteristic involutions are J , $JJJ = J$. In (2) the characteristic involutions are both singular, with their singular points corresponding in J . In (3) the characteristic involutions coincide with the involution J' harmonic to J . Conversely, when the characteristic involution is harmonic to J , the correspondence F must be an involution counted doubly.

5. Closed triads of correspondences F_1, F_2, F_3 .

Let us consider now the ∞^2 family of symmetric correspondences of the second type with branch points i, j . Let F_1, F_2 be two correspondences of this family with the double points p_1, p_2 , and the characteristic involutions I_1, I_2 . The product correspondence $F_2 F_1$ (obtained by performing F_1, F_2 in succession) is then equal to $PI_2 P^{-1} . PI_1 P^{-1}$. Now $P = PJ$, and therefore the $(2, 2)$ -correspondence $P^{-1} P$ is the sum of the identical homography E and the involution J ; in symbols $P^{-1} P = E + J$. Therefore

$$\begin{aligned} F_2 F_1 &= PI_2 P^{-1} PI_1 P^{-1} = PI_2 I_1 P^{-1} + PI_2 J I_1 P^{-1} \\ &= PI_2 I_1 P^{-1} + PJ I_2 J . I_1 P^{-1} \quad (\text{since } P = PJ) \\ &= F_3 + F'_3 \quad (\text{say}). \end{aligned}$$

Thus the product correspondence $F_2 F_1$ splits up into the sum of two correspondences F_3 and F'_3 .

This is a second direct proof of the resolubility of the resultant of F_1, F_2 . We shall now show that, when F_1 and the double point p_2 of F_2 are known, it is possible to choose F_2 uniquely so that one of the factors F_3, F'_3 of the resultant is symmetric. The condition that F_3 is symmetric is that $I_2 I_1$ should be an involution. From Theorem VI it is clear that, when F_2 varies so as to have a given double point p_2 , its characteristic involution I_2 always contains the fixed pair $P^{-1}(p_2)$. Hence it is possible to choose it uniquely so as to be harmonic to I_1 , that is, so that $I_1 I_2 = I_2 I_1 =$ an involution I_3 . With this value of I_2 ,

$$F_3 = PI_2 I_1 P^{-1} = PI_3 P^{-1}$$

becomes a symmetric correspondence of the same family. But there are two characteristic involutions of F_2 , namely $I_2, I'_2 = J I_2, J$, and it is also possible to choose I'_2 uniquely so that $I_1 I'_2$ is an involution. However, it is clear that this involution I'_2 is merely the converse $J I_2 J$ of the I_2 which makes $I_1 I_2$ an involution, so that it leads to the same correspondence F_2 . Thus it has been proved that F_2 can be chosen in only one way so that the resultant of F_1, F_2 has a symmetric factor F_3 . When F_2 has been thus chosen, the three correspondences F_1, F_2, F_3 are of the respective forms $PI_1 P^{-1}, PI_2 P^{-1}, PI_3 P^{-1}$, where every two of I_1, I_2, I_3 are mutually harmonic. This shows that F_1, F_2, F_3 form a symmetrical triad, in which each is the symmetric factor of the resultant of the other two. We call F_1, F_2, F_3 a closed triad of correspondences.

The above determination of the closed triad also shows that F_2 cannot in general be chosen so that both factors of the resultant are

symmetric. For if I_2 is such that both $I_1 I_2$ and $I_1 J I_2 J$ are involutions, then $I_2 = J I_2 J$, so that I_2 is either J or harmonic to J . It was seen in § 4 that in these cases F_2 is respectively the squared identical correspondence and a squared involution. Barring these reducible values of F_2 , it is seen, therefore, that the symmetry of both factors of the resultant is a feature peculiar to the Poncelet case of resolvability.

Our proof of the theorem of closed triads has been based on the principle that a component of the product-correspondence $F_2 F_1$ can be symmetric only if the product of a characteristic involution of F_1 with a characteristic involution of F_2 is an involution. This is, in general, true; for the possible forms of a component of $F_1 F_2$ are $PS_r P^{-1}$ ($r = 1, 2, 3, 4, 5, 6, 7, 8$) where

$$\begin{aligned} S_1 &= I_1 I_2, & S_2 &= I_1 I_2 J, & S_3 &= J I_1 I_2, & S_4 &= J I_1 I_2 J, \\ S_5 &= I_1 J I_2, & S_6 &= I_1 J I_2 J, & S_7 &= J I_1 J I_2, & S_8 &= J I_1 J I_2 J. \end{aligned}$$

The cases in which one of S_1, S_4, S_6, S_7 is an involution are those for which characteristic involutions are harmonic. The other cases, namely those in which S_2 or S_3 (and, therefore, also S_5 or S_8) is an involution, cannot occur unless I_1, I_2, J have a common pair, that is, unless the double points p_1, p_2 of F_1, F_2 coincide. When $p_1 = p_2$ we come back to a special case of Poncelet's theorem, the factors of the resultant being then identically symmetric. For this case, the symmetry of each of the components of $F_1 F_2$ is shown by a modification of our previous proof:

$$\begin{aligned} F_1 F_2 &= P I_1 P^{-1} P I_2 P^{-1} = P_1 I_1 (E + J) I_2 P^{-1} \\ &= P I_1 I_2 P^{-1} + P I_1 J I_2 P^{-1} \\ &= P J I_1 I_2 P^{-1} + P I_1 J I_2 P^{-1} \\ &= P I_3 P^{-1} + P I_3' P^{-1}, \end{aligned}$$

where, since I_1, I_2, J have now a common pair, I_3 and I_3' are both involutions.

6. Properties of closed triads F_1, F_2, F_3 .

The common branch points determining the family of symmetric correspondences of the second type being denoted, as usual, by i, j , we have

THEOREM VII. *Three correspondences F_1, F_2, F_3 , forming a closed triad of the family, can be determined in only one way, when their double points p_1, p_2, p_3 are known.*

For, let I_1, I_2, I_3 be the characteristic involutions of F_1, F_2, F_3 . Then, when F_1, F_2, F_3 vary so as to have the same double points p_1, p_2, p_3 , their characteristic involutions I_1, I_2, I_3 vary so as to have the same pairs $P^{-1}(p_1), P^{-1}(p_2), P^{-1}(p_3)$ in common with the involution J (Theorem VI). Now we know that we can find only two self-polar triangles of a conic whose sides pass severally through three given collinear points. Hence I_1, I_2, I_3 can be chosen only in two ways so that every two are harmonic; if I_1, I_2, I_3 form one such set, it is clear that the other set is given by the other converses JI_1J, JI_2J, JI_3J . These two sets of characteristic involutions therefore lead to only one closed triad F_1, F_2, F_3 , which is the theorem to be proved.

The closure property of F_1, F_2, F_3 implies the existence of ∞^1 point-triads x, y, z , such that x, y correspond in F_1 , y, z in F_2 , and z, x in F_3 . We proceed to show that F_1, F_2, F_3 also admit closure-tetrads of points.

THEOREM VIII. *There exists a quadratic one-parameter system of tetrads $\lambda_0, \lambda_1, \lambda_2, \lambda_3$, such that $\lambda_0, \lambda_1; \lambda_2, \lambda_3$ are two corresponding pairs of F_1 , $\lambda_0, \lambda_2; \lambda_3, \lambda_1$ of F_2 , and $\lambda_0, \lambda_3; \lambda_1, \lambda_2$ of F_3 .*

To prove this, consider the characteristic involutions I_1, I_2, I_3 of F_1, F_2, F_3 , and their converse set JI_1J, JI_2J, JI_3J . They form, as has been seen, two triads of mutually harmonic involutions, and therefore generate two syzygetic pencils of tetrads Γ, Γ' , each of which is the transform of the other by J . Thus, if x_0, x_1, x_2, x_3 be a tetrad of Γ and $J(x_0), J(x_1), J(x_2), J(x_3)$ the corresponding tetrad of Γ' , then $x_0, x_r; x_s, x_t$ are pairs of the involution I_r ($r, s, t = 1, 2, 3$), and therefore also $\{J(x_0), J(x_r)\}, \{J(x_s), J(x_t)\}$ are pairs of the involution JI_rJ ($r, s, t = 1, 2, 3$). If now $P(x_r) = PJ(x_r) = \lambda_r$ ($r = 0, 1, 2, 3$), then, since $\{P^{-1}(\lambda_0), P^{-1}(\lambda_r)\}, \{P^{-1}(\lambda_s), P^{-1}(\lambda_t)\}$ are pairs of I_r [choosing suitably the value of $P^{-1}(\lambda)$ in each case], it follows from the fundamental theorem of uniformization that $(\lambda_0, \lambda_r; \lambda_s, \lambda_t)$ are corresponding pairs of F_r ($r, s, t = 1, 2, 3$). Thus the existence of ∞^1 closure-tetrads $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ is demonstrated. To show that the system of these tetrads is quadratic, we have only to observe that if λ_0 is known, then $\{x_0, J(x_0)\} = P^{-1}(\lambda_0)$ are also known, and so we can find $\lambda_1, \lambda_2, \lambda_3$ in two ways according as we consider x_0 as a point of a tetrad of Γ or of a tetrad of Γ' .

If $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ is a closure-tetrad, it is clear that the other closure-tetrad containing λ_0 will consist of λ_0 and the three correspondents of λ_0 in F_r , other than λ_r , for $r = 1, 2, 3$.

Also, in the above, the syzygetic pencil Γ (or Γ') contains three squared pairs, namely the fixed pairs of I_1, I_2, I_3 . Hence the quadratic

system of closure-tetrads also contains three squared pairs, namely the transforms of these pairs by P . But, by Theorem VI, the transform by P of the fixed pair of the characteristic involution of F is the latent pair of F . Hence we have

THEOREM IX. *The squared latent pairs of F_1, F_2, F_3 belong to the system of closure-tetrads; the latent pair of each of a closed triad of correspondences is a common corresponding pair of the other two.*

Consider now the two closed tetrads which contain the double point p_1 of F_1 . Since the two correspondents of p_1 in F_1 coalesce, the two closed tetrads containing p_1 must also coalesce. Hence we have

THEOREM X. *The double point of any one of three closed correspondences has the same two correspondents in each of the other two.*

The same argument would show that the two closure-tetrads containing either of the branch points i, j also coalesce (though Theorem X would not hold for i or j). Now in respect of a quadratic system of tetrads, there should, in general, exist eight points with the property that the two tetrads of the system containing them coalesce. In the present instance it is easy to see that the eight points are composed of the three double points reckoned twice and the branch points i, j . These properties are theoretically sufficient to determine the quadratic system of the closure-tetrads in terms of p_1, p_2, p_3, i, j . A more direct determination can, however, be effected by the next two theorems.

A quadratic ∞^1 -system of point-sets is, in general, a rational system. Thus the system of closure-tetrads $\lambda_0, \lambda_1, \lambda_2, \lambda_3$, and the quadratic system of corresponding pairs of F_1 , are both rational systems. Further, to each closure-tetrad $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ correspond two corresponding pairs $(\lambda_0, \lambda_1), (\lambda_2, \lambda_3)$ of F_1 , while to each pair (λ_0, λ_1) of F_1 corresponds one closure-tetrad $\lambda_0, \lambda_1, \lambda_2, \lambda_3$. Thus the closure-tetrads are in (1, 2)-correspondence with the pairs of F_1 . Therefore the two pairs of F_1 $(\lambda_0, \lambda_1), (\lambda_2, \lambda_3)$ which correspond to the same closure-tetrad must be in involutory relation. Hence, since the pairs of F_1 constitute a quadratic system, the involution determined by $(\lambda_0, \lambda_1; \lambda_2, \lambda_3)$ must always contain a fixed point-pair (α, β) . To determine (α, β) we have only to consider two particular closure-tetrads. When $\lambda_0 = p_2$, then $\lambda_2 = p_2$ also, and the involution containing $\lambda_0, \lambda_1; \lambda_2, \lambda_3$ is the singular involution determined by p_2 . Similarly, when $\lambda_0 = p_3$, the involution $(\lambda_0, \lambda_1; \lambda_2, \lambda_3)$ is the singular

involution defined by p_3 . Hence the pair (α, β) must be the pair p_2, p_3 . We have thus reached

THEOREM XI. *If $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ be any closure-tetrad of three closed correspondences F_1, F_2, F_3 , with the double points p_1, p_2, p_3 , then*

$$(\lambda_0, \lambda_r; \lambda_s, \lambda_t; p_s, p_t)$$

belong to an involution ($r, s, t = 1, 2, 3$).

If μ_1, μ_2 be the latent pair of F_1 , then by Theorem IX $(\mu_1, \mu_1, \mu_2, \mu_2)$ is a closure-tetrad, so that by the present theorem $(\mu_1, \mu_1; \mu_2, \mu_2; p_2, p_3)$ belong to an involution. We have therefore

THEOREM XII. *The latent pair of each of three closed correspondences separates harmonically the double points of the other two.*

By Theorem VII, F_1, F_2, F_3 can be determined uniquely from p_1, p_2, p_3 . This theorem specifies the latent pairs of F_1, F_2, F_3 ; namely, the latent pair of F_1 is the common pair of the two involutions $(p_2, p_2; p_3, p_3)$ and $(p_1, p_1; i, j)$ (Theorem V)

7. Closed n -ads of correspondences.

If n correspondences $F_r = PI_rP^{-1}$ ($r = 1, 2, \dots, n$) with distinct double points are taken, and if p_{n+1} , the double point of F_{n+1} , is given, then it is possible to choose F_{n+1} so that the resultant of $F_1(x, z_1), F_2(z_1, z_2), \dots, F_n(z_{n-1}, z_n), F_{n+1}(z_n, y)$ has one symmetric factor $F_{n+2}(x, y)$. To prove this, we note that the operative product, in order, of $F_1, F_2, \dots, F_n$ is $PI_1I_2\dots I_nP^{-1} = PSP^{-1}$, where S is a homography. Now it is in general possible to find uniquely an involution I_{n+1} containing a given pair $P^{-1}(p_{n+1})$, such that SI_{n+1} is an involution; I_{n+1} is in fact the involution containing $P^{-1}(p_{n+1})$ and the pair of self-corresponding points of S . Now each of the characteristic involutions $I_1, I_2, \dots, I_n$ has two values, and therefore the homography S has 2^n values, which falls into pairs of the type S, JSJ . Thus the involution I_{n+1} has 2^{n-1} converse pairs of values, and so there are 2^{n-1} correspondences F_{n+1} with the required closure property, in respect of the given correspondences $F_1, F_2, \dots, F_n$ taken in a particular order.

In particular for $n = 1$, F_3 can be chosen uniquely, as we have seen. For $n = 1$, the question of order does not arise.

8. *The companion theorem of Poncelet.*

We shall now interpret our results geometrically by considering the symmetric (2, 2)-correspondences F , of the second type, between the points of a fundamental circle E . The circular points i, j are to be the branch points of F . If x, y be corresponding points of F , it is clear that the envelope of xy is a circle F having contact with E . The point of contact p is the double point of the correspondence F ; also if the common tangents of E and F touch E at μ_1, μ_2 , then μ_1, μ_2 is the latent pair of F .

The companion theorem, which expresses the possibility of constructing closed triads of correspondences, will then read as follows :

THEOREM XIII. *Given any contact-circle F_1 of E , it is possible to find uniquely a circle F_2 touching E at a given point p_2 in such a way that there exists a third contact-circle F_3 of E , with the property of admitting an infinity of inscribed triangles of E whose sides are severally tangent to F_1, F_2, F_3 .*

The contact-circles F_1, F_2, F_3 may be said to form a *closed circle-triad* of E . Theorem VII shows that the closed triad of circles F_1, F_2, F_3 can be uniquely determined from their contacts p_1, p_2, p_3 . Theorem X gives the result :

THEOREM XIV. *The contact of each circle of a closed triad is a centre of similitude of the other two.*

Theorem XII gives still another relation, namely :

THEOREM XV. *The second centre of similitude of E and any one circle of a closed triad lies on the line joining the contacts of the other two.*

Let O be the centre of E and R its radius. Then, if p_1, p_2, p_3 are the contacts of the closed triad F_1, F_2, F_3 , the centre of similitude of E and F_1 (other than the point of contact p_1) lies, by this theorem, on $p_2 p_3$. But it should also lie on the line of centres Op_1 . Hence, if $p_1 O, p_2 O, p_3 O$ intersect the opposite sides at q_1, q_2, q_3 , then q_1, q_2, q_3 are the other centres of similitude of E with F_1, F_2, F_3 respectively. This theorem determines the radii r_1, r_2, r_3 of F_1, F_2, F_3 . If A, B, C are the angles of the triangle $p_1 p_2 p_3$, and r_1, r_2, r_3 are supposed positive if measured inwards along the radius to the point of contact, it may be shown that

$$r_1 = R \tan B \tan C; \quad r_2 = R \tan C \tan A; \quad r_3 = R \tan A \tan B.$$

It also follows from Theorem XII that the centres c_1, c_2, c_3 of F_1, F_2, F_3 form with O a quadrangle which has $p_1 p_2 p_3$ for its harmonic triangle. Hence we have

THEOREM XVI. *The centres of the closed triad $F_1 F_2 F_3$ are the correspondents of O in the three involutory collineations defined by their triangle of contacts.*

Consider now the centres of similitude of the circles F_1, F_2, F_3 , two and two. By Theorem XIV, one centre of similitude of F_s, F_t is p_r ($r, s, t = 1, 2, 3$). Hence, from a property of the centres of similitude, the other centre of similitude of F_s, F_t lies on $p_s p_t$, and is therefore the intersection of $p_s p_t, c_s c_t$. Hence we have

THEOREM XVII. *If L be the polar line of O in respect of the triangle of contacts $p_1 p_2 p_3$, then the other centres of similitude of F_1, F_2, F_3 are the intersections of L with the sides of $p_1 p_2 p_3$.*

Consider next the closure-tetrads $\lambda_0, \lambda_1, \lambda_2, \lambda_3$. By Theorem XI, $\lambda_0 \lambda_r, \lambda_s \lambda_t, p_s p_t$ are concurrent. Hence the harmonic triangle of $\lambda_0 \lambda_1 \lambda_2 \lambda_3$ is inscribed in $p_1 p_2 p_3$, so that the tangents at $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ are the common tangents of E with some conic circumscribed to $p_1 p_2 p_3$. But, on the other hand, the quadratic system of the closure-tetrads contains three squared pairs corresponding to the points q_1, q_2, q_3 (Theorem IX), and is therefore the system of contacts of E with its common tangents with conics having $q_1 q_2 q_3$ for self-polar triangle. Since $q_1 q_2 q_3$ is the harmonic triangle of $p_1 p_2 p_3 O$, it follows therefore that we have

THEOREM XVIII. *The closure-tetrads $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ of F_1, F_2, F_3 are the points of contact of E with its common tangents with conics passing through its centre O and the contacts of F_1, F_2, F_3 .*

The triangle $p_1 p_2 p_3$ determines its circumcircle E , and therefore the closed circle-triad F_1, F_2, F_3 . The whole configuration presented by E and the closed-triad, with their contacts p_1, p_2, p_3 , possesses therefore an added interest from the fact that it can be regarded as a triangle-geometric configuration pertaining to $p_1 p_2 p_3$.

In reference to Poncelet's theorem, the four-point system of conics including the fundamental conic E (which corresponds to a co-discriminant family of symmetric double-quadratics) has the property of containing ∞^2 triads of members which are closed in the first or Poncelet manner. Do there exist, similarly, ∞^1 -systems of contact-circles of a fundamental circle E , which contain ∞^2 closed triads (ex-

cluding, of course, the system of circles with a common contact)? The answer is in the negative. But we can construct, as follows, ∞^1 -systems of contact-circles of E , containing ∞^1 closed triads.

In the previous configuration, $p_1 p_2 p_3$ is the harmonic triangle of $O c_1 c_2 c_3$. Hence the circle E is inpolar to any conic S passing through O, c_1, c_2, c_3 . Hence there are ∞^1 triangles $r_1 r_2 r_3$ which are inscribed to E , and are self-polar in regard to S . The centres c'_1, c'_2, c'_3 of the closed circle-triad F'_1, F'_2, F'_3 , whose contacts are r_1, r_2, r_3 , must lie on S , and be such that $r_1 r_2 r_3$ is the harmonic triangle of $O c'_1 c'_2 c'_3$. It is apparent that the system of triads c'_1, c'_2, c'_3 is a quadratic ∞^1 -system. The system $[F]$ of the contact-circles F'_1, F'_2, F'_3 admits, by definition, ∞^1 closed triads, and is such that its centre locus is the conic S . Also, since $[c'_1, c'_2, c'_3]$ is a quadratic system, each point on S is the centre of two distinct circles F_1, F_2 of $[F]$, which coalesce when $F_1 = F_2 = E$. This indicates that the system $[F]$ is a rational quartic system with the double member E . This is more clearly seen by a three-dimensional representation.

Let every circle F of the plane be put in linear correspondence with a point F of three-dimensional space, so that point-circles correspond to points on a quadric Q . Line-circles then correspond to points on the tangent-plane at a certain point ∞ on Q . The system of contact-circles of E corresponds to the tangent cone from E to Q . The points on S correspond to the points on a rational quartic curve on Q having a double point ∞ . The system of circles with their centres on S accordingly corresponds to the quadric cone which projects this quartic curve from ∞ . Since S passes through O , this cone must pass through E . The system $[F]$ therefore corresponds to the intersection of the tangent cone from E with a second quadric cone which passes through E , and is therefore a rational quartic system with the double member E .

There do not appear to exist ∞^1 -systems of contact-circles, of order less than four, containing an infinite number of closed triads.

A THEOREM CONCERNING THE SUMMABILITY OF SERIES BY BOREL'S METHOD

By T. VIJAYARAGHAVAN.

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1. The series Σa_n is said to be *summable (B) to sum s* if

$$e^{-x} \sum \frac{s_n x^n}{n!} \rightarrow s \quad \left(s_n = \sum_{r=0}^n a_r \right)$$

when $x \rightarrow \infty$. It is to be understood, except when otherwise stated, that $|s| < \infty$. It was proved by Hardy and Littlewood* that if $|n^{\frac{1}{2}} a_n| < \kappa$ then summability (B) implies convergence. It has been recently proved by Schmidt† that the same result remains true even under the condition

$$\lim (s_m - s_n) \geq 0,$$

where $m > n$, and m and n tend to infinity so that $n^{-\frac{1}{2}}(m-n) \rightarrow 0$. In what follows a sequence of the above type will be called a *slowly decreasing sequence*.

The proof given by Schmidt involves the theory of the "moment problems" of Stieltjes and Hamburger, and is much less direct and elementary than the proof given here. I use in the course of the proof the chief results of Hardy and Littlewood's paper mentioned above.

The principle which underlies the proof is, of course, that the r -th differential coefficient of a function may be studied more effectively than the function itself. In the "Abel" problem, it is sufficient to consider a differential coefficient of large fixed order; but this weapon is not suffi-

* *Rend. di Palermo*, 41 (1915), 1-18. For the very much easier case in which $n^{\frac{1}{2}} a_n \rightarrow 0$, see their earlier paper in *Proc. London Math. Soc.* (2), 11 (1913), 1-16.

† *Schriften der Königsberger gelehrten Gesellschaft*, 1 (1925), 205-256.

ciently powerful here, and it is necessary to consider what is in essentials, a combination of the differential coefficients of *all* orders.

There is a large measure of similarity between the arguments which I use here and those of my previous paper on the "Abel" problem*.

2. LEMMA 1. *If a sequence s_n be slowly decreasing and if Σa_n is summable (B), then*

$$s_n = O(1).$$

The proof of this lemma depends on the following lemmas.

LEMMA a. *If $H \rightarrow \infty$, and $x > 1$, then*

$$e^{-x} \sum_{n=x+H\sqrt{x}}^{\infty} \frac{x^n}{n!} \rightarrow 0$$

uniformly in x .

Let

$$T_n = T(n, x) = x^n/n!, \quad N = N(x) = [x+1], \quad M = M(x, H) = [x+H\sqrt{x}],$$

$$M_r = M(x, H, r) = [x+(H+r)\sqrt{x}],$$

where the square brackets are used to denote the integer parts. We shall also suppose that H and x are so large (say $H > 4$, $x > 25$) that

$$(A) \quad M < \frac{1}{2}Hx, \quad (M-N)^2 > \frac{1}{2}H^2x, \quad \frac{x^x}{\Gamma(x+1)} \geq \text{Max}_{n=1, 2, \dots} T_n.$$

Then

$$\begin{aligned} T_M &= T_N \frac{x}{N+1} \frac{x}{N+2} \dots \frac{x}{M} < T_N \left(1 - \frac{1}{N+1}\right) \left(1 - \frac{2}{N+2}\right) \dots \left(1 - \frac{M-N}{M}\right) \\ &< T_N \exp \left(-\frac{1}{N+1} - \frac{2}{N+2} - \dots - \frac{M-N}{M} \right) \\ &< T_N \exp \left(-\frac{1}{M} \frac{1}{2}(M-N)^2 \right) \end{aligned}$$

$$(B) \quad < T_N e^{-H} \leq e^{-H} x^x / \Gamma(x+1)$$

by (A). Also

$$e^{-x} \sum_{n=M}^{\infty} T_n < e^{-x} \left(\sum_M^{M_1} + \sum_{M_1}^{M_2} + \dots \right) \\ < e^{-x} \{ (M_1 - M) T_M + (M_2 - M_1) T_{M_1} + \dots \} < \frac{e^{-x} (\sqrt{x+2}) x^x}{\Gamma(x+1)} \sum_{n=H}^{\infty} e^{-n}$$

by (B). It is evident from Stirling's theorem that if $H \rightarrow \infty$ then the above expression tends to zero uniformly in x . Lemma α is therefore proved.

LEMMA β . If $H \rightarrow \infty$, and $x > H^2$, then

$$e^{-x} \sum_{n=0}^{x-H\sqrt{x}} \frac{x^n}{n!} \rightarrow 0$$

uniformly in x .

Let $A = [x]$, $\theta = x - A$, $1 \leq B \leq A - 1$.

It is evident that

$$T_{A-B} = T_A \left(1 - \frac{\theta}{x}\right) \left(1 - \frac{1+\theta}{x}\right) \dots \left(1 - \frac{B-1+\theta}{x}\right),$$

$$T_{A+B} = T_A \frac{x}{A+1} \left(1 - \frac{1-\theta}{A+2}\right) \left(1 - \frac{2-\theta}{A+3}\right) \dots \left(1 - \frac{B-1-\theta}{A+B}\right).$$

Now if $x > 1$, then

$$1 - \frac{\theta}{x} < 1 < 2 \frac{x}{A+1}, \quad 1 - \frac{1+\theta}{x} < 1 - \frac{1-\theta}{A+2}, \\ \left(1 - \frac{2+\theta}{x}\right) \dots \left(1 - \frac{B-2+\theta}{x}\right) < 1 - \frac{2-\theta}{A+B}, \quad \dots$$

Hence it follows that $T_{A-B} < 2T_{A+B}$.

Combining this with the result of Lemma α we at once deduce Lemma β . Lemmas α and β lead to

LEMMA γ . If $H \rightarrow \infty$, and $x > H^2$, then

$$e^{-x} \sum_{n=x-H\sqrt{x}}^{x+H\sqrt{x}} \frac{x^n}{n!} \rightarrow 1 \quad (1)$$

uniformly in x .

LEMMA δ . If $H \rightarrow \infty$, and M be defined as in Lemma α , then

$$e^{-x} \sum_{n=M}^{\infty} (\sqrt{n} - \sqrt{m}) \frac{x^n}{n!} \rightarrow 0$$

uniformly in x .

Let $M_1, M_2, \dots$ be defined as in Lemma α , so that

$$M_1 - M < \sqrt{x} + 1, \quad M_2 - M_1 < \sqrt{x} + 1, \quad \dots,$$

$$\sqrt{M_1} - \sqrt{M} < 1, \quad \sqrt{M_2} - \sqrt{M} < 2, \quad \dots$$

We then have

$$\begin{aligned} e^{-x} \sum_{n=M}^{\infty} (\sqrt{n} - \sqrt{m}) \frac{x^n}{n!} &< e^{-x} \left(\sum_{M}^{M_1} + \sum_{M_1}^{M_2} + \dots \right) \\ &< e^{-x} (\sqrt{x} + 2) \frac{x^x}{\Gamma(x+1)} \sum_{r=1}^{\infty} r e^{-H-r+1} \rightarrow 0 \end{aligned}$$

uniformly in x . This proves the lemma.

LEMMA ϵ . If s_n be a slowly decreasing sequence, then to every positive number c there corresponds a number $k = k(c)$ such that

$$s_m - s_n > -\{k(\sqrt{m} - \sqrt{n}) + c\},$$

for $m > n$.

It is given that $\lim (s_m - s_n) \geq 0$,

when $m > n$, and $(m-n)/\sqrt{n} \rightarrow 0$. It therefore follows that to every positive number c there corresponds another positive number a such that

$$s_m - s_n > -c$$

whenever

$$(m-n)/\sqrt{n} < 2a.$$

Let $n_s = n(s, a) = [n + sa\sqrt{n}]$ ($s = 0, 1, 2, \dots$)

and let $r = r(n, m)$ be such that

$$n_r < m \leq n_{r+1}.$$

Then $n_{s+1} - n_s = [n + (s+1)a\sqrt{n}] - [n + sa\sqrt{n}] < a\sqrt{n} + 1$,

so that

$$(n_{s+1} - n_s)/\sqrt{n_s} < (a\sqrt{n} + 1)/\sqrt{n} < 2a,$$

when $n > a^{-2}$. Therefore, if $n > a^{-2}$, we have

$$\begin{aligned} s_m - s_n &= (s_m - s_{n_r}) + (s_{n_r} - s_{n_{r-1}}) + \dots + (s_{n_1} - s_n) \\ &> -(r+1)c > -\{ca^{-1}(m-n)/\sqrt{n}+c\}. \end{aligned}$$

Hence we see that, if $k = c/a$, then the lemma will hold for all $n > a^{-2}$. It is plain that k may then be chosen so that the inequality is true for all $m > n$.

3. We now proceed to the proof of Lemma 1. It is easy to see that if $s_n \rightarrow \infty$ or $s_n \rightarrow -\infty$, then there will be a contradiction. Hence it is sufficient to deduce a contradiction from the conditions that

$$\overline{\lim} |s_n| = \infty,$$

and that the sequence is oscillatory. Let

$$s_1(n) = \text{Max}_{r \leq n} s_r, \quad s_2(n) = \text{Max}_{r \leq n} (-s_r).$$

There are two possibilities of behaviour of $s_1(n)$ and $s_2(n)$.

(a) There are an infinity of values of n such that

$$s_1(n) \geq s_2(n),$$

or (b)

$$s_1(n) < s_2(n).$$

for all large n . Clearly the possibilities are exhaustive. Let us now consider the case (a). It is clear that in this case $s_1(n) \rightarrow \infty$. Given $H > 0$, we can find an $N = N(H)$, such that

$$s_N = s_1(N) > 2H, \quad s_1(N) \geq s_2(N).$$

We take the least N satisfying these conditions, and then the least

$$M = M(H) > N,$$

such that $s_M \leq \frac{1}{2}s_N$: $M(H)$ will exist for all large H , since s_n does not tend to infinity. We now take

$$x = x(H) = \frac{1}{2}(N+M)$$

so that $x \rightarrow \infty$ when $H \rightarrow \infty$. By Lemma 2 there is a constant k such that, for all $m > n$,

$$s_m - s_n > -\{k(\sqrt{m} - \sqrt{n}) + 1\}.$$

In particular (taking $m = M$, $n = N$)

$$\sqrt{M} - \sqrt{N} > (H-1)/k \rightarrow \infty$$

as $H \rightarrow \infty$. Therefore it follows that, when $H \rightarrow \infty$, $(M-x)/\sqrt{x} \rightarrow \infty$ and $(x-N)/\sqrt{x} \rightarrow \infty$. Now let

$$I = e^{-x} \sum_{n=0}^{\infty} s_n \frac{x^n}{n!} = \sum_0^{N-1} + \sum_N^{M-1} + \sum_M^{\infty} = I_1 + I_2 + I_3.$$

Then
$$I_1 = e^{-x} \sum_{n=0}^{N-1} s_n \frac{x^n}{n!} > -e^{-x} s_1(N) \sum_{n=0}^{N-1} \frac{x^n}{n!} = \delta_H s_1(N)$$

by Lemma β . Here, and in what follows, δ_H denotes a function of H which is independent of x and which tends to zero when $H \rightarrow \infty$. By using Lemma γ we get

$$I_2 > (1 - \delta_H) \frac{1}{2} s_1(N).$$

Finally
$$I_3 = e^{-x} \sum_M^{\infty} s_n \frac{x^n}{n!} = e^{-x} \sum_M^{\infty} (s_M + s_n - s_M) \frac{x^n}{n!}.$$

We have from Lemma ϵ

$$s_n - s_M > -\{k(n-M)/\sqrt{M+1}\},$$

and
$$s_M > s_{M-1} - k/\sqrt{(M-1)-1} > H - k/\sqrt{(M-1)-1} > 1$$

if H is sufficiently large. Hence

$$I_3 > -e^{-x} \sum_M^{\infty} \{k(n-M)/\sqrt{M}\} \frac{x^n}{n!} = \delta_H$$

by Lemma δ . Hence
$$I_1 + I_2 + I_3 \rightarrow \infty$$

as $H \rightarrow \infty$. This contradiction proves that case (a) is not possible.

In case (b), $s_1(n) < s_2(n)$ for all large n , and clearly $s_2(n) \rightarrow \infty$. Hence, given H , we can find $M = M(H)$, such that

$$s_2(n) > s_1(n) \quad (n \geq M),$$

and
$$s_M = -s_2(M) < -2H.$$

We take the least M satisfying these conditions, and denote by $N = N(H)$ the largest number less than M such that

$$s_{N-1} \geq \frac{1}{2} s_M = -\frac{1}{2} s_2(M).$$

Evidently $(M-N)/\sqrt{N} \rightarrow \infty$ as $H \rightarrow \infty$. We again take

$$x = x(H) = \frac{1}{2}(M+N),$$

so that, when $H \rightarrow \infty$, we have $x \rightarrow \infty$, $(M-x)/\sqrt{x} \rightarrow \infty$, $(x-N)/\sqrt{x} \rightarrow \infty$. Let

$$I = e^{-x} \sum_0^{\infty} s_n \frac{x^n}{n!} = \sum_0^{N-1} + \sum_N^{M-1} + \sum_M^{\infty} = I_1 + I_2 + I_3.$$

Then

$$I_1 = e^{-x} \sum_0^{N-1} s_n \frac{x^n}{n!} < e^{-x} s_1(M) \sum_0^{N-1} \frac{x^n}{n!}$$

$$< s_2(M) e^{-x} \sum_0^{N-1} \frac{x^n}{n!} = \delta_H s_2(M)$$

by Lemma β . Also

$$I_2 = e^{-x} \sum_N^{M-1} s_n \frac{x^n}{n!} < -e^{-x} \frac{1}{2} s_2(M) \sum_N^{M-1} \frac{x^n}{n!}$$

$$< -(\frac{1}{2} - \delta_H) s_2(M)$$

by Lemma γ . Finally

$$I_3 = e^{-x} \sum_M^{\infty} s_n \frac{x^n}{n!} < e^{-x} \sum_M^{\infty} s_2(n) \frac{x^n}{n!}$$

$$< e^{-x} \sum_M^{\infty} \{s_2(M) + k(\sqrt{n} - \sqrt{M}) + c\} \frac{x^n}{n!} \quad (\text{by Lemma } \epsilon)$$

$$= \delta_H s_2(M)$$

by Lemmas α and δ . Hence

$$I = I_1 + I_2 + I_3 \rightarrow -\infty$$

as $H \rightarrow \infty$, and this again contradicts the hypothesis. This completes the proof of Lemma 1.

4. LEMMA 2. If Σa_n is summable (B) and if $s_n = O(1)$, then

$$\lim_{x \rightarrow \infty} \frac{1}{\sqrt{(2\pi x)}} \int_{-\infty}^{\infty} e^{-t^2/2x} f(t+x) dt = s,$$

where

$$f(y) = s_n + (y-n)(s_{n+1} - s_n) \quad (n \leq y \leq n+1),$$

$$f(y) = 0 \quad (y < 0).$$

It is known that, if Σa_n is summable (B) to sum s , and $s_n = O(\sqrt{n})$, then also*

$$\lim_{x \rightarrow \infty} \frac{1}{\sqrt{(2\pi x)}} \int_{-\infty}^{\infty} e^{-t^2/2x} s(t+x) dt = s,$$

where $s(t+x) = s([t+x])$, and $s(y) = 0$ for $y < 0$. It is evident *a fortiori* that this is true for the case under consideration, since, according to our hypothesis,

$$s_n = O(1).$$

Now let

$$\theta(y) = y - [y].$$

Observing that $0 \leq \theta(y) < 1$, $\theta(y)$ has the period unity, $0 \leq e^{-t^2/2x} \leq 1$, and $e^{-t^2/2x}$ is monotonic in $|t|$, we see that, if

$$x > 0, \quad [x] = M, \quad b_n = b(n, x) = \int_{-x+n}^{-x+n+1} e^{-t^2/2x} \theta(t+x) dt,$$

then

$$0 < b_n < 1 \quad (n = 1, 2, \dots),$$

$$b_0 < b_1 < b_2 < \dots < b_M, \quad b_{M+1} > b_{M+2} > \dots$$

Hence

$$\begin{aligned} & \left| \int_{-\infty}^{\infty} e^{-t^2/2x} f(t+x) dt - \int_{-\infty}^{\infty} e^{-t^2/2x} s(t+x) dt \right| = \left| \sum_{n=1}^{\infty} a_n b_{n-1} \right| \\ & \leq \left(\max_{n=1, 2, \dots, M} |s_n - s_0| \right) \left(\max_{n=1, 2, \dots, M} b_n \right) + \left(\max_{n=1, 2, \dots, \infty} |s_{M+n} - s_M| \right) \left(\max_{n=1, 2, \dots, \infty} b_n \right) \\ & < 4 \left(\max_{n=1, 2, \dots, \infty} |s_n| \right) = O(1). \end{aligned}$$

Now, since

$$I(x) = \frac{1}{\sqrt{(2\pi x)}} \int_{-\infty}^{\infty} e^{-t^2/2x} s(t+x) dt \rightarrow s,$$

when $x \rightarrow \infty$, it follows that

$$\frac{1}{\sqrt{(2\pi x)}} \int_{-\infty}^{\infty} e^{-t^2/2x} f(t+x) dt = I(x) + O\left(\frac{1}{\sqrt{(2\pi x)}}\right) \rightarrow s.$$

5. LEMMA 3. If s_n be a sequence of numbers for which the results of Lemmas 1 and 2 hold, then for any a ,

$$\sqrt{\left(\frac{a}{\pi x}\right)} \int_{-\infty}^{\infty} e^{-at^2/x} f(t+x) dt \rightarrow s.$$

* G. H. Hardy and J. E. Littlewood, *loc. cit.*, Theorem 3.4.

Hardy and Littlewood have proved (Lemma 2.21 of the paper cited earlier) that the above conclusion is valid if we assume that

$$f'(x) = O(1/\sqrt{x}),$$

and that Lemma 2 holds. In their proof this condition is used solely to prove that $f(x) = O(1)$. But since, according to our hypothesis, $s_n = O(1)$, and since this implies that

$$f(x) = s_n + (x-n)(s_{n+1} - s_n) = O(1) \quad (n \leq x \leq n+1),$$

we see that the lemma follows from the Hardy-Littlewood lemma.

LEMMA 4. *If δ be any positive constant and if $a \rightarrow \infty$, then*

$$I_\delta = \sqrt{\left(\frac{a}{\pi x}\right)} \int_{-\delta\sqrt{x}}^{\delta\sqrt{x}} e^{-at^2/x} dt \rightarrow 1$$

uniformly in x .

$$\text{For} \quad I = \sqrt{\left(\frac{a}{\pi x}\right)} \int_{-\infty}^{\infty} e^{-at^2/x} dt = 1,$$

$$I - I_\delta = \sqrt{\left(\frac{a}{\pi x}\right)} 2 \left(\int_{\delta\sqrt{x}}^{2\delta\sqrt{x}} + \int_{2\delta\sqrt{x}}^{3\delta\sqrt{x}} + \dots \right)$$

$$< \sqrt{\left(\frac{a}{\pi x}\right)} 2\delta\sqrt{x} \left(\sum_{n=1}^{\infty} e^{-n^2 a \delta^2} \right)$$

$$< \sqrt{\left(\frac{a}{\pi}\right)} 2\delta 2e^{-a\delta^2},$$

provided that a is sufficiently large, say $a > \delta^{-2}$. Hence, when $a \rightarrow \infty$,

$$I - I_\delta = O(a^{\frac{1}{2}} e^{-a\delta^2}) \rightarrow 0$$

uniformly in x . Since $I = 1$, the result at once follows.

We are now in a position to prove the principal result of the paper.

6. THEOREM. *If s_n is slowly decreasing, and if Σa_n is summable (B) to sum s , then $s_n \rightarrow s$.*

We may suppose, without loss of generality, that $s = 0$. We know from Lemma 1 that s_n is bounded. Let H be the upper bound of s_n and let h be such that

$$\overline{\lim} |s_n| > 4h > 0.$$

If there is no h for which this is true, then the theorem is true. It will be shown below that the supposition that a positive h exists leads to a contradiction.

We can determine $c = c(h)$ such that

$$s_m - s_n > -h$$

whenever $n \leq m \leq n + c\sqrt{n}$. We write also

$$F(a) = F(a, h, x) = \sqrt{\left(\frac{a}{\pi x}\right)} \int_{-(c/3)x}^{(c/3)x} e^{-at^2/4} dt, \quad G(a) = 1 - F(a).$$

By Lemma 4, if $a \rightarrow \infty$, then

$$F(a) \rightarrow 1, \quad G(a) \rightarrow 0,$$

uniformly in x , and therefore we can find a value of a so large that

$$3hF(a) - HG(a) > 2h.$$

Since $\overline{\lim} s_n > 4h$, we can find a sequence of numbers $\lambda_1, \lambda_2, \dots$ for which

$$|s_{\lambda_n}| > 4h \quad (n = 1, 2, \dots).$$

We shall suppose, as we may, that all the s_{λ_n} are of the same sign, so that we have either

$$(a) \qquad s_{\lambda_n} > 4h \quad (n = 1, 2, \dots),$$

$$\text{or } (\beta) \qquad s_{\lambda_n} < -4h \quad (n = 1, 2, \dots).$$

These possibilities are exhaustive. Now in the case (a) we have $s_m < 3h$ whenever $\lambda_n \leq m \leq \lambda_n + c\sqrt{\lambda_n}$. Let

$$x_n = \lambda_n + \frac{1}{2}c\sqrt{\lambda_n},$$

so that $f(t+x_n) > 3h$ when

$$-\frac{1}{3}c\sqrt{x_n} \leq t \leq \frac{1}{3}c\sqrt{x_n}.$$

Then it is easy to see that

$$\sqrt{\left(\frac{a}{\pi x_n}\right)} \int_{-\infty}^{\infty} e^{-at^2/4x_n} f(t+x_n) dt > 3hF(a) - HG(a) > 2h.$$

Hence, when $x \rightarrow \infty$ through the values x_n , the above expression does not tend to zero. This contradicts Lemma 3, and therefore (a) is not possible.

If, on the other hand, we take case (β) , we have $s_m < -3h$ whenever $\lambda'_n \leq m \leq \lambda_n$, where λ'_n is defined by the equation*

$$\lambda'_n + c\sqrt{\lambda'_n} = \lambda_n.$$

If we take $x_n = \frac{1}{2}(\lambda_n + \lambda'_n)$, we see, as before, that

$$\sqrt{\left(\frac{a}{\pi x_n}\right)} \int_{-\infty}^{\infty} e^{-t^2/x_n} f(t+x_n) dt < -2h,$$

and this again is a contradiction. Hence the theorem is proved.

When $|s| = \infty$, the methods of this paper do not apply. In fact, the theorem of this paper is not true when $|s| = \infty$. I have discussed elsewhere† this and similar cases in the "Abel" problem.

* It is clear that $(\lambda_n - \lambda'_n)/\sqrt{\lambda_n} \rightarrow c$.

† *Journal London Math. Soc.*, 2 (1927), 215-222.

A THEOREM IN THE THEORY OF SUMMABLE DIVERGENT SERIES

By G. H. HARDY and J. E. LITTLEWOOD.

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1. Introduction.

1.1. This paper arises from a theorem which we published in 1924* and which proved to have important applications in the theory of Fourier series. The theorem ran as follows: *in order that the series*

$$(1.11) \quad \Sigma b_n = b_0 + b_1 + b_2 + \dots$$

should be summable (C, r) , it is necessary and sufficient that the series

$$(1.12) \quad \Sigma c_n = c_0 + c_1 + c_2 + \dots,$$

where

$$(1.13) \quad c_n = \frac{b_n}{n+1} + \frac{b_{n+1}}{n+2} + \dots,$$

* G. H. Hardy and J. E. Littlewood, "Solution of the Cesàro summability problem for power-series and Fourier series", *Math. Zeitschrift*, 19 (1924), 67-96. See also "The allied series of a Fourier series", *Proc. London Math. Soc.* (2), 24 (1925), 211-246.

The substance of the present paper was written in the summer of 1925, and its publication has been delayed for various reasons. The subject is essentially the same as that of the independent researches of Andersen published recently in the *Proceedings*, though there are considerable differences in the point of view and the methods of proof. See A. F. Andersen, "Comparison theorems in the theory of Cesàro summability", *Proc. London Math. Soc.* (2), 27 (1927), 39-71. In particular one of our principal theorems (Theorem 3) is included as a particular case in Andersen's Theorem 2 (stated by him in the form of two theorems 2A and 2B), Andersen considering arbitrary real values of r where we consider only integral values.

We shall also have occasion to refer (under the short title of *Studier*) to Andersen's dissertation "Studier over Cesàro's Summabilitetsmetode", Copenhagen, 1921, which contains the most general account of the theory of summability yet published.

Finally we should add that all these theorems are very closely related to theorems proved earlier by Knopp. See K. Knopp, "Über die Oszillation einfach unbestimmter Reihen", *Sitzungsberichte der Berliner Math. Gesellschaft*, 16 (1917), 45-50, and "Zur Theorie der C - und H -Summierbarkeit", *Math. Zeitschrift*, 19 (1924), 97-113. See also G. H. Hardy, "A theorem concerning summable series", *Proc. Camb. Phil. Soc.*, 20 (1921), 304-307, and W. L. Ferrar, "Necessary and sufficient conditions for summability (C, r) ", *Journal London Math. Soc.*, 1 (1926), 175-179.

should be summable $(C, r-1)$, and the sums of the two series are the same. Here r is a positive integer or zero, and the sum (1.13) by which c is defined is, for $r > 0$, a Cesàro sum of order $r-1$.

This theorem may be stated in a different form which is more illuminating and lends itself more naturally to extension. Suppose, first, that (1.11) is summable (C, r) to sum B , and write

$$b_n = (n+1)a_n.$$

Then Σa_n is summable $(C, r-1)^*$, say to sum A , and

$$c_0 = A, \quad c_n = A - a_0 - a_1 - \dots - a_{n-1} = A - A_{n-1} \quad (n > 0).$$

It follows that

$$A + (A - A_0) + (A - A_1) + \dots$$

is summable $(C, r-1)$ to sum B , or that

$$\sum_{n=0}^{\infty} (A - A_n)$$

is summable $(C, r-1)$ to sum $B-A$. Conversely, if this series is, for any value of A , summable $(C, r-1)$ to sum $B-A$, then

$$A_n \rightarrow A \quad (C, r-1),$$

so that Σa_n is summable $(C, r-1)$ to sum A . If now we write

$$c_n = a_n + a_{n+1} + \dots \quad (C, r-1),$$

then Σc_n is summable $(C, r-1)$ to sum $B-A+A=B$, and so Σb_n is summable (C, r) to the same sum.

We thus obtain

THEOREM 1. *In order that the series*

$$(1.14) \quad \Sigma(n+1)a_n$$

should be summable (C, r) to sum B , it is necessary and sufficient that

$$(1.15) \quad \Sigma(A - A_n),$$

where $A_n = a_0 + a_1 + \dots + a_n$ and A is the sum $(C, r-1)$ of Σa_n , should be summable $(C, r-1)$ to sum $B-A$.

* By a well known theorem due to M. Riesz. See S. Chapman, "On non-integral orders of summability of series and integrals", *Proc. London Math. Soc.* (2), 9 (1911), 369-409 (388); where a more general theorem is proved. A slightly less precise form of the theorem was proved independently by H. Bohr, *Bidrag til de Dirichlet'ske Raekkers Theori*, Copenhagen, 1910.

1.2. We observe now that, if we write generally

$$\Delta u_n = u_n - u_{n+1},$$

then $\Delta(n+1) = -1$. Thus Theorem 1 asserts that the series (1.14) is summable (C, r) if, and only if,

$$\Sigma(A_n - A) \Delta(n+1)$$

is summable $(C, r-1)$, *i.e.* if its order of summability is reduced by unity by an appropriate partial summation.

We are thus led to conjecture that if ϕ_n is any sufficiently regular function of a suitable rate of increase, the series

$$(1.21) \qquad \Sigma a_n \phi_n$$

will be summable (C, r) if, and only if, the series

$$(1.22) \qquad \Sigma(A_n - A) \Delta \phi_n,$$

where A is an appropriate Cesàro sum of Σa_n , is summable $(C, r-1)$. Similarly we are led to suppose that the integral

$$(1.23) \qquad \int_0^\infty a(x) \phi(x) dx,$$

will be summable (C, r) if, and only if, the integral

$$(1.24) \qquad \int_0^\infty \{A(x) - A\} \phi'(x) dx,$$

where

$$(1.25) \qquad A(x) = \int_0^x a(t) dt$$

and A is an appropriate Cesàro sum of

$$(1.26) \qquad \int_0^\infty a(x) dx,$$

is summable $(C, r-1)$. Our object here is to investigate how far these conjectures are true.

1.3. Our conclusions, which it is convenient to state for integrals in the first instance, may be summarized as follows. Our theorem, that the summability $(C, r-1)$ of (1.24) is a necessary and sufficient condition for the summability of (C, r) of (1.23), is true, in the first instance, for

$$\phi(x) = x^a \quad (a > 0).$$

This we prove in § 3. More generally, it is true if $\phi(x)$ is a sufficiently

regular function whose rate of increase is approximately that of a positive power of x . The conditions to be imposed upon $\phi(x)$ could be stated, for any given value of r , as inequalities to be satisfied by $\phi(x)$ and its first $r+1$ derivatives; but it is more convenient to adopt a different point of view. We suppose that $\phi(x)$ is, in the language of Hardy's *Orders of infinity**, a "logarithmico-exponential" function or " L -function", that is to say a function definable by some finite combination of algebraic, logarithmic, or exponential symbols. This being so, the theorem is true if $\phi(x)$ is an L -function of the region

$$x^\delta < \phi < x^\lambda,$$

that is to say an L -function which increases neither more nor less rapidly than all positive powers of x . If $\phi(x)$ lies outside this region, the theorem is false; thus when $\phi(x) = \log x$ our condition is necessary, but not sufficient, while if $\phi(x) = e^x$ it is sufficient, but not necessary.

This we prove in § 5. In § 4 we prove the theorem for series which corresponds to that of § 3; here ϕ_n has the special form

$$\phi_n = \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)},$$

which is more convenient than the substantially equivalent form $(n+1)^\alpha$. The more general theorem corresponding to that of § 5 is discussed briefly in § 6.

There is naturally no really important difference between the proofs of the theorems for series and for integrals, but partial integration is formally a little simpler than partial summation, and we have therefore put integrals first throughout the paper. The formal difficulties are then reduced to a minimum, and the kernel of the argument is clearer, while it is easy to indicate the modifications necessary in the other case.

2. Definitions and lemmas.

2.1. We suppose that $f(x)$ is integrable† over any finite interval $(0, X)$, and we write

(2.11)

$$F_0(x) = F(x) = \int_0^x f(t) dt, \quad F_1(x) = \int_0^x F_0(t) dt, \quad F_2(x) = \int_0^x F_1(t) dt, \quad \dots$$

* G. H. Hardy, *Orders of infinity*, Cambridge Tracts in Mathematics, 12 (2nd ed., 1924), 17.

† In the sense of Lebesgue.

Then

$$(2.12) \quad F_k(x) = \frac{1}{(k-1)!} \int_0^x (x-t)^{k-1} f(t) dt$$

if $k > 0$. If

$$(2.13) \quad F_k(x) \sim F \frac{x^k}{k!}$$

when $x \rightarrow \infty$, we say that

$$(2.14) \quad \int_0^\infty f(x) dx = F \quad (C, k),$$

or that the integral (2.14) is summable (C, k) to sum F . We also say that

$$(2.15) \quad F_0(x) \rightarrow F \quad (C, k).$$

We shall say that the integral (2.14) is summable $(C, -1)$ if it is convergent, i.e. summable $(C, 0)$, and

$$(2.16) \quad f(x) = o\left(\frac{1}{x}\right).$$

We shall also write

$$(2.17) \quad F_0(x) \sim Kx^l, \quad F_0(x) = O(x^l), \quad F_0(x) = o(x^l) \quad (C, k),$$

where $l > -1$, meaning thereby

$$(2.18) \quad F_k(x) \sim K \frac{x^{l+k}}{(l+1)(l+2) \dots (l+k)}, \quad F_k(x) = O(x^{l+k}), \quad F_k(x) = o(x^{l+k})$$

respectively*.

We shall use F generally to denote either the integral (2.14) or its value or sum, if it has one, and F_0 to denote $F_0(x)$ or $F(x)$, or the same function of some other variable. And we shall apply the same system of notation, without further explanation, to letters other than f, F .

2.2. In dealing with series we write

$$(2.21) \quad F_n^0 = F_n = f_0 + f_1 + \dots + f_n, \quad F_n^1 = F_0^1 + F_1^0 + \dots + F_n^0, \quad \dots$$

so that

$$(2.22) \quad F_n^k = \sum_{\nu=0}^n F_{n-\nu}^{k-1} f_\nu,$$

where

$$(2.221) \quad F_n^k = \binom{n+k}{n} = \frac{(n+k)!}{n! k!} = \frac{\Gamma(n+k+1)}{\Gamma(n+1) \Gamma(k+1)}.$$

* The first approximation is to mean Kx^l when $k=0$.

The last equation may be regarded as defining P_n^k for all real values of k greater than -1 .

We shall say that

$$(2.23) \quad \Sigma f_n = f_0 + f_1 + f_2 + \dots = F \quad (C, k),$$

or that Σf_n , or the series F , is summable (C, k) to sum F , if

$$(2.24) \quad F_n^k \sim F P_n^k,$$

or (what is the same thing) if

$$(2.241) \quad F_n^k \sim F \frac{n^k}{k!}.$$

We shall also say in these circumstances that

$$(2.25) \quad F_n^0 \rightarrow F \quad (C, k).$$

If F is convergent, *i.e.* summable $(C, 0)$, and

$$(2.26) \quad f_n = o\left(\frac{1}{n}\right),$$

we shall say that F is summable $(C, -1)$.

Finally, by

$$(2.27) \quad F_n^0 \sim K P_n^l, \quad F_n^0 = O(P_n^l), \quad F_n^0 = o(P_n^l) \quad (C, k)$$

we mean

$$(2.28) \quad F_n^k \sim K P_n^{l+k}, \quad F_n^k = O(P_n^{l+k}), \quad F_n^k = o(P_n^{l+k}).$$

respectively. Here l is any real number greater than -1 .

2.3. LEMMA 1. If

$$(2.31) \quad F_0(x) \sim K x^l \quad (C, k),$$

where $l > -1$, then

$$(2.32) \quad x^{-s} F_s(x) \sim K \frac{x^l}{(l+1)(l+2)\dots(l+s)} \quad (C, k-s)$$

for $s = 0, 1, 2, \dots, k$. In particular, if

$$(2.33) \quad F_0(x) \rightarrow F \quad (C, k),$$

then

$$(2.34) \quad x^{-s} F_s(x) \rightarrow \frac{F}{s!} \quad (C, k-s).$$

If

$$(2.35) \quad F_0(x) = O(x^l) \quad (C, k),$$

then

$$(2.36) \quad x^{-s} F_s(x) = O(x^l) \quad (C, k-s);$$

and similarly with $o(x^l)$.

It is sufficient to prove that (2.32) follows from (2.31), and, since the conclusion is obvious when $k=0$ or $k=s$, we have to prove that

$$(2.37) \quad \frac{1}{(k-s-1)!} \int_0^x F_s(t) t^{-s} (x-t)^{k-s-1} dt \\ \sim \frac{K}{(l+1)(l+2) \dots (l+s)} \frac{x^{l+k-s}}{(l+1)(l+2) \dots (l+k-s)}$$

for $1 \leq s \leq k-1$. Integrating the left-hand side $k-s$ times by parts, we obtain

$$(2.38) \quad x^{-s} F_k(x) + \frac{(-1)^{k-s}}{(k-s-1)!} \int_0^x F_k(t) \left(\frac{d}{dt} \right)^{k-s} \left\{ \frac{(x-t)^{k-s-1}}{t^s} \right\} dt,$$

the terms integrated out vanishing until the last partial integration. If we effect the differentiations, we obtain a sum of multiples of integrals of the type

$$\int_0^x F_k(t) (x-t)^{l-s-1-\lambda} t^{-k+\lambda} dt,$$

where $0 \leq \lambda \leq k-s-1$, and each of these is asymptotically equivalent to the integral obtained by replacing $F_k(t)$ by

$$K \frac{t^{l+k}}{(l+1)(l+2) \dots (l+k)}.$$

It follows that our original integral is asymptotically equivalent to that obtained by replacing $F_k(t)$ by its asymptotic value; and this, on reversing the process of partial integration, becomes

$$\frac{1}{(k-s-1)!} \int_0^x K \frac{t^{l+s}}{(l+1) \dots (l+s)} t^{-s} (x-t)^{k-s-1} dt \\ = \frac{K}{(l+1) \dots (l+s)} \frac{x^{l+k-s}}{(l+1) \dots (l+k-s)}.$$

2.4. We shall say that $\psi(x)$ is a *regular convergence factor* in (ξ, ∞) if (a) it is indefinitely differentiable in (ξ, ∞) , and (b) $(-1)^s \psi^{(s)}(x)$, where $s \geq 0$ and $\psi^{(s)}$ is the s -th derivative of ψ , is positive from a certain

point onwards (this point in general depending upon s) and tends to zero when $x \rightarrow \infty$. Thus x^{-a} ($a > 0$) is a regular convergence factor in (ξ, ∞) if $\xi > 0$, but not in $(0, \infty)^*$.

LEMMA 2. *If*

$$(2.41) \quad F_0(x) = O(x^l) \quad (C, k),$$

where $l > -1$, and $\psi(x)$ is a regular convergence factor in $(0, \infty)$, then

$$(2.42) \quad F_0(x) \psi(x) = o(x^l) \quad (C, k).$$

We shall be concerned in the argument only with a finite number of terms $(-)^s \psi^{(s)}(x)$, and these will be positive from a certain point on, the same for all. There is, therefore, no real loss of generality in supposing that each derivative which occurs is of fixed sign for all positive x .

We may suppose $k > 0$. We have

$$(2.43) \quad \int_0^x F_0 \psi(x-t)^{k-1} dt \\ = (k-1)! F_k \psi + (-1)^k \int_0^x F_k \left(\frac{d}{dt} \right)^k \{ (x-t)^{k-1} \psi \} dt,$$

by k partial integrations†, or

$$(2.44) \quad \int_0^x F_0 \psi(x-t)^{k-1} dt \\ = (k-1)! F_k \psi + \sum_{s=1}^k \binom{k}{s} (k-1)(k-2) \dots s \int_0^x F_k (-1)^s \psi^{(s)}(x-t)^{s-1} dt.$$

Every function which occurs here as a coefficient of F_k is positive. Hence the order of magnitude of the right-hand side is not greater than that of the function obtained by replacing F_k by x^{l+k} or t^{l+k} ; and this function is transformed, by reversing the process of partial integration, into a constant multiple of

$$\int_0^x \psi t^l (x-t)^{l-1} dt = o(x^{l+l}),$$

which proves the lemma.

* [The fact that x^{-a} has a singularity at the origin introduces some small complications which have no bearing on the essence of the problem; see, for example, Lemma 3 and its application in § 3.3. For this reason we now regret that we did not work throughout with $(x+1)^a$ instead of x^a ; the latter factor is more convenient for printing, but its disadvantages outweigh its advantages. It is not possible to make the change now without great inconvenience. (Added Dec. 1926).]

† It is to be understood that when a function, such as F_k , occurs without an argument under the integral sign, the argument is the variable of integration.

We add a corollary, which will be useful :

LEMMA 3. *We may suppose, in Lemma 2, that $\psi(x) = x^{-\beta}$, where $\beta > 0$, provided that $l > \beta - 1$ and that $F_0(x) = o(x^{\beta-1})$ when $x \rightarrow 0$.*

We have only to observe that

$$F_1\psi = o(1), \quad F_2\psi' = o(1), \quad F_3\psi'' = o(1), \quad \dots,$$

when $x \rightarrow 0$, so that the partial integrations are still legitimate. The rest of the proof goes as before, provided that $l - \beta > -1$.

2.5. The corresponding lemmas for series are as follows.

LEMMA 4. *If*

$$(2.51) \quad F_n^0 \sim KP_n^l \quad (C, k),$$

where $l > -1$, then

$$(2.52) \quad F_n^s/P_n^s \sim K(P_n^l/P_n^l) \quad (C, k-s)$$

for $s = 0, 1, \dots, k$. In particular, if

$$(2.53) \quad F_n^0 \rightarrow F \quad (C, k),$$

then

$$(2.54) \quad F_n^s/P_n^s \rightarrow F \quad (C, k-s).$$

There are also results for O and o corresponding to those of Lemma 1.

We say that ψ_n is a regular convergence factor if it and each of its differences is positive from a certain point onwards and tends to zero when $n \rightarrow \infty$.

LEMMA 5. *If*

$$(2.55) \quad F_n^0 = O(P_n^l) \quad (C, k),$$

where $l > -1$, and ψ_n is a regular convergence factor, then

$$(2.56) \quad F_n^0\psi_n = o(P_n^l) \quad (C, k).$$

2.6. The proofs of Lemmas 4 and 5 are the same in principle as those of Lemmas 1 and 2, and we need only indicate the formal differences in the argument. If

$$G_n^0 = F_n^s/P_n^s,$$

we have

$$G_n^{k-s} = \sum_{j=0}^n P_{n-j}^{k-s-1} G_j^0 = \sum_{j=0}^n F_j^s (P_{n-j}^{k-s-1}/P_j^s).$$

Now Abel's formula for partial summation may be written

$$\sum_{j=0}^n u_j v_j = \sum_{j=0}^n U_j^0 \Delta v_j,$$

if we adopt the convention that any v whose suffix exceeds n is to be regarded as zero, so that Δv_n means v_n . Hence, with this convention,

$$(2.61) \quad G_n^{k-s} = \sum_{j=0}^n F_j^k \Delta^{k-s} (P_{n-j}^{k-s-1} / P_j^s).$$

This sum corresponds to (2.38). It follows, as in the proof of Lemma 1, that G_n^{k-s} is asymptotic to the sum obtained by replacing, in (2.61), F_j^k by $K P_j^{l+k}$. We thus obtain a dominant sum which, on reversing the process of partial summation, reduces to

$$K \sum_{j=0}^n P_j^{l+s} (P_{n-j}^{k-s-1} / P_j^s) \sim K (P_n^{k-s+l} / P_s^l);$$

and this proves (2.52).

We should add that the most important part of Lemma 4, that is to say that (2.54) follows from (2.53), has been proved, in a more general form, by Andersen*.

To prove Lemma 5 we observe that, if

$$G_n^0 = F_n^0 \psi_n,$$

$$\text{we have} \quad G_n^k = \sum_{j=0}^n P_{n-j}^{k-1} G_j^0 = \sum_{j=0}^n F_j^0 P_{n-j}^{k-1} \psi_j = \sum_{j=0}^n F_j^k \Delta^k (P_{n-j}^{k-1} \psi_j),$$

with the same convention as before regarding ψ 's whose rank exceeds n . This formula may be written

$$(2.62) \quad G_n^k = \sum_{j=0}^n F_j^k \sum_{s=1}^k P_s^{k-s} P_{n-j}^{s-1} \Delta^s \psi_{j+k-s}.$$

The formula (2.62) is, in the first instance, subject to the convention; but the convention may be ignored, since the terms which it suppresses are $o(n^{l+k})$; and from this point the argument proceeds as before.

3. Integrals: the case $\phi(x) = x^a$.

3.1. THEOREM 2. If

$$(3.11) \quad \phi(x) = x^a \quad (a > 0),$$

* See A. F. Andersen, *Studier*, 64 (where the result is stated without proof).

and $a(x)$ is integrable over any finite interval $(0, X)$, then the necessary and sufficient condition that

$$(3.12) \quad \int_0^\infty a(x) \phi(x) dx$$

should be summable (C, r) , to sum B , is that

$$(3.13) \quad \int_0^\infty \{A_0(x) - A\} \phi'(x) dx$$

should be summable $(C, r-1)$ to sum $-B$. Here r is a positive integer or zero; A is the sum (C, r) of the integral

$$(3.14) \quad \int_0^\infty a(x) dx;$$

and the sufficiency of the criterion is to be understood as implying that, if (3.13) is summable $(C, r-1)$ for any value of A , then (3.12) and (3.14) are summable (C, r) , and A is necessarily the value of the latter integral.

It is to be observed that (3.14) is certainly summable (C, r) if (3.12) is so*.

There is no real loss of generality in supposing throughout that $A = 0$. If this were not so, we could replace $a(x)$ by the function

$$\bar{a}(x) = a(x) - A \quad (0 \leq x \leq 1), \quad \bar{a}(x) = a(x) \quad (x > 1).$$

We therefore adopt this simplification.

We write

$$(3.15) \quad b(x) = a(x) \phi(x),$$

$$(3.16) \quad p(x) = A_0(x) \phi'(x),$$

$$(3.17) \quad q(x) = -B_0(x) \frac{d}{dx} \left\{ \frac{1}{\phi(x)} \right\},$$

so that when, as here, $\phi(x) = x^a$, we have

$$b = x^a a, \quad p = ax^{a-1} A_0, \quad q = ax^{-a-1} B_0.$$

It may be verified at once that

$$(3.18) \quad P_0(x) = Q_0(x) \phi(x) = A_0(x) \phi(x) - B_0(x).$$

* See G. H. Hardy, "Notes on some points in the integral calculus (30)", *Messenger of Math.*, 40 (1911), 108-112. The theorem there proved is the analogue for integrals of what Andersen calls the "Bohr-Hardy'ske Sætning".

3.2. We dispose first of the case $r = 0$. We have then to show that (3.12) is convergent if and only if (i) (3.13) is convergent and (ii)

$$(3.21) \quad A_0 \phi' = o\left(\frac{1}{x}\right).$$

Suppose first that (3.12) is convergent, so that $B_0 \rightarrow B$. Then $A_0 \rightarrow A = 0$, and so

$$Q_0 = A_0 - \frac{B_0}{\phi} \rightarrow 0.$$

Now

$$(3.22) \quad P_0 = x^a Q_0 = ax^a \int_0^x t^{-a-1} B_0 dt,$$

and

$$Q = a \int_0^\infty t^{-a-1} B_0 dt$$

is convergent, so that $Q_0 \rightarrow Q$. It follows that $Q = 0$, and that we may write (3.22) in the form

$$(3.23) \quad P_0 = -ax^a \int_x^\infty t^{-a-1} B_0 dt.$$

Thus

$$P_0 \sim -aBx^a \int_x^\infty t^{-a-1} dt = -B,$$

which proves (i). Also

$$A_0 \phi' = ax^{a-1} A_0 = \frac{a}{x} (B_0 + P_0) = o\left(\frac{1}{x}\right),$$

which proves (ii).

The converse implication is trivial, since (ii) gives $A_0 = o(x^{-a})$ and

$$B_0 + P_0 = x^a A_0 = o(1).$$

3.3. Passing to the general case, we suppose first that (3.12) is summable (C, r) , so that

$$(3.31) \quad B_r(x) \sim B \frac{x^r}{r!}.$$

Then

$$\begin{aligned} (3.32) \quad Q_0 &= -\int_0^x B_0 \frac{d}{dt} \left(\frac{1}{\phi} \right) dt = -B_1 \frac{d}{dx} \left(\frac{1}{\phi} \right) + \int_0^x B_1 \frac{d^2}{dt^2} \left(\frac{1}{\phi} \right) dt \\ &= \dots = \sum_{s=1}^r (-1)^s B_s \frac{d^s}{dx^s} \left(\frac{1}{\phi} \right) + (-1)^{r+1} \int_0^x B_r \frac{d^{r+1}}{dt^{r+1}} \left(\frac{1}{\phi} \right) dt \\ &= \sum_{s=1}^r j_s + J_{r+1}, \end{aligned}$$

say. Hence, by (3.18),

$$(3.33) \quad P_0 = \sum_{s=1}^r \phi j_s + \phi J_{r+1}.$$

Now

$$(3.34) \quad x^{-s} B_s \rightarrow \frac{B}{s!} \quad (C, r-s),$$

by Lemma 1, and

$$(3.35) \quad j_s = a(a+1) \dots (a+s-1) x^{-a-s} B_s \rightarrow 0 \quad (C, r-s),$$

by Lemmas 2 and 3*. Also the integral

$$(3.36) \quad Q = (-1)^{r+1} \int_0^\infty B^r \frac{d^{r+1}}{dx^{r+1}} \left(\frac{1}{\phi} \right) dx$$

is convergent. It follows from (3.32), (3.35), and (3.36) that

$$Q_0 \rightarrow Q \quad (C, r-1).$$

But $Q_0 = A_0 - B_0/\phi$, and A_0 and B_0/ϕ tend to zero (C, r) , the first by hypothesis and the second by Lemmas 2 and 3. Thus Q must be zero, and we may write (3.33) in the form

$$(3.37) \quad P_0 = \sum_{s=1}^r \phi j_s + \phi I_{r+1},$$

where

$$(3.38) \quad I_{r+1} = (-1)^r \int_x^\infty B_r \frac{d^{r+1}}{dt^{r+1}} \left(\frac{1}{\phi} \right) dt.$$

3.4. Now

$$(3.41) \quad \phi j_s = a(a+1) \dots (a+s-1) x^{-s} B_s \rightarrow \frac{a(a+1) \dots (a+s-1)}{s!} B \quad (C, r-s),$$

by (3.34), and

$$(3.42) \quad \begin{aligned} \phi I_{r+1} &= -a(a+1) \dots (a+r) x^a \int_x^\infty B_r t^{-a-r-1} dt \\ &\sim -\frac{a(a+1) \dots (a+r)}{r!} B x^a \int_x^\infty t^{-a-1} dt = -\frac{(a+1) \dots (a+r)}{r!} B. \end{aligned}$$

* The application of Lemma 3 requires a word of explanation. We have to take

$$F_0 = x^{-s} B_s, \quad l=0, \quad k=r-s, \quad \beta=a.$$

Since $b = x^a a$, and a is integrable down to 0, $B_0 = o(x^a)$ and $B_s = o(x^{a+s})$, so that $F_0 = o(x^a) = o(x^{a-1})$. Hence the second special condition of Lemma 3 is satisfied.

The first condition ($l > \beta - 1$) is only satisfied if $a < 1$. If $a > 1$ we must use the lemma several times in succession. If e.g. $1 \leq a < 2$, we may use the lemma first, with $\beta = \frac{1}{2}a$, to show that $x^{-\frac{1}{2}a-s} B_s \rightarrow 0 \quad (C, r-s)$. We then take this function as a new F_0 , and apply the lemma a second time, again with $\beta = \frac{1}{2}a$. The first condition is evidently satisfied in both applications.

Finally, from (3.37), (3.41), and (3.42),

$$P_0 \rightarrow \left\{ a + \frac{a(a+1)}{2!} + \dots + \frac{a(a+1) \dots (a+r-1)}{r!} - \frac{(a+1) \dots (a+r)}{r!} \right\} B \\ = -B \quad (C, r-1).$$

3.5. Next, we suppose that (3.13), with $A = 0$, is summable $(C, r-1)$, so that

$$(3.51) \quad P_{r-1} \sim P \frac{x^{r-1}}{(r-1)!}.$$

We have
$$B_0 = q \frac{\phi^2}{\phi'},$$

$$(3.52) \quad B_1 = \int_0^x q \frac{\phi^2}{\phi'} dt = Q_0 \frac{\phi^2}{\phi'} - \int_0^x Q_0 \frac{d}{dt} \left(\frac{\phi^2}{\phi'} \right) dt \\ = P_0 \frac{\phi}{\phi'} - \int_0^x P_0 \psi dt,$$

where

$$(3.53) \quad \psi = \frac{1}{\phi} \frac{d}{dt} \left(\frac{\phi^2}{\phi'} \right).$$

In the present case this formula takes a particularly simple form, since

$$(3.54) \quad \frac{\phi}{\phi'} = \frac{x}{a}, \quad \psi = \frac{1+a}{a},$$

$$(3.55) \quad B_1 = \frac{x}{a} P_0 - \frac{1+a}{a} P_1;$$

from which we deduce

$$(3.56) \quad B_r = \frac{x}{a} P_{r-1} - \frac{r+a}{a} P_r \\ \sim \frac{P}{(r-1)! a} \left\{ x^r - (r+a) \int_0^x t^{r-1} dt \right\} = -P \frac{x^r}{r!}.$$

This completes the proof of the theorem.

$$4. \text{ Series: the case } \phi_n = \frac{\Gamma(n+1+a)}{\Gamma(n+1)}.$$

4.1. THEOREM 3. If

$$(4.11) \quad \phi_n = \frac{\Gamma(n+1+a)}{\Gamma(n+1)} \quad (a > 0),$$

then the necessary and sufficient condition that

$$(4.12) \quad \sum_0^{\infty} a_n \phi_n$$

should be summable (C, r) , to sum B , is that

$$(4.13) \quad \sum_0^{\infty} (A_n^0 - A) \Delta \phi_n$$

should be summable $(C, r-1)$ to sum $B - A\phi_0$. Here A is the sum (C, r) of Σa_n ; and the sufficiency of the criterion is to be interpreted as in Theorem 1.

It is to be observed, as in § 3.1, that Σa_n is certainly summable (C, r) if $\Sigma a_n \phi_n$ is so, and that there is no loss of generality in supposing $A = 0$.

4.2. The proof is substantially the same as that of Theorem 2. We write

$$(4.21) \quad b_n = a_n \phi_n, \quad p_n = A_n^0 \Delta \phi_n, \quad q_n = -B_n^0 \Delta \left(\frac{1}{\phi_n} \right),$$

so that when, as here, $\phi_n = \frac{\Gamma(n+1+a)}{\Gamma(n+1)}$, we have

$$b_n = a_n \frac{\Gamma(n+1+a)}{\Gamma(n+1)}, \quad p_n = -a A_n^0 \frac{\Gamma(n+1+a)}{\Gamma(n+2)}, \quad q_n = -a B_n^0 \frac{\Gamma(n+1)}{\Gamma(n+2+a)}.$$

Then

$$(4.22) \quad P_n^0 = Q_n^0 \phi_{n+1} = -A_n^0 \phi_{n+1} + B_n^0.$$

We dispose first of the case $r = 0$; the proof in this case is so like that of § 3.2 that we need hardly repeat it. Passing to the general case, we suppose first that (4.12) is summable (C, r) , so that

$$(4.23) \quad B_n^r \sim B P_n^r = B \binom{n+r}{n}.$$

Then

$$\begin{aligned} (4.24) \quad Q_n^0 &= -\sum_0^n B_j^0 \Delta \left(\frac{1}{\phi_j} \right) = -B_n^1 \Delta \left(\frac{1}{\phi_n} \right) - \sum_0^{n-1} B_j^1 \Delta^2 \left(\frac{1}{\phi_j} \right) \\ &= \dots = -\sum_{s=1}^r B_{n-s+1}^s \Delta^s \left(\frac{1}{\phi_{n-s+1}} \right) - \sum_0^{n-r} B_j^r \Delta^{r+1} \left(\frac{1}{\phi_j} \right) \\ &= -\sum_{s=1}^r \sigma_s - S_{r+1}, \end{aligned}$$

say.

We now prove, much as in § 3.3, that the series

$$\sum_0^{\infty} B_n^r \Delta^{r+1} \left(\frac{1}{\phi_n} \right)$$

converges to zero. The essential point is that

$$\sigma_s \rightarrow 0 \quad (C, r-s);$$

and

$$\sigma_s = B_{n-s+1}^s \Delta^s \left(\frac{1}{\phi_{n-s+1}} \right) = \frac{\alpha(\alpha+1)\dots(\alpha+s-1)}{s!} \frac{\Gamma(n+2)}{\Gamma(n+2+\alpha)} \frac{B_{n-s+1}^s}{P_{n-s+1}^s},$$

which tends to zero $(C, r-s)$ by Lemmas 4 and 5.

We have therefore

$$(4.25) \quad P_n^0 = Q_n^0 \phi_{n+1} = - \sum_{s=1}^r \phi_{n+1} \sigma_s + \phi_{n+1} T_{r+1},$$

where

$$T_{r+1} = \sum_{n-r+1}^{\infty} B_j^r \Delta^{r+1} \left(\frac{1}{\phi_j} \right),$$

and the proof may now be completed as in § 3.4.

4.3. The converse inference is (as with integrals) easier. We have

$$\begin{aligned} (4.31) \quad q_n &= -B_n^0 \Delta \left(\frac{1}{\phi_n} \right) = B_n^0 \frac{\Delta \phi_n}{\phi_n \phi_{n+1}}, \\ B_n^1 &= \sum_0^n B_j^0 = \sum_0^n q_j \frac{\phi_j \phi_{j+1}}{\Delta \phi_j} \\ &= Q_n^0 \frac{\phi_n \phi_{n+1}}{\Delta \phi_n} + \sum_0^{n-1} Q_j^0 \Delta \left(\frac{\phi_j \phi_{j+1}}{\Delta \phi_j} \right) \\ &= P_n^0 \frac{\phi_n}{\Delta \phi_n} + \sum_0^{n-1} P_j^0 \psi_j, \end{aligned}$$

where

$$(4.32) \quad \psi_n = \frac{1}{\phi_{n+1}} \Delta \left(\frac{\phi_n \phi_{n+1}}{\Delta \phi_n} \right).$$

But here

$$(4.33) \quad \frac{\phi_n}{\Delta \phi_n} = -\frac{n+1}{\alpha}, \quad \psi_n = \frac{1+\alpha}{\alpha},$$

so that (4.31) becomes

$$(4.34) \quad B_n^1 = -\frac{n+1}{\alpha} P_n^0 + \frac{1+\alpha}{\alpha} P_{n-1}^1.$$

From this it follows, as in § 3.5, that $P_n^0 \rightarrow P$ ($C, r-1$) implies $B_n^0 \rightarrow P$ (C, r). This completes the proof of Theorem 3.

4.4. We have already mentioned the particular case of Theorem 3 in which $\alpha = 1$, $\phi_n = n+1$. Suppose now that we specialize by making $r = 1$. Then the necessary and sufficient condition that

$$\Sigma b_n = B \quad (C, 1)$$

is that

$$(4.41) \quad \sum_{j=0}^{n-1} (A_j^0 - A) \Delta \phi_j \rightarrow B - A \phi_0.$$

But
$$B_n^0 = \sum_{j=0}^n a_j \phi_j = \sum_{j=0}^{n-1} (A_j^0 - A) \Delta \phi_j + A \phi_0 + (A_n^0 - A) \phi_n.$$

Hence (4.41) takes the form

$$(4.42) \quad B_n^0 + (A - A_n^0) \phi_n \rightarrow B,$$

or

$$(4.43) \quad B_n^0 + \phi_n \left(\frac{b_{n+1}}{\phi_{n+1}} + \frac{b_{n+2}}{\phi_{n+2}} + \dots \right) \rightarrow B,$$

where the series in brackets is a $(C, 1)$ sum.

If $\alpha \geq 1$, and Σb_n is summable $(C, 1)$, the series in brackets is convergent. We thus obtain

THEOREM 4. *The necessary and sufficient condition that Σb_n should be summable $(C, 1)$, to sum B , is that*

$$b_0 + b_1 + b_2 + \dots + b_n + \phi_n \left(\frac{b_{n+1}}{\phi_{n+1}} + \frac{b_{n+2}}{\phi_{n+2}} + \dots \right) \rightarrow B,$$

where $\alpha \geq 1$ and
$$\phi_n = \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)}.$$

This theorem is due to Knopp. It is to be observed that it remains true for $0 < \alpha < 1$, provided that the series in brackets is regarded as a summable series only.

5. Integrals: the general case.

5.1. We suppose now that $\phi(x)$ is an L -function* of the range (x^0, x^A) . Then $\phi(x)$ possesses derivatives of every order, all of which

* For an account of the principal properties of L -functions see Hardy's tract quoted on p. 330. We refer to this tract as *O.I.*

are continuous and monotonic from a certain value of x onwards*, and we may suppose, as we are at any moment considering a finite number of derivatives only, that these conditions are satisfied for $x > 0$. Then†

$$\phi = x^a \chi,$$

where χ is an L -function for which

$$x^{-\delta} < \chi < x^{\delta}.$$

We shall suppose, in order to avoid trivial complications, that χ and its derivatives are continuous also for $x = 0$. In these circumstances we shall say that ϕ is an L -function of the range (x^{δ}, x^{Δ}) regular for $x > 0$.

The orders of magnitude of the derivatives of ϕ and χ , or of any elementary combinations of them, may be calculated by the rules laid down in Chap. 5 of Hardy's tract‡.

If η is an L -function which tends to zero, then either η or $-\eta$ is a regular convergence factor in the sense of § 2. 4. If, e.g., η is ultimately positive, then η' is ultimately negative and tends to zero, and the argument may evidently be repeated.

THEOREM 5. *If $\phi(x)$ is any L -function of the range (x^{δ}, x^{Δ}) , regular for $x > 0$, then the necessary and sufficient condition that the integral (3. 12) should be summable (C, r) , to sum B , is that the integral (3. 13) should be summable $(C, r-1)$ to sum $-B + A\phi(0)$.*

We may suppose, as before, that $A = 0$; and we may confine ourselves to pointing out where the argument of §§ 3. 3-3. 5 requires modification.

Very little change is necessary in §§ 3. 3-3. 4. The formulae (3. 32) and (3. 33) are unaltered, but we must reconsider the behaviour of the various derivatives of $1/\phi$. We have

$$\frac{1}{\phi} = x^{-a} \omega,$$

where ω is an L -function of the range $(x^{-\delta}, x^{\delta})$; and

$$\frac{d}{dx} \left(\frac{1}{\phi} \right) = -ax^{-a-1} \omega \left(1 - \frac{x\omega'}{a\omega} \right) = -ax^{-a-1} \omega(1+\eta),$$

* *O.I.*, 17-20.

† *O.I.*, 21-22.

‡ *O.I.*, 32-43.

where η is an L -function which tends to zero*. Similarly

$$\left(\frac{d}{dx}\right)^s \left(\frac{1}{\phi}\right) = (-1)^s a(a+1) \dots (a+s-1) x^{-a-s} \omega(1+\eta),$$

where η has the same significance. Thus every function which arises is asymptotically equivalent to the corresponding function of §§ 3.3-3.4, and its dominant term leads to an identical result. The error term differs from the dominant term in every case by a factor η , and the corresponding Cesàro limit is zero, by Lemma 2. The conclusion is therefore unaltered.

5.2. Rather more alteration is required in the argument of § 3.5, since the function there denoted by ψ is no longer a constant. We have now

$$\begin{aligned} (5.21) \quad B_1 &= P_0 \frac{\phi}{\phi'} - \int_0^x P_0 \psi dt = P_0 \frac{\phi}{\phi'} - P_1 \psi + \int_0^x P_1 \psi' dt \\ &= P_0 \frac{\phi}{\phi'} + \sum_{s=1}^{r-1} (-1)^s P_s \psi^{(s-1)} + (-1)^r \int_0^x P_{r-1} \psi^{(r-1)} dt. \end{aligned}$$

Now

$$(5.22) \quad \frac{\phi}{\phi'} = \frac{x}{a} (1+\eta), \quad \frac{\phi^2}{\phi'} = \frac{x^{a+1}}{a} X (1+\eta),$$

$$(5.23) \quad \psi = \frac{1}{\phi} \frac{d}{dx} \left(\frac{\phi^2}{\phi'} \right) = \frac{1+a}{a} + \eta,$$

$$(5.24) \quad \psi^{(s-1)} = x^{-s+1} \eta \quad (s > 1),$$

where η is in each case an L -function which tends to zero. Thus we may write (5.21) in the form

$$(5.25) \quad B_1 = \frac{x}{a} P_0 - \frac{1+a}{a} P_1 + \sum_{s=1}^{r-1} \mu_s + \nu_r,$$

where

$$(5.26) \quad \mu_s = x^{-s+1} \eta P_s, \quad \nu_r = (-1)^r \int_0^x P_{r-1} t^{-r+1} \eta dt.$$

5.3. Now

$$P_{r-1} \sim P \frac{x^{r-1}}{(r-1)!},$$

so that

$$(5.31) \quad P_1 \sim Px \quad (C, r-2).$$

* *O.I.*, 36.

It follows from Lemma 1, with $l = 1$, that

$$\frac{P_2}{x} \sim \frac{1}{2}Px \quad (C, r-3),$$

and so
$$\frac{1}{x} \int_0^x tP_0 dt = P_1 - \frac{P_2}{x} \sim \frac{1}{2}Px \quad (C, r-2),$$

$$xP_0 \sim Px \quad (C, r-1),$$

$$(5.32) \quad \frac{x}{a} P_0 - \frac{1+a}{a} P_1 \sim -Px \quad (C, r-1).$$

It also follows from (5.31) and Lemma 1 that

$$x^{-s+1}P_s \sim \frac{Px}{s!} \quad (C, r-s-1)$$

for $s = 1, 2, \dots, r-1$; and from Lemma 2 that

$$(5.33) \quad \mu_s = o(x) \quad (C, r-s-1).$$

Finally

$$(5.34) \quad \nu_r = \int_0^\infty O(t^{r-1}) o(t^{-r+1}) dt = o(x);$$

and from (5.25), (5.32), (5.33), and (5.34) we obtain

$$B_1 \sim -Px \quad (C, r-1)$$

or

$$B_0 \rightarrow -P \quad (C, r).$$

5.4. It is natural to ask whether the result of Theorem 5 is valid when $\phi(x)$ is an L -function of the ranges $(1, x^\delta)$ or (x^Δ, ∞) . An examination of the argument shows that the hypothesis $\phi < x^\delta$ is used only in the argument from P to B , and the hypothesis $\phi < x^\Delta$ only in the argument from B to P . If $\phi < x^\delta$

$$\phi \frac{d}{dx} \left(\frac{1}{\phi} \right) = -\frac{\phi'}{\phi} < \frac{1}{x},$$

and generally

$$\phi \left(\frac{d}{dx} \right)^s \left(\frac{1}{\phi} \right) < \frac{1}{x^s}.$$

In this case the argument of §§ 3.3-3.4 succeeds as though a were zero. The only point of the proof which requires special comment is that involved in (3.38) and (3.42). We have to show that $\phi I_{r+1} \rightarrow -B$,

and this reduces to showing that

$$\frac{(-1)^{r-1}}{r!} \phi \int_x^\infty t^r \left(\frac{d}{dt} \right)^{r+1} \left(\frac{1}{\phi} \right) dt \rightarrow 1.$$

But, if $\psi = 1/\phi$, we have*

$$\frac{(-1)^r}{r!} \psi^{(r+1)} \sim x^{-r} \psi' = -x^{-r} \frac{\phi'}{\phi^2},$$

and so
$$\frac{(-1)^{r-1}}{r!} \phi \int_x^\infty t^r \left(\frac{d}{dt} \right)^{r+1} \left(\frac{1}{\phi} \right) dt \sim \phi \int_x^\infty \frac{\phi'}{\phi^2} dt = 1.$$

If $\phi \succ x^\lambda$ this argument fails; but that of § 3.5, which depends essentially on the fact that ψ tends to a limit when $x \rightarrow \infty$, succeeds, since

$$\psi = 2 - \frac{\phi \phi''}{\phi'^2} \rightarrow 1.$$

We have, in fact,

THEOREM 6. *If ϕ is an L-function of the range $1 < \phi < x^\delta$, then the summability $(C, r-1)$ of (3.13) is a necessary, but not a sufficient, condition for the summability (C, r) of (3.12). If ϕ is an L-function of the range $\phi \succ x^\lambda$, then the condition is sufficient but not necessary.*

It is easy to give examples to prove the negative assertions of the theorem. If

$$\phi(x) = e^x, \quad a(x) = e^{aix - x} \quad (a > 0),$$

then

$$\int_0^\infty a(x) \phi(x) dx = \int_0^\infty e^{aix} dx$$

is summable $(C, 1)$, but

$$\int_0^\infty \{A_0(x) - A\} \phi'(x) dx = \frac{1}{ai-1} \int_0^\infty e^{aix} dx$$

is not convergent. On the other hand, if

$$\phi(x) = \log x, \quad a(x) = \frac{x^{-1-ai}}{\log x} \quad (x \geq 2), \quad a(x) = 0 \quad (x < 2),$$

then

$$A_0(x) = \int_2^x \frac{t^{-1-ai}}{\log t} dt = A - \frac{x^{-ai}}{ia \log x} + O\left(\frac{1}{(\log x)^2}\right),$$

* O.I., 36-37 (see especially 37, top).

and $\int_2^\infty \{A_0(x) - A\} \frac{dx}{x} = -\frac{1}{ia} \int_2^\infty \frac{x^{-1-ai}}{\log x} dx + \int_2^\infty O\left(\frac{1}{x(\log x)^2}\right) dx$

is convergent. But

$$\int_2^\infty a(x) \log x dx = \int_2^\infty x^{-1-ai} dx$$

is not summable by any Cesàro mean.

6. Series : the general case.

6.1. It will hardly be necessary now that we should enter into details concerning the proof of the theorem for series corresponding to Theorem 5. Suppose first that n , in Theorem 3, is replaced by $n+1$, so that n runs over 1, 2, 3, ... Then ϕ_n may be replaced by n^a or by any function of the form

$$n^a \left(1 + \frac{u_1}{n} + \frac{u_2}{n^2} + \dots\right),$$

where the series is convergent or asymptotic. In fact, if

$$\psi_n = \phi_n \left(1 + \frac{u_1}{n} + \frac{u_2}{n^2} + \dots\right),$$

then the series

$$\sum a_n \phi_n, \quad \sum a_n \psi_n$$

are summable or non-summable, whatever the order, together*.

Finally, if we observe that, when ϕ_n is an L -function of n , of the range $n^\delta < \phi_n < n^\Delta$, then all the differences of ϕ_n are also L -functions, and are asymptotically equivalent to multiples of the corresponding derivatives, we may at once enunciate

THEOREM 7. *The result of Theorem 3 remains valid when ϕ_n is any L -function of the range $n^\delta < \phi_n < n^\Delta$.*

* The materials necessary for a formal proof will be found in Andersen, *Studier*, Kap. 3.

MATHEMATICAL NOTES (5): ON THE FUNCTION $1/\zeta(1+ti)$.

By J. E. LITTLEWOOD.

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1. In a recent paper* in the Proceedings of this Society I proved the following inequalities for Riemann's function $\zeta(s)$. Let C be Euler's constant and $c = e^C$. Then

$$(1.1) \quad \overline{\lim}_{t \rightarrow \infty} \frac{|\zeta(1+ti)|}{\log \log t} \leq 2c,$$

$$(1.2) \quad \overline{\lim}_{t \rightarrow \infty} \frac{|\hat{\zeta}(1+ti)|}{\log \log t} \geq c;$$

of which (1.1) is proved subject to the truth of the Riemann hypothesis, (1.2) without any hypothesis. The point of the results consists in the clear-cut factor 2 which distinguishes the two right-hand sides: similar results with other constants were known before.

The number

$$(1.3) \quad \overline{\lim} \frac{|1/\hat{\zeta}(1+ti)|}{\log \log t}$$

is known (on the Riemann hypothesis) to be finite; indeed, in proving (1.1) we show at the same time that (1.3) does not exceed $2c\dagger$. (1.3) is known also to be positive. Bohr and Landau‡ have recently discovered a remarkable method of overcoming the formidable difficulties of Diophantine approximation that this problem raises. Thus (1.3) lies

* J. E. Littlewood, 1. (See the list of papers at the end.)

† J. E. Littlewood, 1 (189, Theorem 7).

‡ H. Bohr and E. Landau, 2 and 3.

between two positive absolute constants. §§ 2-6 below set out what can be done by adaptations of existing arguments to make these constants precise. My main object in the note is not so much this discussion as the general review of the subject (in § 7) of which it forms the text.

2. We use A to denote a positive absolute constant, not always the same from one occurrence to another. ϵ, δ denote "small" arbitrary positive constants. p denotes a typical prime, p_n the n -th prime in order of magnitude ($p_1 = 2$). $\Lambda(n)$ denotes, as usual, $\log p$ if n is a prime power p^m , and otherwise denotes zero. We write $\Lambda_1(n)$ for $\Lambda(n)/\log n$. c denotes e^C as in § 1, and we write b for the constant $6c/\pi^2$.

LEMMA 1. *We have, for large x ,*

$$\log \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} = \log(c \log x) + o(1);$$

$$\log \prod_{p \leq x} \left(1 + \frac{1}{p}\right) = \log(b \log x) + o(1).$$

The first result, due to Mertens, is proved in Landau's "Handbuch", page 140. Another proof has been given recently by Hardy*.

For the second we have

$$\begin{aligned} \log \prod_{p \leq x} \left(1 + \frac{1}{p}\right) &= \log \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} - \log \prod_p \left(1 - \frac{1}{p^2}\right)^{-1} + \log \prod_{p > x} \left(1 - \frac{1}{p^2}\right)^{-1} \\ &= \log(c \log x) + o(1) - \log(\sum n^{-2}) + o(1) = \log(b \log x) + o(1). \end{aligned}$$

LEMMA 2. *Let $x_1 \geq x$. Then, as $x \rightarrow \infty$, we have, uniformly in x_1 ,*

$$\sum_{x < p \leq x_1} \log \left(1 - \frac{1}{p}\right)^{-1} = \log \left(\frac{\log x_1}{\log x}\right) + o(1).$$

This is an immediate corollary of Lemma 1.

3. LEMMA 3. *For*

$$\frac{1}{2} < \frac{1}{2} + \delta < \sigma \leq \frac{9}{8}, \quad e^{-\sqrt{t}} < y < 1,$$

* G. H. Hardy, 1.

we have, on the Riemann hypothesis,

$$(3.1) \quad -\frac{\hat{\xi}'(s)}{\hat{\xi}(s)} = \Sigma \Lambda(n) n^{-s} e^{-ny} + O\{y^{\sigma-\frac{1}{2}-\epsilon-\delta}(\log t)^{1+\epsilon}\}.$$

This is obtained by taking $\sigma_1 = \frac{1}{2} + \delta$ in the formula (2.53) of my paper 1, observing that [by (1.12) of the same paper] $\nu(\sigma_1) \leq 1^*$.

LEMMA 4. For

$$(3.2) \quad e^{-y^t} < y \leq (\log t)^{-2-\epsilon}$$

we have, on the Riemann hypothesis,

$$\log \hat{\xi}(1+ti) = \Sigma \log(1-p^{-1-ti}e^{-ny})^{-1} + o(1).$$

We take in Lemma 3,

$$\sigma = \epsilon < \frac{1}{24}, \quad e^{-y^t} < y \leq (\log t)^{-2-12\epsilon}, \quad 1 \leq \sigma \leq \frac{9}{8}.$$

Then the O -term in (3.1) is $o(1)$, and

$$-\frac{\hat{\xi}'(s)}{\hat{\xi}(s)} = \Sigma \Lambda(n) n^{-s} e^{-ny} + o(1) \quad (1 \leq \sigma \leq \frac{9}{8}).$$

Integrating this with respect to σ from 1 to $\frac{9}{8}$ we obtain

$$\log \hat{\xi}(1+ti) = \Sigma \Lambda_1(n) n^{-1-ti} e^{-ny} + \log \hat{\xi}(\frac{9}{8}+ti) - \Sigma \Lambda_1(n) n^{-\frac{9}{8}-ti} e^{-ny} + o(1).$$

The second and third terms on the right-hand side contribute

$$\Sigma \Lambda_1(n) n^{-\frac{9}{8}-ti} (1-e^{-ny}) = O\{\Sigma n^{-\frac{9}{8}}(1-e^{-ny})\} = o(1).$$

Hence

$$\begin{aligned} \log \hat{\xi}(1+ti) &= \Sigma_{p,m} \frac{1}{m} p^{-m(1+ti)} e^{-p^m y} + o(1) \\ &= \Sigma_{p,m} \frac{1}{m} p^{-m(1+ti)} e^{-m p y} + \Sigma_{p,m>1} \frac{1}{m} p^{-m(1+ti)} (e^{-p^m y} - e^{-m p y}) + o(1)^\dagger. \end{aligned}$$

Now the second double sum has a modulus not exceeding

$$\Sigma_{p,m>1} p^{-m} (e^{-m p y} - e^{-p^m y}).$$

* Theorem 3 (of paper 1) in whose context (2.53) occurs, is incorrectly (too weakly) stated. This and some other slips are corrected in *Journal London Math. Soc.*, 2 (1927), 45. It happens that (2.53) itself is correct.

† $e^{-p^m y} = e^{-m p y}$ when $m = 1$.

This last (double) series is evidently uniformly convergent for $y \geq 0$, the summand being less than p^{-n} . Since each term tends to zero with y the sum is $o(1)$. Hence

$$\log \zeta(1+ti) = \sum_{p, m} \frac{1}{m} p^{-m(1+ti)} e^{-mpy} + o(1) = \sum_p \log(1 - p^{-1-ti} e^{-py})^{-1} + o(1),$$

and this is the desired result with 12ϵ in place of ϵ .

4. LEMMA 5. *For $x > x_0(\epsilon)$ there exists a $t = t(x, \epsilon)$ satisfying*

$$t < \exp(x^{2+2\epsilon}),$$

for which $\cos(t \log p_n) < -1 + x^{-\epsilon}$ ($n \leq x$).

This is the result of pushing the argument of Bohr and Landau as far as it will go: only minor alterations are required. The argument on page 75 of their paper **2** proceeds unaltered (except that they denote our x , which may clearly be supposed integral, by N) down to line 14 of page 76. We observe now that

$$\left| \sum_{\nu=1}^n (l_\nu - l'_\nu) \log p_\nu \right| = \left| \log \frac{m}{n} \right|,$$

where $m = \prod p_\nu^{l_\nu}$, $n = \prod p_\nu^{l'_\nu}$ and $m \neq n$. Since $\sum l_\nu = \sum l'_\nu = k$, m and n do not exceed p_x^k ; also we have, supposing, say, $m > n$,

$$\left| \log \frac{m}{n} \right| = \log \left(1 - \frac{m-n}{n} \right)^{-1} > \frac{m-n}{m} \geq \frac{1}{m} \geq \frac{1}{p_x^k};$$

and we obtain in place of Bohr and Landau's (11)

$$(4.1) \quad \frac{1}{\Omega} \int_0^\Omega |F(t)|^{2k} dt > (x+1)^{2k} k^{-4x} \left(1 - \frac{2p_x^k k^{4x}}{\Omega} \right).$$

We choose now $k = x^{2+\frac{3}{2}\epsilon}$, $\Omega = \exp(x^{2+2\epsilon})$,

in which case the fraction with denominator Ω on the right-hand side of (4.1) is less than $\frac{1}{2}$ (for large x). Then

$$\begin{aligned} \left(\frac{1}{\Omega} \int_0^\Omega |F(t)|^{2k} dt \right)^{1/2k} &> (x+1) \exp \left(- \frac{4x \log k + \log 2}{2k} \right) \\ &> (x+1) \left\{ 1 - \frac{1}{2}(x+1)^{-1-\epsilon} \right\} \quad [x > x_0(\epsilon)] \\ &> x+1 - \frac{1}{2}x^{-\epsilon}. \end{aligned}$$

The argument now concludes, subject to the obvious modifications, as in the original.

LEMMA 6. Any number t with the properties of Lemma 5 satisfies $t \geq \log^3 x$ when x is large.

The proof follows the lines of the one on page 188 of my paper 1. If $t \leq \log^3 x$ we should have, for $u = 2/\log(x-1)$,

$$\begin{aligned} -\Re \zeta(1+u+ti) &= \left(\sum_{n \leq x} + \sum_{n > x} \right) n^{-1-u} \cos(-t \log n) \\ &\geq \sum_{n \leq x} n^{-1-u} (1-x^{-\epsilon}) - \sum_{n > x} n^{-1-u} \\ &> (1-x^{-\epsilon}) \sum_{n=1}^{\infty} n^{-1-u} - 2 \sum_{n > x} n^{-1-u} > \frac{1}{2} \zeta(1+u) - 2 \int_{x-1}^{\infty} v^{-1-u} dv \\ &> \frac{1}{2} \frac{1}{u} - A - \frac{2}{u} (x-1)^{-u} = \left(\frac{1}{2} - 2e^{-2} \right) \frac{1}{u} - A > A \log x > At^{\frac{1}{3}}, \end{aligned}$$

which contradicts $\zeta(\sigma+ti) = O(\log t)$ ($\sigma > 1$).

5. THEOREM 1. We have, on the Riemann hypothesis,

$$\left| \frac{1}{\zeta(1+ti)} \right| < \{1+o(1)\} 2b \log \log t.$$

We take $y = (\log t)^{-2-\epsilon}$ in Lemma 4. Then

$$\begin{aligned} -\log |\zeta(1+ti)| &= \sum_p \Re \log(1-p^{-1-ti} e^{-py}) + o(1) \\ &\leq \sum_p \log(1+p^{-1} e^{-py}) + o(1) \\ &< \sum_{p \leq \varpi} \log(1+p^{-1} e^{-py}) + \sum_{\varpi+1}^{\infty} \log(1+n^{-1} e^{-ny}) + o(1), \end{aligned}$$

where $\varpi = [y^{-1-\epsilon}]$. The second term is

$$O\left(\sum_{\varpi}^{\infty} e^{-ny}\right) = o(1),$$

and the first does not exceed

$$\sum_{p \leq \varpi} \log(1+p^{-1}) = \log(b \log \varpi) + o(1),$$

* We shall see in § 6 that much more than this is true.

by Lemma 1. Thus

$$-\log |\zeta(1+ti)| < \log(2b \log \log t) + \log \{(1+\frac{1}{2}\epsilon)(1+\epsilon)\} + o(1),$$

and the term involving ϵ may be omitted, since ϵ is arbitrary. The result of the theorem follows by taking exponentials.

6. Let $\mathfrak{S}$ be the lower bound of the numbers h with the property: Given $\epsilon > 0$, then, for x greater than some $x_0(\epsilon, h)$, there exists a t satisfying

$$t < \exp(x^{h+2\epsilon})$$

for which

$$\cos(t \log p_n) < -1 + x^{-\epsilon} \quad (n \leq x).$$

$\mathfrak{S}$ is a non-negative absolute constant, and Lemma 5 shows that

$$(6.1) \quad \mathfrak{S} \leq 2.$$

Let

$$\mathfrak{S}_1 = \text{Max}(\mathfrak{S}, \frac{1}{2}).$$

THEOREM 2. We have, on the Riemann hypothesis,

$$\overline{\lim}_{t \rightarrow \infty} \frac{1/|\zeta(1+ti)|}{\log \log t} \geq \frac{b}{2\mathfrak{S}_1} \geq \frac{1}{8}b.$$

Let $h > \mathfrak{S}$, $\epsilon < 1$, and let x be a large integer. There exists a t satisfying

$$(6.2) \quad t < \exp(x^{h+2\epsilon})$$

for which

$$(6.3) \quad \cos(t \log p) < -1 + x^{-\epsilon} \quad (p \leq x).$$

Let y be defined by

$$y^{-1} = \text{Max} \{x^{1+\epsilon}, (\log t)^{2+\epsilon}\},$$

and let $\varpi = [y^{-1-\epsilon}]$. It is easily verified from Lemma 6, (6.2), and (6.1), that y satisfies (3.2). Hence, by Lemma 4,

$$\begin{aligned} \log |\zeta(1+ti)| &< \left(\sum_{p \leq x} + \sum_{p > x} \right) \log |1 - p^{-1-ti} e^{-py}|^{-1} + o(1) \\ (6.4) \quad &< \sum_{p \leq x} \log(1+p^{-1})^{-1} + \sum_{p \leq x} \log(1+\eta_p) + \sum_{x+1}^{\varpi} \log(1-p^{-1}e^{-py})^{-1} \\ &\quad + \sum_{\varpi+1}^{\infty} \log(1-p^{-1}e^{-py})^{-1} + o(1), \end{aligned}$$

$$\text{where } 1+\eta_p = \left| \frac{1-p^{-1-ti}e^{-py}}{1+p^{-1}} \right|^{-1} = \left| 1 - \frac{p^{-1}(1-e^{-py}e^{-(t \log p - \pi)i})}{1+p^{-1}} \right|^{-1}.$$

$$\text{Now } \eta_p < Ap^{-1}(|1-e^{-py}| + |e^{-(t \log p - \pi)i} - 1|) < Ap^{-1}(py + Ax^{-\epsilon}).$$

The second term in (6.4) is therefore less than

$$A \sum_{p \leq x} (y+x^{-\epsilon}p^{-1}) < Axy + o(1) = o(1).$$

The third term is less than

$$\sum_{x < p < \varpi} \log(1-p^{-1})^{-1} < \log \left(\frac{\log \varpi}{\log x} \right) + o(1)$$

by Lemma 2,

$$\begin{aligned} &< \log \left(\frac{\text{Max} \{ (1+\epsilon)^2 \log x, (1+\epsilon)(2+\epsilon) \log \log t \}}{\log x} \right) + o(1) \\ &< \text{Max} [2 \log(1+\epsilon), \log(h+2\epsilon) + \log \{ (1+\epsilon)(2+\epsilon) \}] + o(1) \\ &< \text{Max} [0, \log(2h)] + A\epsilon = \log(2h_1) + A\epsilon. \end{aligned}$$

The fourth term is

$$O \left(\sum_{n > \varpi} e^{-ny} \right) = o(1).$$

Collecting these results, and applying Lemma 1 to the first term of (6.4), we obtain finally

$$\begin{aligned} \log |\zeta(1+ti)| &< -\log(b \log x) + \log(2h_1) + A\epsilon + o(1) \\ &< -\log \left(\frac{b}{2h_1(h+2\epsilon)} \log \log t \right) + A\epsilon + o(1). \end{aligned}$$

Since ϵ and $h-\mathfrak{S}$ are arbitrarily small we may replace ϵ by 0 and h by $\mathfrak{S}$ in this, when we obtain the result of the theorem.

THEOREM 3. *On the Riemann hypothesis we have*

$$\mathfrak{S}_1 = \mathfrak{S} \geq \frac{1}{2}$$

(and we may write $\mathfrak{S}^2$ for $\mathfrak{S}\mathfrak{S}_1$ in Theorem 2).

For if $\mathfrak{S} < \frac{1}{2}$, Theorems 1 and 2 are incompatible.

7. It is interesting to compare the results (1.1) and Theorem 1, on the one hand, and (1.2) and Theorem 2 on the other. (1.1) and Theorem 1 correspond exactly, but there are several important differences between (1.2) and Theorem 2. Let us call (1.2) Problem A and Theorem 2 Problem B. In the first place B, unlike A, depends on the Riemann

hypothesis. What we can prove without the hypothesis adds nothing of real interest to what is known already.

Secondly, Theorem 2 and (1.2) differ in a numerical factor 8^* . Of this a factor 2 is accounted for by the fact that $\mathfrak{S}$ may be as great as 2, while the corresponding number in Problem A is known to be at most 1 (Dirichlet's theorem on Diophantine approximation). The remaining factor 4 arises from the sum

$$(7.1) \quad \sum_x^{\varpi} \log |1 - p^{-1-ti} e^{-xy}|^{-1}.$$

In Problem A it is possible to "average" over $t, 2t, \dots, Nt$ (where N is large but fixed). The average of the expression corresponding in A to (7.1) in B is, on the face of it, capable of being appreciably negative, but cannot, as it happens, be appreciably *positive*, and the whole sum can be ignored. But the signs are reversed in Problem B, and the averaging argument fails.

There is another point connected with the averaging argument. It works in Problem A even when ϖ is replaced by ∞ . But, this being so, it is no longer necessary in A to use the machinery of Lemmas 3 and 4 to *terminate* our series at $p = \varpi$, and when we modify the detailed argument by replacing e^{-ny} by n^{-y} (and working with the infinite series) we are merely using the ordinary Dirichlet's series for $\log \zeta$ with $\sigma = 1+y > 1$. This, in general terms, is why the Riemann hypothesis is not required in Problem A.

In my paper 1 the inequality (1.1) is extended by replacing the number $2c$ by $2\beta(1)c$, where $\beta(1)$ lies between $\frac{1}{2}$ and 1 inclusive. The arguments of the present note can be extended in a corresponding way. The final results are that, on the Riemann hypothesis,

$$(7.2) \quad 2\beta(1)\mathfrak{S} \geq 1,$$

$$(7.3) \quad \left| \frac{1}{\zeta(1+ti)} \right| \leq \{1+o(1)\} 2\beta(1)b \log \log t$$

for all large t , and

$$(7.4) \quad \left| \frac{1}{\zeta(1+ti)} \right| \geq \{1+o(1)\} \frac{b}{2\beta(1)\mathfrak{S}^2} \log \log t$$

for some arbitrarily large values of t .

* This 8 has no connexion with the factor 2 by which the right-hand sides of (1.1) and (1.2) differ: *this* 2 is untouched by anything said in the text. To reduce it we must prove something new either about $\mathfrak{S}$ or about the number $\beta(1)$ mentioned below. If, however, it were known that $\mathfrak{S} = 1$ and $\beta(1) = \frac{1}{2}$, then not only would the 2 be disposed of, but all our results would be best possible.

Landau* has made the important observation that some of the arguments used to discuss the behaviour of $\zeta(s)$ for fixed σ and large t apply *mutatis mutandis* to the function $L(s)$ for fixed real s and large k , where

$$L(s) = \sum \frac{\chi(n)}{n^s} \quad (\sigma > 0),$$

and χ is a non-principal Dirichlet's character to modulus k . I have recently discovered that analogues of this kind exist in several new directions; in fact, all the problems which we have been discussing here possess them†. The point of interest at the moment is that, whereas the t -problems are generally the easier, the problems here in question are (except, perhaps, in the matter of the unproved hypotheses) easier in k -form. The factor 8 by which the B result falls short of the A result can be completely disposed of, analogues being available both for a *two-sided* averaging argument and for the result " $\mathfrak{N} = 1$ ". [Thus the results for $1/L(1)$ stand on precisely the same footing as those for $L(1)$.] We are generally in the position of being encouraged by a t -theorem to attack the corresponding k -problem; here the suggestion is in the reverse direction; we are encouraged to attack both the problem of "averaging" for $1/\zeta(1+ti)$ and that of reducing from 2 to 1 the upper bound of $\mathfrak{N}$.

List of papers referred to.

- H. Bohr and E. Landau, **1**. "Über das Verhalten von $\zeta(s)$ und $\zeta_c(s)$ in der Nähe der Geraden $\sigma = 1$ ", *Göttinger Nachrichten*, 1910, 303-330.
 — **2**. "Über das Verhalten von $1/\zeta(s)$ auf der Geraden $\sigma = 1$ ", *Göttinger Nachrichten*, 1923, 71-80.
 — **3**. "Nachtrag zu unseren Abhandlungen aus den Jahrgängen 1910 und 1923", *Göttinger Nachrichten*, 1924, 168-172.
 G. H. Hardy, **1**. "Note on a theorem of Mertens", *Journal London Math. Soc.*, 2 (1927), 70-71.
 E. Landau, **1**. "Über das Nichtverschwinden der Dirichletschen Reihen, welche komplexen charakteren entsprechen", *Math. Annalen*, 70 (1911), 69-78.
 J. E. Littlewood, **1**. "On the Riemann zeta-function", *Proc. London Math. Soc.* (2), 24 (1924), 175-201.

* E. Landau, **1**.

† See the paper immediately following the present one.

ON THE CLASS-NUMBER OF THE CORPUS $P(\sqrt{-k})$

By J. E. LITTLEWOOD.

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1. We owe to Landau† the important observation that some of the arguments used to discuss the behaviour of the Riemann Zeta-function $\zeta(s) = \zeta(\sigma + ti)$ for fixed σ and large t apply *mutatis mutandis* to the function $L(s)$ for fixed real s and large k , where

$$L(s) = \sum \frac{\chi(n)}{n^s} \quad (\sigma > 0),$$

and χ is a non-principal Dirichlet's character to modulus k . In two recent papers‡ in the *Proceedings* of this Society I proved the following inequalities for the function $\zeta(s)$. Let C be Euler's constant, and let

$$c = e^C, \quad b = \frac{6}{\pi^2} c = \frac{6}{\pi^2} e^C.$$

Then

$$(1.1) \quad \frac{1+o(1)}{2b \log \log t} < |\zeta(1+ti)| < \{1+o(1)\} 2c \log \log t,$$

$$(1.2) \quad |\zeta(1+ti)| > \{1+o(1)\} c \log \log t,$$

$$(1.3) \quad |\zeta(1+ti)| < \frac{1+o(1)}{\frac{1}{4}b \log \log t}.$$

Of these (1.1) holds for all large values of t , and (1.3) holds for some

† E. Landau, 1 (391). [See the list of papers at the end.]

‡ J. E. Littlewood, 1 (175–194), 2. Some relevant corrections to Paper 1 are made in *Journal London Math. Soc.*, 2 (1927), 45.

arbitrarily large values of t , in each case on the assumption that the Riemann hypothesis is true; while (1.2) holds for some arbitrarily large values of t independently of any hypothesis. The results involving c are evidently final except for a certain factor 2. I showed also that on a certain further hypothesis (which there is, perhaps, no good reason for believing) this factor 2 disappears.

The proofs involve, on the one hand, arguments to which, after the work of Landau, there is a straightforward k -analogue, and, on the other, arguments analogues of which are not foregone conclusions. Actually (1.1) has a straightforward analogue, but (1.2) and (1.3) involve arguments of the latter class. Of these there are four of special importance.

(a) In the first place there is that special case of Dirichlet's theorem which asserts the existence of a t between τ and τq^N such that for appropriate integers h

$$|t \log n - 2\pi h_n| < \frac{2\pi}{q} \quad (0 < n \leq N).$$

Some theorem of this kind is essential in proving any inequality of type " $|\zeta(1+ti)| > \phi(t)$ for some large t "†.

(b) Kronecker's theorem asserts that if $x_1, x_2, \dots, x_N$ are linearly independent and $\alpha_1, \alpha_2, \dots, \alpha_N$ are any real numbers, then there exist arbitrarily large t such that, for suitable integers h ,

$$|tx_n - \alpha_n - 2\pi h_n| < \frac{2\pi}{q} \quad (0 \leq n \leq N).$$

When all the α_n are zero the theorem is included in Dirichlet's theorem. The general theorem can assign no upper bound to t such as q^N , but Bohr and Landau‡ have proved the remarkable result that when $x_n = \log p_n$, where p_n is the n -th prime, a t can be found, to put the matter roughly, less than an upper bound not differing very drastically from q^N . Some result of this kind is essential to the proof of any inequality of type " $|1/\zeta(1+ti)| > \psi(t)$ for some large t ".

† It was used first by Bohr in 1910 to prove that $|\zeta(\sigma+ti)|$ is unbounded in $\sigma > 1$. His proof is given in §1 of H. Bohr and E. Landau, 1. The joint authors then prove further (among other things) that $|\zeta(1+ti)| > A \log \log t$ for some A and some arbitrarily large values of t .

‡ H. Bohr and E. Landau, 2. See also 3.

(c) To obtain the precise constant c of (1.2) we require to use a certain "averaging" argument. The machinery for this is set out in Lemma 4 of my Paper 1.

(d) The result (1.2) is proved, in the first instance, with $\zeta(\sigma+ti)$ for $\zeta(1+ti)$ and $\sigma > 1$ (and, of course, for *some* values of t and corresponding values of σ). A theorem of a classical type enables us to make the inference to $\zeta(1+ti)$.

It will appear that k -analogues can be found for (a), (b), and (c); an analogue of the result just mentioned concerning $\zeta(\sigma+ti)$ can accordingly be established. (This is not in itself of great interest, and it has not been included in the paper.) But I have not succeeded with (d) (which, indeed, has every appearance of great difficulty), and proof of an *exact* k -analogue of (1.2) is wanting. Fortunately there is an alternative proof of (1.2): this makes no use of (d), but [like (1.1) and (1.3)] assumes the Riemann hypothesis. The analogue can accordingly be proved on condition of assuming the analogous hypothesis (which consequently is assumed *everywhere*).

k -theorems are generally more difficult than t -theorems when more than the simplest kind of analogue is involved. It is therefore of interest to observe that in two respects the k -theorems of the present paper are more successful than the corresponding t -theorems. In the first place the k -analogue of (b) is twice as strong as the best version of the original; in the second (c) works only one-sidedly in t , and is not available at all for (1.3), but works two-sidedly in the k -analogue. Each improvement involves a factor 2, and we can get rid of the factor $\frac{1}{2}$ multiplying b in (1.3) (leaving finally only a factor 2 to be accounted for). As a result of all this the inequalities proved for $L(1)$ and $1/L(1)$ stand on precisely the same footing.

The results of the paper encourage the belief that the field of promising k -analogues is not yet exhausted.

2. Let k be a positive integer, $\chi(n)$ a non-principal Dirichlet's character to modulus k , and let

$$L(s) = L(s, \chi)$$

denote the function defined for $\sigma > 1$ by

$$\sum \frac{\chi(n)}{n^s}.$$

We employ the Kronecker symbol $(D|n)$ of quadratic residuacity. In this D is a "discriminant", *i.e.* $D \equiv 0$ or $1 \pmod{4}$ and D is not a perfect square; and the symbol obeys, for positive n , the following propositions:

$$(\alpha) \quad (D|1) = 1.$$

(β) If p is a (positive) prime $(D|p)$ is 0 if p divides D , $+1$ if $(p, D) = 1$ and D is a quadratic residue of $4p$, -1 if $(p, D) = 1$ and D is a non-residue of $4p$.

(γ) For all positive integral m, n ,

$$(D|mn) = (D|m)(D|n).$$

These propositions incidentally determine the values of $(D|n)$ for all positive n .

$(D|n)$ is (obviously) a real character to modulus $|D|$. It is a primitive character if (and only if) D is a "fundamental" discriminant, *i.e.* if there is no dissection $D = Q^2 D'$, where $Q > 1$ and D' is a discriminant. In the general case there is a unique dissection $D = P^2 \Delta$, where Δ is a fundamental discriminant, and then the primitive character "associated" with $(D|n)$ is $(\Delta|n)$.

We write k^* for any k such that $-k$ is a fundamental discriminant, and $\chi^*(n)$ for $(-k^*|n)$. Thus χ^* is primitive. We write also $L^*(s)$ for $L(s, \chi^*)$. Considerable interest attaches to the special characters χ^* , since the class-number of the corpus $P(\sqrt{-k^*})$ is expressible in terms of $L^*(1)$. Some of our results about general χ are valid in the special case $\chi = \chi^*$, and these give direct applications to the class-number problem. But the reader must bear in mind that the direct analogues of (1.2) and (1.3) assert the existence only of *some* large k , and corresponding χ , with certain properties; for class-number applications we require the existence of χ^* with the same properties, and hence have to prove more.

We shall give short names to certain hypotheses. The hypothesis that, for a particular k and χ , $L(s, \chi)$ has no zeros in $\sigma > \frac{1}{2}$ we call $X(k, \chi)$, or simply X . The hypothesis that X is true for all (k and) χ we call R (but we never use this hypothesis). Hypothesis R is true if X is true for (all k and) all primitive χ . The hypothesis that X is true for all χ^* we call R^* .

In what follows we are concerned with *large* k , and shall suppose always that $k > 4$. We denote by $\Lambda(n)$ (as usual) the function that is 0

unless n is a prime power p^m and is then $\log p$. We denote by $\Lambda_1(n)$ the function $\Lambda(n)/\log n$. p always denotes a typical prime. $A(x, y, \dots)$ denotes a positive constant, not always the same from one occurrence to another, that depends only on the parameters shown explicitly. Thus A denotes a positive absolute constant. O 's are absolute, *i.e.* the constants implied in them are of type A .

3. LEMMA 1. *We have, for large x ,*

$$(3.1) \quad \log \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} = \log(e^C \log x) + o(1).$$

This is proved in Landau's *Handbuch*, page 140†.

LEMMA 2. *We have, for large x ,*

$$(3.2) \quad \sum_{n \leq x} \frac{\Lambda_1(n)}{n} = \sum_{p^m \leq x} \frac{\Lambda_1(p^m)}{p^m} = \sum_{p^m \leq x} \frac{1}{mp^m} = \log(e^C \log x) + o(1),$$

$$(3.3) \quad \sum_{p^m \leq x} (-1)^{m-1} \frac{\Lambda_1(p^m)}{p^m} = \log(b \log x) + o(1).$$

Since $\Lambda_1(p^m) = m^{-1}$ the left-hand sides of (3.1) and (3.2) differ at most by

$$\sum_{\substack{p \leq x \\ p^m > x}} \frac{1}{mp^m} \leq \sum_{p \leq x} \frac{1}{p} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots\right) \leq \frac{2\pi(x)}{x} = o(1)^\dagger,$$

and (3.2) follows from (3.1). Similarly the left side of (3.3) differs by $o(1)$ from

$$\log \prod_{p \leq x} \left(1 + \frac{1}{p}\right) = \log \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} + \log \prod_{p \leq x} \left(1 - \frac{1}{p^2}\right),$$

and the second term on the right differs by $o(1)$ from

$$-\log \prod_p \left(1 - \frac{1}{p^2}\right)^{-1} = -\log \left(\sum_1^\infty n^{-2}\right) = \log \frac{6}{\pi^2}.$$

† The result is "elementary", but the proof is curiously elaborate. A proof on different lines in some respects simpler, has recently been published by G. H. Hardy, 1.

‡ The first inequality in line 4, p. 190, of my paper 1 is incorrect, and the argument should be replaced by the present one.

4. LEMMA 3. Let x be a large positive number, $p_1, p_2, \dots, p_v$, the primes not greater than x , $T = 4 \prod_{r=1}^v p_r$. Let $a_1, a_2, \dots, a_v$ be v numbers, each of which is either $+1$ or -1 . Then there exists a k_0 satisfying $x < k_0 \leq T+1$, and such that $-k_0$ is a discriminant and

$$(-k_0 | p_r) = a_r \quad (0 < r \leq v).$$

When the a_r are all $+1$ this is a k -analogue of the theorem of (a), § 1; the general case is an analogue of the theorem in (b). The proof of Lemma 3 is much easier than that of the latter theorem. The result also is stronger; the inequality corresponding strictly to (the best form of) (b) would be no better than $k_0 < T^{2+\epsilon}$.

We define a number β_r for $0 < r \leq v$ as follows. If $a_r = +1$ we take for β_r (any) one of the quadratic residues of $4p_r$; if $a_r = -1$ we take one of the non-residues. The congruences

$$y \equiv 1 \pmod{4}, \quad y \equiv \beta_1 \pmod{8}^\dagger,$$

$$y \equiv \beta_r \pmod{4p_r} \quad (1 < r \leq v),$$

have a solution $y = y_0$, which is necessarily prime to T ; moreover, since $T \equiv 0 \pmod{4p_r}$, $y_0 + mT$ is also a solution. Let $-k_0$ be the number congruent to $y_0 \pmod{T}$ and satisfying

$$(4.3) \quad 1 < k_0 \leq T+1.$$

Evidently $-k_0$ is a discriminant and $(-k_0 | p_r) = a_r$ ($0 < r \leq v$). Since k_0 is prime to T it is not divisible by any prime less than or equal to x ; hence, by (4.3), it satisfies $x < k_0 \leq T+1$. This completes the proof[†].

5. LEMMA 4. Suppose that, in $|s - s_0| < r$, $f(s)$ is regular and satisfies

$$\Re \{f(s) - f(s_0)\} \leq U.$$

Then for $|s - s_0| = \rho < r$,

$$|f'(s)| < \frac{AUr}{(r-\rho)^2} \S.$$

[†] β_1 is clearly 1 or 5 according as a_1 is $+1$ or -1 .

[‡] In the case $a_r = 1$ ($0 < r \leq v$) we have only to take $k_0 = T-1$. Cp. H. T. J. Norton, 1 (57).

[§] In point of fact we may take $A = 2$.

This is almost a classical theorem. Let

$$f(s) - f(s_0) = \sum_1^{\infty} a_n (s - s_0)^n \quad (|s - s_0| < r).$$

By a theorem of Borel†,

$$|a_n| r^n \leq AU.$$

For $|s - s_0| = \rho < r$, therefore,

$$|f'(s)| \leq \sum_1^{\infty} n |a_n| \rho^{n-1} \leq \frac{AU}{r} \sum_1^{\infty} n \left(\frac{\rho}{r}\right)^{n-1} = \frac{AUr}{(r-\rho)^2}.$$

6. LEMMA 5. Let $\epsilon > 0$ and let χ be a non-principal character to modulus k . Then on hypothesis $X(k, \chi)$

$$\left| \frac{L'(s)}{L(s)} \right| < A(\epsilon) \{ \log(|t| + 2) + \log k \} \quad \left(\frac{1}{2} + \epsilon < \sigma \leq 2 \right).$$

Let $S(u) = \sum_{1 \leq n \leq u} \chi(n)$. Then $S(k) = 0$ and $|S(u)| \leq k$. Hence for $\sigma > 0$

$$\begin{aligned} L(s) &= \sum_{n=1}^{\infty} \frac{S(n) - S(n-1)}{n^s} = \sum_1^{\infty} S(n) \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) \\ &= \sum_1^{\infty} S(n) s \int_n^{n+1} \frac{du}{u^{s+1}} = s \int_1^{\infty} \frac{S(u) du}{u^{s+1}}. \end{aligned}$$

Hence

$$|L(s)| < A(|t| + 2) \int_1^{\infty} \frac{k du}{u^{\frac{3}{2}}} < \lambda = Ak(|t| + 2) \quad \left(\frac{1}{2} \leq \sigma \leq 2 \right),$$

and so

$$\Re \log L(s) < \log \lambda \quad \left(\frac{1}{2} \leq \sigma \leq 2 \right),$$

whence

$$(6.1) \quad \Re \log L(z) < A \log \lambda \quad \{ |z - (2 + ti)| \leq \tfrac{3}{2} \}.$$

We may suppose $\epsilon < 1$. Since $f(s) = \log L(s)$ is regular in $\sigma > \frac{1}{2}$ (on hypothesis X) we have, in virtue of (6.1) and Lemma 4 with

$$s_0 = 2 + ti, \quad r = \tfrac{3}{2}, \quad \rho = \tfrac{3}{2} - \epsilon,$$

$$\left| \frac{L'(z)}{L(z)} \right| < \frac{A \log \lambda}{\epsilon^2} \quad (|z - s_0| \leq \tfrac{3}{2} - \epsilon).$$

† E. Borel, **1** (365).

‡ See E. Landau, **2** (391).

In this we take $z = \sigma + ti$, $\frac{1}{2} + \epsilon \leq \sigma \leq \frac{7}{2} - \epsilon$, obtaining the result of the lemma.

7. LEMMA 6. Let χ be a non-principal character to modulus k , $\epsilon > 0$, $1 \leq \sigma \leq \frac{9}{8}$, $0 < y \leq 1$. Then on hypothesis $X(k, \chi)$,

$$(7.1) \quad \left| -\frac{L'(\sigma)}{L(\sigma)} - \sum_1^\infty \frac{\Lambda(n)\chi(n)}{n^\sigma} e^{-ny} \right| < A(\epsilon) y^{\sigma-\frac{1}{2}-\epsilon} \log k.$$

The proof follows straightforwardly the model of Theorem 3 of my paper 1. We may suppose $\epsilon < \frac{1}{2}$. Mellin's well known formula

$$e^{-y} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} y^{-z} \Gamma(z) dz \quad (\Re y > 0)$$

$$\text{gives} \quad \sum \frac{\Lambda(n)\chi(n)}{n^\sigma} e^{-ny} = \sum \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Lambda(n)\chi(n) n^{-\sigma} (ny)^{-z} \Gamma(z) dz$$

$$(7.2) \quad = -\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{L'(\sigma+z)}{L(\sigma+z)} y^{-z} \Gamma(z) dz.$$

In virtue of Lemma 5 and the asymptotic behaviour of $\Gamma(z)$ the integrand tends exponentially to zero when $|\Im z| \rightarrow \infty$ and $|\Re z|$ is bounded, and we may replace the contour of integration by a line from $\frac{1}{2} + \epsilon - \sigma - i\infty$ to $\frac{1}{2} + \epsilon - \sigma + i\infty$, taking account of the residue at $z = 0$, the only relevant singularity. This gives

$$-\frac{L'(\sigma)}{L(\sigma)} - \sum \frac{\Lambda(n)\chi(n)}{n^\sigma} e^{-ny} = \frac{1}{2\pi i} \int_{\frac{1}{2}+\epsilon-\sigma-i\infty}^{\frac{1}{2}+\epsilon-\sigma+i\infty}$$

$$\left| -\frac{L'(\sigma)}{L(\sigma)} - \sum \frac{\Lambda(n)\chi(n)}{n^\sigma} e^{-ny} \right|$$

$$< A \int_{-\infty}^{\infty} \left| \frac{L'(\frac{1}{2}+\epsilon+i\eta)}{L(\frac{1}{2}+\epsilon+i\eta)} \right| y^{\sigma-\frac{1}{2}-\epsilon} |\Gamma(\frac{1}{2}+\epsilon+i\eta-\sigma)| d\eta$$

$$< A \cdot y^{\sigma-\frac{1}{2}-\epsilon} \int_0^\infty A(\epsilon) \{ \log k + \log(\eta+2) \} A(\epsilon) e^{-A\eta} d\eta$$

by Lemma 5,

$$< A(\epsilon) y^{\sigma-\frac{1}{2}-\epsilon} \log k.$$

LEMMA 7. With the hypotheses of Lemma 6 we have as $y \rightarrow 0$,

$$(7.8) \quad \log L(1) = \sum \frac{\Lambda_1(n) \chi(n)}{n} e^{-ny} + O\{A(\epsilon) y^{\frac{1}{2}-\epsilon} \log k\} + o(1),$$

uniformly in k and χ .

Integration with respect to σ from 1 to $\frac{9}{8}$ in Lemma 6 gives

$$\begin{aligned} \log L(1) = \sum \frac{\Lambda_1(n) \chi(n)}{n} e^{-ny} + \log L\left(\frac{9}{8}\right) - \sum \frac{\Lambda_1(n) \chi(n)}{n^{\frac{8}{9}}} e^{-ny} \\ + O\{A(\epsilon) y^{1-\frac{1}{2}-\epsilon} \log k\}. \end{aligned}$$

The desired result follows since

$$\begin{aligned} \left| \log L\left(\frac{9}{8}\right) - \sum \frac{\Lambda_1(n) \chi(n)}{n^{\frac{8}{9}}} e^{-ny} \right| &= \left| \sum \frac{\Lambda_1(n) \chi(n)}{n^{\frac{8}{9}}} (1 - e^{-ny}) \right| \\ &\leq \sum n^{-\frac{8}{9}} (1 - e^{-ny}) = o(1). \end{aligned}$$

8. LEMMA 8. Suppose that χ is a non-principal character to modulus k , and that

$$(8.1) \quad y = x^{-3}, \quad A \log k < x < A \log k.$$

Then as $x \rightarrow \infty$ we have, on hypothesis X,

$$(8.2) \quad \log L(1) = \sum_{p^m \leq x} \frac{1}{m} \left(\frac{\chi(p)}{p} \right)^m + \sum_{x < p < x^*} \frac{\chi(p)}{p} e^{-py} + o(1),$$

uniformly in k and χ .

Taking $\epsilon = \frac{1}{7}$ in Lemma 7 we have [subject to (8.1)]

$$\log L(1) = \sum \frac{\Lambda_1(n) \chi(n)}{n} e^{-ny} + o(1).$$

Hence

$$\begin{aligned} \log L(1) - \sum_{n \leq x} \frac{\Lambda_1(n) \chi(n)}{n} \\ &= - \sum_{n \leq x} \frac{\Lambda_1(n) \chi(n)}{n} (1 - e^{-ny}) \\ &\quad + \sum_{x < n < x^*} \frac{\Lambda_1(n) \chi(n)}{n} e^{-ny} + \sum_{n \geq x^*} \frac{\Lambda_1(n) \chi(n)}{n} e^{-ny} + o(1) \\ (8.3) \quad &= R_1 + R_2 + R_3 + o(1). \end{aligned}$$

Now

$$(8.4) \quad R_1 = O\left(\sum_{n \leq x} \frac{1}{n} ny\right) = O(xy) = o(1);$$

$$R_2 - \sum_{x < p < x^4} \frac{1}{p} e^{-py} = \sum_{m=2}^{\infty} \sum_{x < p^m < x^4} \frac{1}{mp^m} e^{-p^m y} = \sum_{m=2}^4 + \sum_{m=5}^{\infty}$$

$$(8.5) \quad \leq \sum_{p > x^{\frac{1}{2}}} \frac{1}{p^2} + \sum_{p > x^{\frac{1}{3}}} \frac{1}{p^3} + \sum_{p > x^{\frac{1}{4}}} \frac{1}{p^4} + \sum_{p \leq x^{\frac{1}{4}}} \frac{1}{p} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots\right) = o(1);$$

$$(8.6) \quad R_3 = O\left(\sum_{n \geq x^4} e^{-ny}\right) = O(y^{-1} e^{-x^4 y}) = o(1).$$

The lemma follows from (8.3) to (8.6) and the equation

$$\sum_{n \leq x} \frac{\Lambda_1(n) \chi(n)}{n} = \sum_{p^m \leq x} \frac{\chi(p^m)}{mp^m} = \sum_{p^m \leq x} \frac{1}{m} \left(\frac{\chi(p)}{p}\right)^m.$$

9. THEOREM 1. If χ is a real non-principal character to modulus k and the hypothesis $X(k, \chi)$ is true, then as $k \rightarrow \infty$

$$(9.1) \quad \frac{1+o(1)}{2b \log \log k} < L(1) < \{1+o(1)\} 2c \log \log k,$$

where $c = e^C$, $b = \frac{6}{\pi^2} c = \frac{6}{\pi^2} e^C$,

and C is Euler's constant. The right-hand inequality in (9.1) is true also for a complex character if we replace $L(1)$ by $|L(1)|$.

We take $\epsilon < \frac{1}{4}$, $y = (\log k)^{-2+7\epsilon}$ in Lemma 7. Then in (7.3) the O -term becomes $o(1)$; further, writing $v = (\log k)^{2+8\epsilon}$,

$$\left| \sum_{n \geq v} \frac{\Lambda_1(n) \chi(n)}{n} e^{-ny} \right| < \sum_{n \geq v} e^{-ny} < A y^{-1} e^{-vy} \\ < A (\log k)^4 \exp \{- (\log k)^\epsilon\} = o(1).$$

Thus for any non-principal character

$$(9.2) \quad |\log L(1)| < o(1) + \sum_{n < v} \frac{\Lambda_1(n)}{n};$$

and, if χ is real (and non-principal),

$$\begin{aligned} -\log L(1) &< o(1) - \sum_{p^m < \nu} \frac{\Lambda_1(p^m) \chi(p^m)}{p^m} e^{-p^m y} = o(1) - \sum_{p^m < \nu} \frac{1}{m} \left(\frac{\chi(p)}{p} \right)^m e^{-p^m y} \\ &\leq o(1) - \sum_{p < \nu} \frac{\chi(p)}{p} e^{-py} - \sum_{\substack{p^m < \nu \\ m \geq 1}} \frac{1}{m} \left(\frac{\chi(p)}{p} \right)^m e^{-p^m y} \\ &< o(1) - \sum_{p < \nu} \frac{\chi(p)}{p} e^{-py} - \sum_{\substack{p^m < \nu \\ m \geq 1}} \frac{1}{m} \left(\frac{\chi(p)}{p} \right)^m + O \left(\sum_{\substack{p^m < \nu \\ m \geq 1}} \frac{1}{p^m} (1 - e^{-p^m y}) \right). \end{aligned}$$

The O -term is $O \left(\sum_{\substack{p^m < \nu \\ m \geq 1}} y \right) = O(y \nu^{\frac{1}{2}}) = o(1)$.

Hence

$$(9.3) \quad -\log L(1) < o(1) - \sum_{p < \nu} \left(\frac{\chi(p)}{p} e^{-py} + \sum_{m=2}^M \frac{\chi^m(p)}{m p^m} \right) = o(1) - \sum_{p < \nu} S_p,$$

say, where $M = M(p) = [\log \nu / \log p]$. Now if $\chi(p) = 0$ or 1 , $S_p \geq 0$. If $\chi(p) = -1$, then $\chi^m(p) = (-1)^m$ and

$$(9.4) \quad S_p \geq -\frac{1}{p} + \frac{1}{2p^2} - \frac{1}{3p^3} + \dots \pm \frac{1}{Mp^M}.$$

Since the right-hand side is negative (9.4) holds in any case, and (9.3) gives

$$\begin{aligned} (9.5) \quad -\log L(1) &< o(1) + \sum_{p < \nu} \left(\frac{1}{p} - \frac{1}{2p^2} + \dots \mp \frac{1}{Mp^M} \right) \\ &= o(1) + \sum_{p^m < \nu} (-1)^{m-1} \frac{\Lambda_1(p^m)}{p^m}. \end{aligned}$$

Since $\sum_{n < \nu} \frac{\Lambda_1(n)}{n} < \log(c \log \nu) + o(1)$,

since $\sum_{p^m < \nu} (-1)^{m-1} \frac{\Lambda_1(p^m)}{p^m} < \log(b \log \nu)$,

by Lemma 2, and since

$$\log \nu < (1 + 4\epsilon) 2 \log k,$$

where ϵ is arbitrarily small, (9.2) and (9.5) lead at once to the results of the theorem.

10. THEOREM 2. *On hypothesis R^* there exist characters χ^* , of arbitrarily large k^* , for which*

$$L^*(1) > \{1 + o(1)\} c \log \log k^*,$$

and others for which

$$L^*(1) < \frac{1 + o(1)}{b \log \log k^*}.$$

In Lemma 3 we take, firstly all the a_r to be $+1$, secondly all the a_r to be -1 . In either case let the number x of Lemma 3 be an integer, and let

$$k_r = rT + k_0 \quad (0 < r \leq x^4),$$

$$L_r = L(1),$$

where $L(1)$ belongs to the character $(-k_r | n)$. Then

$$\log T = \log 4 + \sum_{p \leq x} \log p.$$

Since the right-hand side has an Ax for upper and for lower bound, and since

$$T < k < (x^4 + 1)(T + 1),$$

we have clearly

$$A \log k_r < x < A \log k_r.$$

We take now $k = k_r$, $\chi = (-k_r | n)$ in Lemma 8, and average the result over $r = 1, 2, \dots, x^4$. Observing that in the two cases we have respectively $(-k_r | p) = +1$ for $p \leq x$, and $(-k_r | p) = -1$ for $p \leq x$, we find respectively

$$(10.1) \quad \frac{1}{x^4} \sum_{r=1}^{x^4} \log L_r = \sum_{p^m \leq x} \frac{1}{m} \left(\frac{\pm 1}{p} \right)^m + o(1) \\ + \sum_{x < p < x^4} \frac{e^{-p/x}}{p} \frac{1}{x^4} \sum_{r=1}^{x^4} (-rT - k_0 | p).$$

+ This argument is the k -analogue of (c) of § 1.

Now p does not divide T ; hence $(-rT - k_0 | p)$ is, for fixed p and varying r , a (primitive and so) non-principal character $\chi(r)$ to modulus p ; hence

$$\left| \sum_{r=1}^{x^4} (-rT - k_0 | p) \right| \leq p,$$

and the last term in (10.1) is

$$O\left(\frac{1}{x^4}\right) \sum_{p > x} e^{-py} = O\left(\frac{1}{x^4 y}\right) = o(1),$$

and may be omitted. If now we use Lemma 2 to evaluate the (new) right-hand side of (10.1) we find that the average of the $\log L_r$ exceeds $\log(c \log x) + o(1)$ in the first case, and is less than $-\log(b \log x) + o(1)$ in the second. It follows that some one of the $x^4 \log L_r$'s, let us say $\log L_q$, has respectively the same property. Since $k_q < e^{Ax}$ it follows that $\log L_q$ respectively exceeds $\log(c \log \log k_q) + o(1)$ and is less than $-\log(b \log \log k_q) + o(1)$.

If $-k_q$ is a *fundamental* discriminant we have obtained what we want. If it is not† we have $k_q = P^2 k^*$, where $P > 1$ and $-k^*$ is a fundamental discriminant. On the one hand k^* exceeds unity, being congruent to $-1 \pmod{4}$; on the other it is divisible by no $p \leq x$. Hence k^* is large with x . Also, if $\chi^*(n) = (-k^* | n)$,

$$\frac{L^*(1)}{L_q} = \Pi \left(1 - \frac{\chi^*(p)}{p}\right) = \varpi,$$

the product being taken over all primes p that divide P . Each factor in ϖ is $1 + O(x^{-1})$, since k_q is prime to T and so to any prime less than x . Also, if ν is the number of factors in ϖ , $P \geq x^\nu$, and so

$$\nu < \frac{\log P}{\log x} < \frac{\log k_q}{\log x} < \frac{Ax}{\log x}.$$

Thus

$$\varpi = \left\{1 + O\left(\frac{1}{x}\right)\right\}^{O\left[\frac{x}{\log x}\right]} = 1 + o(1),$$

and the conclusions about L_q are true also for $L^*(1)$.

† Mr. Ingham, to whom I communicated a rough draft of the paper, pointed out that I had overlooked this possibility, and showed how to make good the omission. I am indebted to Mr. Ingham also for various improvements in the exposition of other parts of the paper.

11. If $-k$ is a fundamental discriminant and $k > 4$, the class-number $h(k)$ of the corpus $P(\sqrt{-k})$ is given by

$$h = \frac{\sqrt{k}}{\pi} L^*(1) = \frac{\sqrt{k}}{\pi} \sum_{n=1}^{\infty} \frac{(-k|n)}{n}.$$

The results of Theorems 1 and 2 provide direct applications to the order of $h(k)$. It is not necessary to set these out at length, since they may be read off at once from the parent theorems.

12. I conclude with some remarks on the unproved hypotheses employed throughout the paper. Hypothesis X, without modification, is essential in proving the results of Theorem 1. In Theorem 2, however, it is possible to replace hypothesis R^* by a very much slighter one. It is enough, in fact, to assume that, for all large $k = k^*$ and *some* $A_1 \dagger$, $L^*(s)$ has no zero in

$$\sigma > 1 - \frac{A_1 \log \log k}{\log k} = 1 - \eta(k).$$

The proof follows the same general lines as before, but it is naturally rather more complicated in detail. I give only a bare summary.

The O -term in Lemma 7 we replace by

$$(12.1) \quad O(y^{\frac{1}{2}\eta(k)} \log k).$$

In Lemma 8 we replace $y = x^{-3}$ by $y = \exp(-A_2 x)$, choosing A_2 large enough; then (12.1) becomes $o(1)$, and we obtain in place of (8.2)

$$(12.2) \quad \log L(1) = \sum_{p^m \leq x} \frac{1}{m} \left(\frac{\chi(p)}{p} \right)^m + \sum_{x < p^2 < y^{-2}} \frac{\chi(p)}{p} e^{-py} + o(1).$$

We now average $\log L_r$ in Theorem 2 over $r = 1, 2, \dots, R$, where $R = [y^{-3}]$, instead of over $r = 1, 2, \dots, x^4$. This gives

$$\frac{1}{R} \sum_{r=1}^R \log L_r = \sum_{p^m \leq x} \frac{1}{m} \left(\frac{\pm 1}{p} \right)^m + o(1),$$

the average being the same as before. Since we still have

$$Ax < \log k_1 \leq \log k_q \leq \log k_R \leq \log(y^{-3} k_0) < Ax + \log k_0 < Ax$$

the rest of the argument may stand unaltered.

† The propriety here of the " A " notation is a delicate point of pedantry.

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THE LONDON MATHEMATICAL SOCIETY.

BALANCE SHEET, 31st October, 1927.

LIABILITIES.		ASSETS.	
	£ s. d.		£ s. d.
Lieut.-Col. Campbell's Fund ...	499 7 5	London, Midland & Scottish Railway £385 4 per Cent. Guaranteed Stock at 80½ ...	309 18 6
Sir A. B. Kempe's Fund ...	200 0 0	India Stock £700. 7s. 1d. 3 per Cent. at 60¼ ...	425 9 4
Donations ...	357 10 0	Funding Loan £3000 4 per Cent. at 85½ ...	2576 5 0
Life Compositions Fund ...	3064 12 0	Conversion Loan £1700 3½ per Cent. at 75½ ...	1283 10 0
Reserve Fund in 1926 ...	356 17 2	<i>The above-mentioned stocks are valued at the present market price.</i>	
Appreciation of Investments	74 12 1	Cash at Bank ...	515 8 11
Reserve Fund ...	431 9 3	Cash in hand ...	3 6 9
Sundry Creditors ...	21 11 8		
Revenue Account balance in hand ...	539 8 2		
			<u>£5113 18 6</u>

REVENUE ACCOUNT for the year 1st November, 1926, to 31st October, 1927.

	YEAR 1925-26.				YEAR 1925-26		
	£ s. d.	£ s. d.	£ s. d.		£ s. d.	£ s. d.	£ s. d.
Balance from last year ...	566 9 1	316 17 7		Printing and Circulating Proceedings ...	967 8 3		
Income tax returned ...	—	20 8 8		Ditto <i>Journal</i> ...	420 11 0	1195 12 6	
Dividends on—				General Printing ...	107 7 6		
Lord Rayleigh's Fund ...	38 11 11	42 9 8		Purchases and Binding for Library ...	43 11 1	31 5 0	
The Society's Investments ...	203 16 3	189 3 10		Rent ...	30 0 0	36 0 0	
Entrance Fees ...	48 6 3	39 18 0		Postages and Sundries ...	9 6 3	6 3 5	
Annual Subscriptions ...	459 6 0	441 2 5		Balance in hand on 31st October, 1927 ...	539 8 2	566 9 1	
Sales of <i>Proceedings</i> ...	544 1 3	464 4 4					
Ditto <i>Journal</i> ...	59 16 6						
Advertisements revenue ...	47 5 0	15 5 0					
Grant from Royal Society ...	150 0 0	300 0 0					
Reserve for current year's printing ...	—	—					
	<u>£2117 12 3</u>	<u>1829 9 6</u>			<u>£2117 12 3</u>	<u>1829 9 6</u>	

REVENUE ACCOUNT of the DE MORGAN MEDAL FUND.

	£ s. d.		£ s. d.
Balance in hand 30th October, 1926 ...	9 2 3	Messrs. J. Moore, Ltd., De Morgan medals ...	9 0 9
Dividends received ...	2 8 0	Balance in hand ...	3 1 6
Income tax returned ...	12 0		
	<u>£12 2 3</u>		<u>£12 2 3</u>

STATEMENT OF TRUST FUNDS, 31st October, 1927.

Lord Rayleigh's Fund consists of the following Investments:—

Great Indian Peninsula Railway £50. 8s. 8d. Annuity, Class B.
£229. 13s. 4d. 4 per Cent. Funding Stock.

This Fund is held in trust to apply the income as part of the general income of the Society.

The De Morgan Medal Fund consists of Great Western Railway £60 5 per Cent. Preference Stock.

10th November, 1927.

Signed on behalf of the Council:—

G. H. HARDY, *President*.

A. E. WESTERN, *Treasurer*.

I report to the Members that I have obtained all the information and explanation I required as Auditor, and that I have examined the above Accounts with the books, and that, to the best of my information and of the explanations given to me, such Accounts are properly drawn up so as to exhibit a true and correct view of the state of the affairs of the Society as shown by the books of the Society.

15th November, 1927.

A. E. JOLLIFFE, *Auditor*.

ON MEAN VALUES OF FUNCTIONS OF A COMPLEX AND
OF A REAL VARIABLE

By A. S. BESICOVITCH.

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LET $f(z)$ be an analytic function of $z = \sigma + it$, which is regular at all points of the strip $a < \sigma < b$, $-\infty < t < +\infty$. We denote the limit

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} |f(\sigma_0 + it)| dt,$$

if it exists, by $M\{|f(\sigma_0 + it)|\}$, and we call it the mean value of the function $f(z)$ on the line $\sigma = \sigma_0$. In their paper "Theorems concerning mean values of analytic functions", Hardy, Ingham, and Pólya* (H.I.P.) consider various questions connected with the existence of the mean value of a function. They prove the following theorem:—

THEOREM A. *If the function $f(z)$ satisfies the following conditions:*

(i) *the mean value $M\{|f(\sigma + it)|^2\}$ exists on an enumerable set of lines in the strip*

$$\sigma = a_1, \sigma = a_2, \dots, \sigma = a_n, \text{ where } a_n \rightarrow c \quad (a < c < b);$$

$$(ii) \quad \int_t^{t+1} |f(\sigma + ti)|^2 dt < C \quad (C \text{ some constant})$$

for all σ, t such that $a < \sigma < b$, $-\infty < t < +\infty$;

then the mean value $M\{|f(\sigma + it)|^2\}$ exists on all lines in the strip $a < \sigma < b$.

It follows then that, if the function $f(z)$ has no zeros in the strip and is bounded in the strip, the same theorem holds for the existence of the

* *Proc. Royal Soc. (A)*, 113 (1927), 542–569. In what follows we shall denote this paper by "H.I.P.".

mean value not only of the function $[f(z)]^2$, but also for any positive power $[f(z)]^p$, whether integral or not, of $f(z)$, since then we can select any branch of the function $[f(z)]^{\frac{1}{p}}$, and this will be a one-valued analytic function in the strip. In this paper I show that this last theorem is also valid in the case when the function has zeros in the strip, but the number of zeros with modulus smaller than n is $o(n)$ (Landau's notation).

Prof. G. H. Hardy has pointed out to me that by using an argument analogous to one employed in the proof of Theorem A, he proved the following theorem (not yet published):—

THEOREM B. *If the function $f(x)$ of a real variable x satisfies the following conditions:*

$$(i) \quad f(x) \geq 0;$$

$$(ii) \quad \frac{1}{T} \int_{-T}^{+T} [f(x)]^p dx < C$$

for all $T \geq 1$ (C, p positive constants);

(iii) *The mean value $M\{[f(x)+a]^p\}$ exists for*

$$a = a_1, a_2, \dots, a_n, \dots, \text{ where } a_n \rightarrow c > 0.$$

then the mean value $M\{[f(x)+a]^p\}$ exists for all $a > 0$.

I shall now show that in the hypotheses of these two theorems it is necessary to include not only the existence of an infinite set of lines on which mean values exist, but also the stipulation that, in the case of Theorem A, the set $\sigma_1, \sigma_2, \dots$ has a limit-point inside the interval (a, b) , and that, in the case of Theorem B, the set $a_1, a_2, \dots$ has a limit-point which is greater than zero. I do this by giving examples of functions, for which the mean values exist for an enumerable set of values of σ or of a , while they do not exist for the remaining values.

I begin with the construction of a regular function for which the mean value exists on any finite number of lines of the strip, but not on others. Independently, and at the same time, Prof. M. Riesz has constructed by a quite different method a function for which the mean value exists on two lines of the strip, but not on others. Later, by a suitable modification of Riesz's method, Hardy, Ingham, and Pólya have constructed a function possessing mean values on any finite number of lines of the strip, but on no others*.

* H.I.P., 565-569.

1. Let $f(z)$ be an analytic function of $z = \sigma + it$ defined in the strip $0 \leq \sigma \leq 1$, and let

$$b_1, b_2, \dots, b_n, \dots \quad (b_n = \beta_n + i\gamma_n, |\gamma_n| \leq |\gamma_{n+1}|) \quad (1)$$

be its singular points in this strip. The function $f(z)$ may be many-valued. We then divide the strip into rectangles by the lines $t = \gamma_1, t = \gamma_2, \dots$. We make the function $f(z)$ one-valued in the strip (except on the lines $t = \gamma_1, t = \gamma_2, \dots$) by taking it to be equal to any branch of $f(z)$ inside each of these rectangles. This function $f(z)$ is not generally an analytic function in the whole strip.

Let $f(z)$ satisfy the following conditions :

$$(i) \quad \int_{\sigma+it}^{\sigma+i(t+1)} |f(z)|^2 |dz| < 1 \quad (2)$$

for $0 \leq \sigma \leq 1, -\infty < t < +\infty$;

(ii) if the sequence (1) is infinite, then

$$\frac{n}{\gamma_n} \rightarrow 0; \quad (3)$$

(iii) the limit, as $T \rightarrow \infty$, of the expression

$$\frac{1}{2T} \int_{\sigma-iT}^{\sigma+iT} |f(\sigma+it)|^2 dt \quad (4)$$

exists for

$$\sigma = a_1, a_2, \dots, a_n, \dots; \quad a_n \rightarrow a \quad (0 < a < 1);$$

then the limit exists for every σ of the interval $0 < \sigma < 1$.

Let us denote by $l(T)$ the set of those points of the open interval $(-T, +T)$ which are outside all the intervals

$$(\gamma_1-1, \gamma_1+1), (\gamma_2-1, \gamma_2+1), \dots;$$

$l(T)$ is evidently a finite set of open intervals. Let $\beta_n + i\gamma_n$ be the last point of (1) for which $|\gamma_n| < T+1$. We have

$$\begin{aligned} 0 &< \frac{1}{2T} \int_{\sigma-iT}^{\sigma+iT} |f(z)|^2 |dz| - \frac{1}{2T} \int_{l(T)} |f(\sigma+it)|^2 dt \\ &\leq \frac{1}{2T} \sum_{k=1}^n \int_{\gamma_{k-1}}^{\gamma_k+1} |f(\sigma+it)|^2 dt < \frac{2n}{2T} = \frac{n}{T}. \end{aligned}$$

This difference tends to zero as $T \rightarrow \infty$. For if the sequence (1) is finite, then

$n/T \rightarrow 0$; if not, then we have $n/T < n/|\gamma_n| - 1$, which also tends to zero by (3). Thus the limits of the integral $I(\sigma, T) = \frac{1}{2T} \int_{i(T)} |f(\sigma + it)|^2 dt$ and of (4) exist for the same values of σ and are always equal. The rest of the proof is an application of the Hardy-Ingham-Pólya argument.

Let $\bar{f}(z)$ be the conjugate function of $f(z)^*$. Then we have

$$I(\sigma, T) = \frac{1}{2T} \int_{i(T)} f(\sigma + it) \bar{f}(\sigma - it) dt.$$

Now let us take

$$\sigma = x + iy, \quad 0 < x < 1, \quad -\frac{1}{2} < y < \frac{1}{2}.$$

The function $I(\sigma, T)$ evidently exists, and is regular for all these values of σ ; and, for these values of σ and all integral values of T ,

$$|I(\sigma, T)| < 1.$$

From the existence of the limit, as $T \rightarrow \infty$, of the analytic function of $\sigma : I(\sigma, T)$ for $\sigma = a_1, a_2, \dots$ we deduce, by Vitali's theorem, that the limit exists for every σ in $(0, 1)$.

The preceding argument is not applicable to the general case of any distribution of singular points. For in the general case the measure of the set of intervals

$$(\gamma_1 - 1, \gamma_1 + 1), \quad \dots, \quad (\gamma_n - 1, \gamma_n + 1),$$

is not infinitely small in comparison with the length of the interval $(-T, +T)$. On the other hand, if instead of these intervals, we exclude the intervals $(\gamma_1 - \delta_1, \gamma_1 + \delta_1), (\gamma_2 - \delta_2, \gamma_2 + \delta_2), \dots$ such that

$$\lim_{n \rightarrow \infty} \frac{\delta_1 + \dots + \delta_n}{\gamma_n} = 0,$$

then $\liminf \delta_n = 0$. Consider now the function $f(\sigma + it) \bar{f}(\sigma - it)$ in the region

$$\sigma = x + iy, \quad 0 < x < 1, \quad -\delta < y < +\delta, \quad \gamma_{n-1} + \delta_{n-1} < t < \gamma_n - \delta_n,$$

where δ is some fixed positive number, and $\delta_n < \delta$. For values of t in

* If the analytic function $f(z)$ is obtained from the series $\sum_{n=0}^{\infty} A_n(z-a)^n$, then the function obtained from the series $\sum_{n=0}^{\infty} \bar{A}_n(z-\bar{a})^n$, where $\bar{A}_n, \bar{a}$ are the conjugate numbers of A_n, a , is called "the conjugate function $\bar{f}(z)$ ".

the interval $(\gamma_n - \delta, \gamma_n + \delta_n)$ the function has singular points in the above region of values of σ , and, therefore, in this region the integral

$$\frac{1}{2T} \int_{l(T)} f(\sigma + it) \bar{f}(\sigma - it) dt,$$

considered as a function of σ , may not be regular.

2. We now construct an analytic function which is bounded and regular in a strip, and which possesses a mean value only on a finite set of lines of the strip.

$$\text{Let} \quad f_1(z) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{c'_n}{[z - (\sigma'_n + nli)]^2},$$

where, for every integer k ,

$$\begin{aligned} c'_{2k} &= x_1, & \sigma'_{2k} &= 0, \\ c'_{2k+1} &= y_1, & \sigma'_{2k+1} &= -2; \end{aligned}$$

and x_1, y_1 are positive numbers not greater than 1. The function $f_1(z)$ is regular, periodic with period $2li$, and bounded in the half-plane $\sigma \geq \sigma_0 > 0$. Let us consider the integral

$$\int_{\sigma - \frac{1}{2}li}^{\sigma + \frac{1}{2}li} |f_1(z)| |dz|;$$

its value lies between

$$\frac{x_1}{\pi} \int_{\sigma - \frac{1}{2}li}^{\sigma + \frac{1}{2}li} \frac{|dz|}{|z|^2} - \frac{1}{\pi} \sum'_{n=-\infty}^{\infty} \int_{\sigma - \frac{1}{2}li}^{\sigma + \frac{1}{2}li} \frac{c'_n |dz|}{|z - (\sigma'_n + nli)|^2}$$

$$\text{and} \quad \frac{x_1}{\pi} \int_{\sigma - \frac{1}{2}li}^{\sigma + \frac{1}{2}li} \frac{|dz|}{|z|^2} + \frac{1}{\pi} \sum'_{n=-\infty}^{\infty} \int_{\sigma - \frac{1}{2}li}^{\sigma + \frac{1}{2}li} \frac{c'_n |dz|}{|z - (\sigma'_n + nli)|^2},$$

where Σ' denotes that the value $n = 0$ is omitted. Now

$$\begin{aligned} \frac{1}{\pi} \sum'_{n=-\infty}^{\infty} \int_{\sigma - \frac{1}{2}li}^{\sigma + \frac{1}{2}li} \frac{c'_n |dz|}{|z - (\sigma'_n + nli)|^2} &= \frac{1}{\pi} \sum'_{n=-\infty}^{\infty} \int_{\sigma - (n + \frac{1}{2})li}^{\sigma - (n - \frac{1}{2})li} \frac{c'_n |dz|}{|z - \sigma'_n|^2} \\ &= \frac{1}{\pi} \sum'_{n=-\infty}^{\infty} \int_{-(n + \frac{1}{2})l}^{-(n - \frac{1}{2})l} \frac{c'_n dt}{[(\sigma - \sigma'_n)^2 + t^2]} < \frac{1}{\pi} \sum'_{n=-\infty}^{\infty} \int_{-(n + \frac{1}{2})l}^{-(n - \frac{1}{2})l} \frac{dt}{t^2} = \frac{2}{\pi} \int_{\frac{1}{2}l}^{\infty} \frac{dt}{t^2} = \frac{4}{\pi l} \end{aligned}$$

and

$$\frac{x_1}{\pi} \int_{\sigma - \frac{1}{2}li}^{\sigma + \frac{1}{2}li} \frac{|dz|}{|z|^2} = \frac{x_1}{\pi} \int_{-\frac{1}{2}l}^{+\frac{1}{2}l} \frac{dt}{\sigma^2 + t^2} = \frac{x_1}{\pi} \int_{-\infty}^{+\infty} \frac{dt}{\sigma^2 + t^2} - \frac{2x_1}{\pi} \int_{\frac{1}{2}l}^{\infty} \frac{dt}{\sigma^2 + t^2} = \frac{x_1}{\sigma} - \frac{4\theta_1}{\pi l};$$

θ_1 being a number between -1 and 1 . We use $\theta_2, \theta_3, \dots$, likewise, to denote numbers between -1 and 1 in the following equations. Thus we have

$$\int_{\sigma-\frac{1}{2}li}^{\sigma+\frac{1}{2}li} |f_1(z)| |dz| = \frac{x_1}{\sigma} + \frac{8\theta_2}{\pi l}.$$

In the same way we obtain

$$\int_{\sigma+\frac{3}{2}li}^{\sigma+\frac{5}{2}li} |f_1(z)| |dz| = \frac{y_1}{\sigma+2} + \frac{8\theta_3}{\pi l}.$$

The function $f_1(z)$ is periodic with the period $2li$, and therefore

$$\begin{aligned} M\{|f_1(\sigma+it)|\} &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} |f_1(\sigma+it)| dt = \frac{1}{2l} \int_{\sigma-\frac{1}{2}li}^{\sigma+\frac{3}{2}li} |f_1(z)| |dz| \\ &= \frac{1}{2l} \left(\frac{x_1}{\sigma} + \frac{y_1}{\sigma+2} + \frac{16\theta_4}{l} \right). \end{aligned}$$

Now let

$$f_2(z) = \frac{1}{\pi} \sum_{n=-\infty}^{+\infty} \frac{c_n''}{[z - (\sigma_n'' + nli)]^2}$$

where, for any integer k ,

$$\begin{aligned} c_{2k}'' &= x_2, & \sigma_{2k}'' &= -1, \\ c_{2k+1}'' &= y_2, & \sigma_{2k+1}'' &= -3; \end{aligned}$$

and x_2, y_2 are positive numbers not greater than 1 . We obtain, in precisely the same way as before,

$$M\{|f_2(\sigma+it)|\} = \frac{1}{2l} \left(\frac{x_2}{\sigma+1} + \frac{y_2}{\sigma+3} + \frac{16\theta_5}{l} \right),$$

and hence

$$M\{|f_1(\sigma+it)|\} - M\{|f_2(\sigma+it)|\} = \frac{1}{2l} \left(\frac{x_1}{\sigma} - \frac{x_2}{\sigma+1} + \frac{y_1}{\sigma+2} - \frac{y_2}{\sigma+3} + \frac{32\theta_6}{l} \right). \quad (1)$$

Let us now give x_1, x_2, y_1, y_2 such values that the expression

$$\frac{x_1}{\sigma} - \frac{x_2}{\sigma+1} + \frac{y_1}{\sigma+2} - \frac{y_2}{\sigma+3} \quad (2)$$

is zero for $\sigma = 1, 2, 3$. The equations to be satisfied are

$$\begin{aligned} x_1 - \frac{1}{2}x_2 + \frac{1}{3}y_1 - \frac{1}{4}y_2 &= 0, \\ \frac{1}{2}x_1 - \frac{1}{3}x_2 + \frac{1}{4}y_1 - \frac{1}{5}y_2 &= 0, \\ \frac{1}{3}x_1 - \frac{1}{4}x_2 + \frac{1}{5}y_1 - \frac{1}{6}y_2 &= 0, \end{aligned}$$

so that we may take

$$x_1 = \frac{1}{30}, \quad x_2 = \frac{2}{5}, \quad y_1 = 1, \quad y_2 = \frac{2}{3}.$$

The expression on the left of (1) is evidently a continuous function of σ for $\sigma > 0$. Let $\delta < 1$ be an arbitrarily small positive number, and let a be a positive constant greater than $3 + \delta$; finally let ϵ be the lower bound of the modulus of (2) in that part of the interval $(0, a)$ which is left after removing the intervals $(1 - \delta, 1 + \delta)$, $(2 - \delta, 2 + \delta)$, $(3 - \delta, 3 + \delta)$, and let us choose $l > 32/\epsilon$; then the expression on the right of (1) has the same signs in the intervals $(0, 1 - \delta)$, $(1 + \delta, 2 - \delta)$, $(2 + \delta, 3 - \delta)$, $(3 + \delta, a)$ as the expression (2), *i.e.* $+$, $-$, $+$, $-$ respectively, and so, being a continuous function, has at least one zero in each of the intervals $(1 - \delta, 1 + \delta)$, $(2 - \delta, 2 + \delta)$, $(3 - \delta, 3 + \delta)$. Thus, if $0 < \sigma < a$, the expressions

$$M\{|f_1(\sigma + it)|\}, \quad M\{|f_2(\sigma + it)|\}$$

are equal for values $\sigma_1, \sigma_2, \sigma_3$ (which are very near to the numbers 1, 2, 3), and are different from one another throughout the intervals $(0, 1 - \delta)$, $(1 + \delta, 2 - \delta)$, $(2 + \delta, 3 - \delta)$, $(3 + \delta, a)$. It can be shown that for large values of l there exist only three roots of the equation

$$M\{|f_1(\sigma + it)|\} = M\{|f_2(\sigma + it)|\},$$

but we shall not dwell on that.

Let us divide the imaginary axis into intervals

$$\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_{-1}, \lambda_{-2}, \dots,$$

where λ_0 is the interval of length l with centre at $z = 0$; $\lambda_1, \lambda_2, \dots$ are successive intervals on the positive part of the imaginary axis, and $\lambda_{-1}, \lambda_{-2}, \dots$ on its negative part, and let us take $\lambda_n = \lambda_{-n} = n! \, i$ ($n = 1, 2, \dots$)*.

Let us consider the function

$$f(z) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{c_n}{[z - (\sigma_n + lni)]^2},$$

where c_n and σ_n are defined in the following way :

When lni belongs to the interval λ_{2j} ($j = 0, 1, 2, \dots$),

$$\begin{aligned} c_n &= x_1, & \sigma_n &= 0 & (n \text{ even}), \\ c_n &= y_1, & \sigma_n &= -2 & (n \text{ odd}); \end{aligned}$$

* We use λ_n to denote also the length of the interval.

when lni belongs to the interval λ_{2j+1} ,

$$c_n = x_2, \quad \sigma_n = -1 \quad (n \text{ even}),$$

$$c_n = y_2, \quad \sigma_n = -3 \quad (n \text{ odd}).$$

Let the numbers $a_n i$, $b_n i$ denote the ends of the interval λ_n , so that $\lambda_n = (a_n i, b_n i)$. It is easy to see that (for $\sigma > 0$) to an arbitrary positive number ϵ there corresponds a positive number d , such that, for

$$a_n + d < t < b_n - d,$$

(for all values of n so large that $\lambda_n > 2d$)

$$\left. \begin{aligned} |f(z) - f_1(z)| &< \epsilon \quad (n \text{ even}) \\ |f(z) - f_2(z)| &< \epsilon \quad (n \text{ odd}) \end{aligned} \right\}. \quad (3)$$

From (3), and from the mode of increase of λ_n , it is easy to see that, as $T \rightarrow \infty$, the upper limit of the expression

$$\frac{1}{2T} \int_{-T}^{+T} |f(\sigma + ti)| dt \quad (4)$$

is equal to the greater of the numbers $M\{|f_1(\sigma + it)|\}$ and $M\{|f_2(\sigma + it)|\}$, and its lower limit to the smaller. Hence the mean value $M\{|f(\sigma + it)|\}$ exists for $\sigma = \sigma_1, \sigma_2, \sigma_3$, and does not exist when σ lies in the intervals $(0, 1 - \delta)$, $(1 + \delta, 2 - \delta)$, $(2 + \delta, 3 - \delta)$, $(3 + \delta, a)$.

Thus $f(z)$ is the function which we wished to construct. In precisely the same way we can construct a function possessing mean values on any number of lines.

We shall now construct the function $F(z)$ which satisfies the following conditions:

(i) it is regular in the strip $a < \sigma < b$;

(ii) $\int_t^{t+1} |F(\sigma + ti)| dt < C$ for $a < \sigma < b$ and $-\infty < t < +\infty$;

(iii) $M\{|F(\sigma + it)|\}$ exists for an enumerable set of values of σ : $\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3, \dots$ in the interval (c, d) , but does not exist for other values of σ in this interval.

In this case, in virtue of Theorem A, the set of points $\bar{\sigma}_1, \bar{\sigma}_2, \dots$ can have its limit-points only at the ends of the interval (a, b) .

We can obtain such a function in the following way.

We begin with the above defined functions $f_1(z)$, $f_2(z)$, and consider the interval $\frac{1}{2} < \sigma < a$. We have proved that the function

$$D(\sigma) = M\{|f_1(\sigma + it)|\} - M\{|f_2(\sigma + it)|\}$$

changes its sign at least three times in the interval $(\frac{1}{2}, a)$. Let

$$\sigma_1 > \sigma_2 > \dots > \sigma_s$$

be all the zeros of the function $D(\sigma)$ in the interval $(\frac{1}{2}, a)$ at which it changes its sign, and let

$$(\sigma_1 - \epsilon_1, \sigma_1 + \epsilon_1), \quad \dots, \quad (\sigma_s - \epsilon_s, \sigma_s + \epsilon_s)$$

be non-overlapping intervals inside the interval $(\frac{1}{2}, a)$, which are such that the function $D(\sigma)$ has different signs at the ends of each interval. Let $\eta_1, \eta_2, \dots, \eta_s$ be positive numbers such that

$$|D(\sigma_j \pm \epsilon_j)| > \eta_j$$

for $j = 1, 2, \dots, s$.

Define now the functions $\phi_1(z), \phi_2(z), \dots$ as follows:

$$\phi_k(z) = \alpha_k \sum_{n=-\infty}^{\infty} \frac{1}{(z - e_k - nli)^2},$$

where $e_k < \frac{1}{2}$ and $\alpha_k < 1/2^k$, and also define the functions

$$D_1(\sigma) = M\{|f_1(\sigma + ti)|\} - M\{|f_2(\sigma + ti) + \phi_1(\sigma + ti)|\},$$

$$D_2(\sigma) = M\{|f_1(\sigma + it) + \phi_2(\sigma + it)|\} - M\{|f_3(\sigma + it) + \phi_1(\sigma + it)|\},$$

$$D_3(\sigma) = M\{|f_1(\sigma + it) + \phi_2(\sigma + it)|\} - M\{|f_2(\sigma + it) + \phi_1(\sigma + it) + \phi_2(\sigma + it)|\},$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

and so on.

Take α_1 so small that the function $D_1(\sigma)$ has the same sign as $D(\sigma)$ at all points $\sigma_j \pm \epsilon_j$ and satisfies the inequalities

$$|D_1(\sigma_j \pm \epsilon_j)| > \eta_j \quad (j = 1, 2, \dots, s)$$

for all values of $e_1 < \frac{1}{2}$. We know that $D(\frac{1}{2}) > 0$. Define e_1 so that $D_1(\frac{1}{2}) = -\frac{1}{2}$. Then the function $D_1(\sigma)$ will change its sign at least once in the interval $(\frac{1}{2}, \sigma_s - \epsilon_s)$. Let σ_{s+1} be a zero of $D_1(\sigma)$ which belongs to the interval $(\frac{1}{2}, \sigma_s - \epsilon_s)$ and at which $D_1(\sigma)$ changes its sign. We can then define the numbers $\epsilon_{s+1}, \eta_{s+1}$ so that the interval

$$(\sigma_{s+1} - \epsilon_{s+1}, \sigma_{s+1} + \epsilon_{s+1})$$

does not overlap any of the previous intervals, and is included in $(\frac{1}{2}, a)$, and so that $D_1(\sigma)$ has different signs at its ends and satisfies the inequalities

$$|D_1(\sigma_{s+1} \pm \epsilon_{s+1})| > \eta_{s+1}.$$

In the same way we define the numbers $a_2, e_2, e_{s+2}, \eta_{s+2}$ so that $D_2(\frac{1}{2}) = \frac{1}{4}$ and that $D_2(\sigma)$ changes its sign at the ends of non-overlapping intervals $(\sigma_j - \epsilon_j, \sigma_j + \epsilon_j)$ ($j = 1, 2, \dots, s+2$) and satisfies the inequalities

$$|D_2(\sigma_j \pm \epsilon_j)| > \eta_j \quad (j = 1, 2, \dots, s+2).$$

Proceeding in this way we obtain an enumerable set of non-overlapping intervals $(\sigma_j - \epsilon_j, \sigma_j + \epsilon_j)$ ($j = 1, 2, \dots$) and two periodic functions

$$F_1(z) = f_1(z) + \phi_2(z) + \phi_4(z) + \dots,$$

$$F_2(z) = f_2(z) + \phi_1(z) + \phi_3(z) + \dots,$$

which are such that the equation

$$M\{|F_1(\sigma + it)|\} = M\{|F_2(\sigma + it)|\}$$

is satisfied inside each of the intervals, and the inequality

$$\int_t^{t+1} |F_p(\sigma + it)| dt < C \quad (p = 1, 2),$$

where C is some constant, is satisfied for $\frac{1}{2} < \sigma < a$, $-\infty < t < +\infty$.

Define now the functions $\psi_k(z)$ ($k = 1, 2, \dots$) as follows:

$$\psi_k(z) = a_k \sum_{n=-\infty}^{\infty} \frac{1}{(z - e_k - nli)^2},$$

where the summation is extended over those values of nli which belong to the intervals $\lambda_0, \lambda_2, \lambda_{-2}, \lambda_4, \lambda_{-4}, \dots$, when k is even, and to the intervals $\lambda_1, \lambda_{-1}, \lambda_3, \lambda_{-3}, \dots$, when k is odd. Let

$$F(z) = f(z) + \psi_1(z) + \psi_2(z) + \dots$$

Then to any number $\epsilon > 0$ there corresponds a number d such that one of the two inequalities

$$|F(z) - F_1(z)| < \epsilon, \quad |F(z) - F_2(z)| < \epsilon$$

is satisfied in any interval $a_n + d < t < b_n - d$ (a_n, b_n being the ends of the interval λ_n), namely the first or the second of them according as n is even or odd. Further, there exists some positive number C_1 such that

$$\int_t^{t+1} |F(\sigma + it)| dt < C_1$$

for $\frac{1}{2} < \sigma < a$, $-\infty < t < +\infty$.

Then it is easy to see that for any σ of the interval $(\frac{1}{2}, a)$ the upper and the lower limits of

$$\frac{1}{2T} \int_{-T}^{+T} |F(\sigma + it)| dt, \quad \text{as } T \rightarrow +\infty,$$

are equal to the numbers

$$M\{|F_1(\sigma + it)|\}, \quad M\{|F_2(\sigma + it)|\}.$$

Thus $M\{|F(\sigma + it)|\}$ exists only for values of σ for which

$$M\{|F_1(\sigma + it)|\} = M\{|F_2(\sigma + it)|\},$$

i.e. for an enumerable set of values.

3. The last question which we shall discuss is the construction of a real function $f(x)$, bounded and greater than or equal to zero in the whole range $-\infty < x < +\infty$, for which the mean-value $M\{[f(x) + a]^\nu\}$ exists for an enumerable set of values of a in the interval $(0 < a < 1)$ and does not exist at any other point of this interval. By Theorem B the set of numbers a for which the mean value exists must have the single limit-point zero.

Let $\phi(x)$, $\psi(x)$ be two positive bounded functions,

$$C > \phi(x) > c > 0, \quad C > \psi(x) > c > 0,$$

given in the interval $(0, 1)$, and let $1 < p < 2$. Let the functions $\phi(x)$ and $\psi(x)$ satisfy the conditions

$$\int_0^1 [\phi(x)]^p dx = \int_0^1 [\psi(x)]^p dx, \quad (1a)$$

$$\int_0^1 [\phi(x)]^{p-1} dx = \int_0^1 [\psi(x)]^{p-1} dx, \quad (1b)$$

$$\int_0^1 [\phi(x) + 1]^p dx - \int_0^1 [\psi(x) + 1]^p dx = 1. \quad (1c)$$

A proof of the existence of such functions evidently would not present any difficulty. By Taylor's theorem

$$\begin{aligned} & [\phi(x) + a]^\nu - [\psi(x) + a]^\nu \\ &= \left([\phi(x)]^\nu + p[\phi(x)]^{\nu-1}a + \frac{p(p-1)}{1 \cdot 2} a^2 [\phi(x) + \theta a]^{\nu-2} \right) \\ &- \left([\psi(x)]^\nu + p[\psi(x)]^{\nu-1}a + \frac{p(p-1)}{1 \cdot 2} a^2 [\psi(x) + \theta a]^{\nu-2} \right) \quad (0 < \theta < 1). \quad (2) \end{aligned}$$

Integrating (2) we have, from (1a) and (1b),

$$\left| \int_0^1 [\phi(x) + a]^p dx - \int_0^1 [\psi(x) + a]^p dx \right| < b_1 a^2, \quad (3)$$

where, for $0 < a \leq 1$, b_1 is some fixed positive number. We now define the functions $\phi(x)$ and $\psi(x)$ in the intervals

$$(1, \frac{3}{2}), (\frac{3}{2}, \frac{7}{4}), (\frac{7}{4}, \frac{15}{8}), \dots \quad (4)$$

by means of their values in the interval $(0, 1)$ as follows. We take the sequence of linear functions

$$2x-2, \quad 4x-6, \quad 8x-14, \quad \dots,$$

so that the first of them runs from 0 to 1 in the first interval (4), the second runs through these values in the second interval, and so on, and now, taking some sequence of constants

$$a_1, \quad a_2, \quad \dots,$$

which will be defined later, we write

$$\phi(x) = a_1 \psi(2x-2), \quad \psi(x) = a_1 \phi(2x-2) \quad \text{for } 1 < x \leq \frac{3}{2}; \quad (4, 1)$$

$$\phi(x) = a_2 \phi(4x-6), \quad \psi(x) = a_2 \psi(4x-6) \quad \text{for } \frac{3}{2} < x \leq \frac{7}{4}; \quad (4, 2)$$

$$\phi(x) = a_3 \psi(8x-14), \quad \psi(x) = a_3 \phi(8x-14) \quad \text{for } \frac{7}{4} < x \leq \frac{15}{8}; \quad (4, 3)$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

Then, for any positive integer n , we have

$$\int_{2-(\frac{3}{2})^{n-1}}^{2-(\frac{3}{2})^n} [\phi(x)]^p dx = \int_{2-(\frac{3}{2})^{n-1}}^{2-(\frac{3}{2})^n} [\psi(x)]^p dx, \quad (5_a^n)$$

$$\int_{2-(\frac{3}{2})^{n-1}}^{2-(\frac{3}{2})^n} [\phi(x)]^{p-1} dx = \int_{2-(\frac{3}{2})^{n-1}}^{2-(\frac{3}{2})^n} [\psi(x)]^{p-1} dx, \quad (5_b^n)$$

$$\int_{2-(\frac{3}{2})^{n-1}}^{2-(\frac{3}{2})^n} [\phi(x) + a_n]^p dx - \int_{2-(\frac{3}{2})^{n-1}}^{2-(\frac{3}{2})^n} [\psi(x) + a_n]^p dx = (-\frac{1}{2})^n a_n^p. \quad (5_c^n)$$

We observe, further, that, in

$$\left| \int_{2-(\frac{3}{2})^{n-1}}^{2-(\frac{3}{2})^n} [\phi(x) + a]^p dx - \int_{2-(\frac{3}{2})^{n-1}}^{2-(\frac{3}{2})^n} [\psi(x) + a]^p dx \right|, \quad (6)$$

$\phi(x)$ and $\psi(x)$ are of order a_n by (4, n), and therefore this difference tends to zero with a_n uniformly for $0 \leq a \leq 1$.

Taking $n = 1$ in (6) we obtain the inequality

$$\left| \int_1^{\frac{3}{2}} [\phi(x) + a]^p dx - \int_1^{\frac{3}{2}} [\psi(x) + a]^p dx \right| < \beta_1 < \frac{1}{4} \quad (0 \leq a \leq 1) \quad (7)$$

for all sufficiently small values of a_1 . Let a_1 be chosen so small that

$$\frac{1}{2}a_1^3 > 2b_1a_1^2. \quad (8)$$

Then, writing

$$\begin{aligned} I(0, \frac{3}{2}) &= \int_0^{\frac{3}{2}} [\phi(x) + a]^p dx - \int_0^{\frac{3}{2}} [\psi(x) + a]^p dx \\ &= \left\{ \int_0^1 [\phi(x) + a]^p dx - \int_0^1 [\psi(x) + a]^p dx \right\} \\ &\quad + \left\{ \int_1^{\frac{3}{2}} [\phi(x) + a]^p dx - \int_1^{\frac{3}{2}} [\psi(x) + a]^p dx \right\}, \end{aligned}$$

we have

$$\left. \begin{aligned} I(0, \frac{3}{2}) &= 0 && \text{for } a = 0, \text{ by (1}_a\text{) and (5}_a^1\text{)} \\ &< -b_1a_1^2 && \text{for } a = a_1, \text{ by (3), (5}_c^1\text{), and (8)} \\ &> 1 - \beta_1 && \text{for } a = 1, \text{ by (1}_c\text{) and (7)} \end{aligned} \right\}. \quad (I_1)$$

Now from (5_a¹) and (5_b¹) we have

$$\left| \int_0^{\frac{1}{2}} [\phi(x) + a]^p dx - \int_0^{\frac{1}{2}} [\psi(x) + a]^p dx \right| < b_2 a^2 \quad (b_2 \text{ constant}) \quad (9)$$

just as with (3).

Taking $n = 2$ in (6) we have

$$\left| \int_{\frac{1}{2}}^{\frac{1}{2}} [\phi(x) + a]^p dx - \int_{\frac{1}{2}}^{\frac{1}{2}} [\psi(x) + a]^p dx \right| < \beta_2 \quad (0 \leq a \leq 1), \quad (10)$$

where, for sufficiently small values of a_2 , we can choose

$$\beta_2 < \frac{1}{4}\beta_1 \quad \text{and} \quad \beta_2 < \frac{1}{4}b_1a_1^2. \quad (11)$$

We also choose a_2 so small that

$$a_2 < a_1 \quad \text{and} \quad \frac{1}{4}a_2^3 > 2b_2a_2^2. \quad (12)$$

Then, writing

$$\begin{aligned} I(0, \frac{7}{4}) &= \int_0^{\frac{7}{4}} [\phi(x) + a]^p dx - \int_0^{\frac{7}{4}} [\psi(x) + a]^p dx \\ &= I(0, \frac{3}{2}) + \left\{ \int_1^{\frac{7}{4}} [\phi(x) + a]^p dx - \int_1^{\frac{7}{4}} [\psi(x) + a]^p dx \right\}, \end{aligned}$$

we have, in virtue of (I_1) ,

$$\left. \begin{aligned} I(0, \frac{7}{4}) &= 0 && \text{for } a = 0, \text{ by } (5_a^2) \text{ and } (I_1) \\ &> b_2 a_2^2 && \text{for } a = a_2, \text{ by } (9), (12), \text{ and } (5_c^2) \\ &< -b_1 a_1^2 + \beta_2 && \text{for } a = a_1, \text{ by } (10) \text{ and } (I_1) \\ &> 1 - \beta_1 - \beta_2 && \text{for } a = 1, \text{ by } (10) \text{ and } (I_1) \end{aligned} \right\} \quad (I_2)$$

Applying this argument successively to

$$I(0, \frac{3}{2}), \quad I(0, \frac{7}{4}), \quad \dots, \quad I(0, 2 - \frac{1}{2^n}), \quad \dots,$$

and writing

$$I(0, 2) = \int_0^2 [\phi(x) + a]^p dx - \int_0^2 [\psi(x) + a]^p dx,$$

we obtain finally

$$\begin{aligned} I(0, 2) &> 1 - \beta_1 - \beta_2 - \beta_3 - \dots && \text{for } a = 1, \\ &< -b_1 a_1^2 + \beta_2 + \beta_3 + \beta_4 + \dots && \text{for } a = a_1, \\ &> b_2 a_2^2 - \beta_3 - \beta_4 - \beta_5 - \dots && \text{for } a = a_2, \\ &< -b_3 a_3^2 + \beta_4 + \beta_5 + \beta_6 + \dots && \text{for } a = a_3, \\ \dots & \dots && \dots \end{aligned}$$

and so by inequalities (7), (11), we have

$$\begin{aligned} \int_0^2 [\phi(x) + a]^p dx - \int_0^2 [\psi(x) + a]^p dx &> \frac{2}{3} && \text{for } a = 1, \\ &< -\frac{2}{3} b_2 a_1^2 && \text{for } a = a_1, \\ &> \frac{2}{3} b_2 a_2^2 && \text{for } a = a_2, \\ &< -\frac{2}{3} b_3 a_3^2 && \text{for } a = a_3, \\ \dots & \dots && \dots \\ &= 0 && \text{for } a = 0. \end{aligned}$$

Thus we see that the continuous function

$$\int_0^2 [\phi(x) + a]^p dx - \int_0^2 [\psi(x) + a]^p dx$$

takes positive and negative values alternatively for a certain monotonic

sequence of values of a :

$$1 > a_1 > a_2 > a_3 \dots \quad (13)$$

Therefore the equation

$$\int_0^2 [\phi(x) + a]^p dx - \int_0^2 [\psi(x) + a]^p dx = 0 \quad (14)$$

has a root c_i between any two consecutive numbers a_{i-1} , a_i of this sequence, so that

$$1 > c_1 > a_1 > c_2 > a_2 > c_3 > a_3 > \dots$$

We now define the functions $\phi(x)$ and $\psi(x)$ for the whole interval $-\infty < x < +\infty$ by taking them to be both periodic with period 2.

Finally we define the even function $f(x)$ for the range

$$-\infty < x < +\infty$$

as follows :

$$\begin{aligned} f(x) &= \phi(x) & \text{for } 0 \leq x < 2 & \text{ and for } (2k)! \leq x < (2k+1)! \quad (k \geq 1), \\ &= \psi(x) & \text{for } (2k+1)! \leq x < (2k+2)! \end{aligned}$$

Then the limit as $T \rightarrow \infty$ of the expression

$$\frac{1}{2T} \int_{-T}^{+T} [f(x) + a]^p dx = \frac{1}{T} \int_0^T [f(x) + a]^p dx \quad (15)$$

exists for all values of a satisfying equation (14), and does not exist for any other value of a .

For consider the integral

$$\int_{2l}^{2l+2} [f(x) + a]^p dx, \quad (16)$$

where l is a positive integer; $f(x)$ is equal either to $\phi(x)$ or to $\psi(x)$ throughout the interval $(2l, 2l+2)$. Since both the functions $\phi(x)$ and $\psi(x)$ have period 2 the integral (16) is equal to

$$\int_0^2 [\phi(x) + a]^p dx \quad \text{or else to} \quad \int_0^2 [\psi(x) + a]^p dx. \quad (17)$$

If now a is a root of (14), then the integrals (17) are equal, and if T is an even positive integer we have

$$\int_0^T [f(x) + a]^p dx = \frac{T}{2} \int_0^2 [\phi(x) + a]^p dx.$$

Since $f(x)$ is positive, we immediately conclude that

$$\lim \frac{1}{T} \int_0^T [f(x)+a]^p dx = \frac{1}{2} \int_0^2 [\phi(x)+a]^p dx.$$

Now let us assume that a is not a root of equation (14), so that the two integrals (17) have different values. Then, since $f(x)$ is bounded, we have

$$\lim_{k \rightarrow \infty} \left\{ \frac{1}{(2k+1)!} \int_0^{(2k+1)!} [f(x)+a]^p dx - \frac{1}{(2k+1)!} \int_{(2k)!}^{(2k+1)!} [f(x)+a]^p dx \right\} = 0.$$

Also

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{1}{(2k+1)!} \int_{(2k)!}^{(2k+1)!} [f(x)+a]^p dx &= \lim_{k \rightarrow \infty} \frac{(2k+1)! - (2k)!}{(2k+1)! \cdot 2} \int_0^2 [\phi(x)+a]^p dx \\ &= \frac{1}{2} \int_0^2 [\phi(x)+a]^p dx, \end{aligned}$$

and thus

$$\lim_{k \rightarrow \infty} \frac{1}{(2k+1)!} \int_0^{(2k+1)!} [f(x)+a]^p dx = \frac{1}{2} \int_0^2 [\phi(x)+a]^p dx. \quad (18)$$

In the same way

$$\lim_{k \rightarrow \infty} \frac{1}{(2k)!} \int_0^{(2k)!} [f(x)+a]^p dx = \frac{1}{2} \int_0^2 [\psi(x)+a]^p dx. \quad (19)$$

The equations (18) and (19) show that the integral (15) tends to two different values as T takes infinitely increasing values of the form $(2k+1)!$ or of the form $(2k)!$. Consequently the limit of (15) as $T \rightarrow \infty$ does not exist for any value of a which is not a root of (14).

Thus the limit
$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T [f(x)+a]^p dx$$

exists for infinitely many values of a in the interval $(0, 1)$, but not for all values in this interval.

We assumed that p is less than 2. But we can construct a function of the same kind for any non-integral p . If, for instance, $2 < p < 3$, then we need only add to (1a), (1b), (1c) the additional condition

$$\int_0^1 [\phi(x)]^{p-2} dx = \int_0^1 [\psi(x)]^{p-2} dx.$$

THE ASYMPTOTIC EXPANSION OF GENERALIZED HYPERGEOMETRIC FUNCTIONS

By C. Fox.

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The functions studied in this paper are given by

$$\begin{aligned} {}_p\mathcal{F}_q\{z\} &= {}_p\mathcal{F}_q\left\{\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}, \dots, \frac{\alpha_p}{\beta_p}; \frac{\rho_1}{\sigma_1}, \frac{\rho_2}{\sigma_2}, \dots, \frac{\rho_q}{\sigma_q}; z\right\} \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1+n\beta_1) \Gamma(\alpha_2+n\beta_2) \dots \Gamma(\alpha_p+n\beta_p)}{\Gamma(\rho_1+n\sigma_1) \Gamma(\rho_2+n\sigma_2) \dots \Gamma(\rho_q+n\sigma_q)} \frac{z^n}{n!}. \end{aligned}$$

The parameters β_r and σ_r are assumed to be positive real numbers, but α_r and ρ_r may be complex, and are only subject to the restriction that $\alpha_r+n\beta_r$ is not a negative integer for any positive integral value of n .

Watson* has studied the more general function given by

$$f(z) = \sum_{n=0}^{\infty} \frac{e^{gn} \phi(n)}{\Gamma(an+1)} z^n,$$

where $R(a) > 0$, $\phi(s/a)$ is analytic to the right of the line $R(s) = h$ and has the asymptotic expansion

$$\exp(-\beta \log s) \left[a_0 + \frac{a_1}{s} + \frac{a_2}{s^2} + \dots + \frac{a_n}{s^n} + O\left(\frac{1}{s^{n+1}}\right) \right]$$

whenever $|\arg(s/a)| < \pi$. For certain values of $\arg z$, $f(z)$ is of algebraic order when $|z|$ is large (in general it is of exponential order), and for such values of $\arg z$ Watson's method does not give the asymptotic expansion of $f(z)$ in detail.

In this paper we obtain explicit asymptotic expansions for ${}_p\mathcal{F}_q\{z\}$ for rational values of $\sum_{r=1}^q \sigma_r - \sum_{r=1}^p \beta_r$, covering all ranges of values of the complex variable z . The methods used differ from those of Watson

* "A class of integral functions defined by Taylor's series", *Trans. Camb. Phil. Soc.* 22 (1913), 15-27.

mainly in that no appeal is made to the properties of inverse factorial series*.

When the β 's and σ 's are unity, ${}_p\mathfrak{F}_q\{z\}$ reduces to a function studied by Barnes†. Again, when his function is of algebraic order for large values of $|z|$, Barnes's method does not always give the asymptotic expansion.

Throughout this paper we shall write

$$\mu = q - p + 1, \quad \mathfrak{S} = \sum_{r=1}^p \alpha_r - \sum_{r=1}^q \rho_r + \frac{1}{2}(\mu - 1),$$

$$\kappa = \sum_{r=1}^q \sigma_r - \sum_{r=1}^p \beta_r + 1, \quad k = \prod_{r=1}^p \beta_r^{\beta_r} / \prod_{r=1}^q \sigma_r^{\sigma_r},$$

$$g = (2\pi)^{\frac{1}{2}(\mu-1)} \prod_{r=1}^q \sigma_r^{\rho_r - \frac{1}{2}} / \prod_{r=1}^p \beta_r^{\alpha_r - \frac{1}{2}},$$

$$G(t) = \frac{g}{k^t} \frac{\Gamma(\alpha_1 + \beta_1 t) \Gamma(\alpha_2 + \beta_2 t) \dots \Gamma(\alpha_p + \beta_p t)}{\Gamma(\sigma_1 + \sigma_1 t) \Gamma(\sigma_2 + \sigma_2 t) \dots \Gamma(\sigma_q + \sigma_q t)},$$

$$T(s) = \sum_{n=0}^{\infty} \frac{\Gamma(n-s)}{\Gamma(n+1)} G(n).$$

$\mathfrak{F}_q\{z\}$ is an integral function of z if $\kappa > 0$, and it will appear that the form of its asymptotic expansions depends on the values of κ and $\arg z$. We first obtain the asymptotic expansion when $\kappa = 1$, and deduce the expansion when κ has any fixed rational value. The final result is given in Theorem 3, and the values of $\arg z$ for which the asymptotic expansion is of exponential or algebraic order are given at the end of the paper.

LEMMA 1. *If the moduli of the arguments of t and $\alpha + \beta t$ are less than π , then*

$$\Gamma(\alpha + \beta t) = (2\pi)^{\frac{1}{2}} (\beta t)^{\alpha + \beta t - \frac{1}{2}} e^{-\beta t} \psi(t),$$

where
$$\psi(t) = 1 + \frac{a_1}{t} + \frac{a_2}{t^2} + \dots + \frac{a_m}{t^m} + O(t^{-m-1}),$$

the a 's being polynomials in α .

This is a slight extension of a well known result.

* G. N. Watson, "The transformation of an asymptotic series into a convergent series of inverse factorials", *Rend. di Palermo*, 34 (1912), 41-88.

† "The asymptotic expansion of integral functions", *Proc. London Math. Soc.* (2), 5 (1907), 59-118.

LEMMA 2. *If the moduli of the arguments of $-t$, $-t+a$, $-t+s+1$ are all less than π , then*

$$\frac{\Gamma(-t+a)}{\Gamma(-t+s+1)} = (-t)^{a-s-1} \left\{ 1 + \frac{\lambda_1}{t} + \frac{\lambda_2}{t^2} + \dots + \frac{\lambda_m}{t^m} \right\} + O(t^{a-s-m-2}),$$

the λ 's being polynomials in s .

This follows without difficulty from Lemma 1.

LEMMA 3. *If $\kappa = 1$ and the moduli of the arguments of t , $a_r + \beta_r t$, $\rho_r + \sigma_r t$ and $t - \mathfrak{S} + 1$ are all less than π , then*

$$G(t) = \sum_{r=0}^m A_r \frac{\Gamma(t+1)}{\Gamma(t-\mathfrak{S}+r+1)} + O(t^{\mathfrak{S}-m-1}),$$

where the A_r 's are determinate constants and, in particular, $A_0 = 1$.

On applying Lemma 1, it becomes obvious that the coefficients A_r may be determined in succession so that the remainder is of the order indicated.

LEMMA 4. *With the conditions of Lemma 3,*

$$\frac{\Gamma(t-s)}{\Gamma(t+1)} G(t) = t^{\mathfrak{S}-s-1} \left\{ 1 + \frac{\mu_1}{t} + \dots + \frac{\mu_m}{t^m} \right\} + O(t^{\mathfrak{S}-s-m-2}),$$

where the μ 's are polynomials in s . The value of μ_r , when $s = \mathfrak{S} - r$, is A_r as defined by Lemma 3.

The first part follows from Lemma 1. In this insert the value of $G(t)$, given by Lemma 3, and then put $s = \mathfrak{S} - r$. If $r \leq m$, we have, for sufficiently large real values of t ,

$$\sum_{r=0}^m A_r \frac{\Gamma(t-\mathfrak{S}+r)}{\Gamma(t-\mathfrak{S}+p+1)} = t^{r-1} \left\{ 1 + \frac{\mu'_1}{t} + \dots + \frac{\mu'_m}{t^m} \right\} + O(t^{r-m-2}),$$

where μ'_m is the value of μ_m when $s = \mathfrak{S} - r$. Hence μ'_r is the coefficient of t^{-1} in the expansion of the left-hand side. Since the arguments of the Gamma functions differ by an integer, this coefficient is easily seen to be A_r .

LEMMA 5. *If $\kappa = 1$, $T(s)$ may be continued for all finite values of s , and is analytic except for simple poles: (i) at $s = m$ ($m = 0, 1, 2, \dots$), where the residue is*

$$\sum_{n=0}^m \frac{(-1)^{m-n+1} G(n)}{(m-n)! n!},$$

and (ii) at $s = \mathfrak{S} - m$ ($m = 0, 1, 2, \dots$), where the residue is A_m .

Let N be a positive integer greater than the greater of $R(s)$ and $R(-a_r/\beta_r)$, and let

$$T_N(s) = \sum_{n=N}^{\infty} \frac{\Gamma(n-s)}{\Gamma(n+1)} G(n).$$

Using Lemma 4, we find that

$$\begin{aligned} T_N(s) = & \xi_N(s-\vartheta+1) + \mu_1 \xi_N(s-\vartheta+2) + \dots \\ & + \mu_m \xi_N(s-\vartheta+m+1) + \sum_{n=N}^{\infty} O(n^{s-s-m-2}), \end{aligned}$$

where $\xi_N(z)$ denotes the sum of all the terms after the N -th in the series for the Zeta function. It accordingly differs from $\zeta(z)$ by an integral function of z , and, in particular, it is analytic for all values of z except $z=1$, where it has a simple pole with residue unity. The series is uniformly convergent provided that $R(s-\vartheta) > -m-1$, and m is an arbitrary positive integer. Hence $T_N(s)$ may be continued for all finite values of s , and its only poles are those specified under (ii) above.

It is obvious that the finite series $T(s) - T_N(s)$ is analytic for all values of s except at the poles specified under (i) above.

LEMMA 6. If $\kappa = 1$ and $R(s) > -a$, where a is any finite positive number, then

$$\sin \pi s T(s) = O(e^s s^{\lambda-\epsilon})$$

for some positive value of λ .

Let M be a positive integer greater than the greater of $R(\vartheta)$ and $R(-a_r/\beta_r)$ and, with the previous notation, let

$$T_M(s) = \sum_{n=M}^{\infty} \frac{\Gamma(n-s)}{\Gamma(n+1)} G(n).$$

Then, provided that $R(s-\vartheta) > 0$,

$$\sin \pi s T_M(s) = \frac{1}{2i} \int_{\gamma} \frac{\Gamma(-t) G(t)}{\Gamma(1-t+s)} dt,$$

where γ is the straight line $R(t) = M - \epsilon$; the integral round a great circular arc closing the contour to the right so as to avoid the poles of $\Gamma(-t)$ is easily proved to tend to zero by Lemma 4. If $R(s) \geq M$, we have

$$(1) \quad \left| \frac{\Gamma(s-M) \Gamma(M-t+1)}{\Gamma(1-t+s)} \right| = \left| \int_0^1 x^{s-M-1} (1-x)^{M-t} dx \right| < H,$$

where H is a constant.

Since $|\arg s| \leq \frac{1}{2}\pi$, we may combine this with Lemma 1, when we get

$$|\sin \pi s T_M(s)| < K \left| \frac{e^s}{s^{s-M-\frac{1}{2}}} \right| \left| \int_{\gamma} \left| \frac{\Gamma(-t) G(t)}{\Gamma(M-t+1)} dt \right| \right|,$$

where K is a constant. On inserting the asymptotic form of the integrand and noting that $|\arg t| \leq \frac{1}{2}\pi$ on γ , the integral is seen to be absolutely convergent.

If $R(s) < M$, we may apply Lemma 3, and then

$$\begin{aligned} \sin \pi s T_M(s) &= \frac{1}{2i} \sum_{r=0}^{\infty} A_r \int_{\gamma} \frac{\Gamma(-t) \Gamma(t+1)}{\Gamma(1-t+s) \Gamma(1+t-\mathfrak{S}+r)} dt \\ &\quad + \frac{1}{2i} \int_{\gamma} \frac{\Gamma(-t)}{\Gamma(1-t+s)} O(t^{s-m-1}) dt. \end{aligned}$$

By an obvious modification of inequality (1) it can be shown that the last integral is $O(e^s s^{-s+M+\frac{1}{2}})$ in s provided that $R(s-\mathfrak{S}) > -m-1$. Denoting the co-factor of A_r by I_r and changing the contour to γ' , the straight line $R(t) = -\epsilon$, making due allowance for the residues at the poles between the contours, we have

$$\begin{aligned} I_r &= -\pi \sum_{n=0}^{M-1} \frac{(-1)^n}{\Gamma(s-n+1) \Gamma(n-\mathfrak{S}+r+1)} \\ &\quad + \frac{(-1)^r}{2i} \int_{\gamma'} \frac{\sin \pi(t-\mathfrak{S})}{\sin \pi t} \frac{\Gamma(-t+\mathfrak{S}-r)}{\Gamma(-t+s+1)} dt. \end{aligned}$$

By Lemma 1 the series is $O(e^s s^{M-\frac{3}{2}-s})$. On γ' we have $|\arg(-t)| < \pi$, and by Lemma 2 the integral round a circular arc closing the contour to the left tends to zero as the radius of this arc tends to infinity. Expressing the Gamma functions in the form given in Lemma 2, and evaluating the residues at the poles to the left of γ' , the term involving the integral gives

$$-\sin \pi \mathfrak{S} \sum_{n=0}^m \lambda_n \xi(s-\mathfrak{S}+n+r+1) + \int_{\gamma'} \frac{\sin \pi(t-\mathfrak{S})}{\sin \pi t} O(t^{s-m-s-2}) dt.$$

The integral is $O(1)$ in s if $R(s-\mathfrak{S}) > -m-1$. If $R(s)$ is finite, $\xi(s) = O(s^N)$ for some finite N . Hence, if $R(s-\mathfrak{S}) > -m-1$ and $R(s) < M$,

$$\begin{aligned} \sin \pi s T_M(s) &= O(e^s s^{M-\frac{3}{2}-s}) + O(s^N) + O(e^s s^{M+\frac{1}{2}-s}) \\ &= O(e^s s^{\lambda-s}). \end{aligned}$$

Finally

$$\begin{aligned} \sin \pi s \{T(s) - T_M(s)\} &= -\pi \sum_{n=0}^{M-1} \frac{(-1)^n G(n)}{\Gamma(s-n+1) \Gamma(n+1)} \\ &= O(e^s s^{M-\frac{3}{2}-s}) \end{aligned}$$

by Lemma 1.

Combining all these results we have the lemma as stated.

THEOREM 1. If $\kappa = 1$ and $|\arg z| < \frac{1}{2}\pi$, then

$${}_v\mathcal{F}_q\left\{\frac{z}{k}\right\} = \frac{e^z z^3}{g} \left\{1 + \frac{A_1}{z} + \dots + \frac{A_N}{z^N} + o(z^{-N})\right\}.$$

Consider

$$\frac{1}{2\pi i} \int_C T(s) z^s ds,$$

where the closed contour C consists of the straight line $R(s - \zeta) = -N - \epsilon$, where N is a positive integer, together with an arc of a circle whose centre is the origin and whose radius is large and such that the distance of any pole of $T(s)$ from the arc is greater than δ ($\delta > 0$).

Writing $s = \rho e^{i\phi}$ and noting that $\operatorname{cosec} \pi s = O(e^{\mp \pi \rho \sin \phi})$ on C , the upper or lower sign being taken according as ϕ is positive or negative, Lemma 6 shows that, on those parts of C for which ρ is sufficiently large, the modulus of the above integral is less than

$$K\rho^{\lambda+1} \int \exp[\rho\{\cos \phi - \cos \phi \log(\rho|z|^{-1}) + \sin \phi(\phi - \arg z \mp \pi)\}] d\phi.$$

If $|\arg z| < \frac{1}{2}\pi$ the integral round that part of C for which $-\frac{1}{2}\pi \leq \phi \leq \frac{1}{2}\pi$ is easily seen to tend to zero as $\rho \rightarrow \infty$. On that part of C for which $\phi > \frac{1}{2}\pi$, we have $\rho \cos \phi \geq R(\zeta) - N - \epsilon$ and

$$\cos \phi + \phi \sin \phi \leq \frac{1}{2}\pi \sin \phi,$$

with similar relations on that part of C for which $\phi < -\frac{1}{2}\pi$. This is sufficient to show that the integral round those parts of C for which ρ is large tends to zero as $\rho \rightarrow \infty$. The integral along the straight part of C is therefore absolutely convergent; it is $O(z^3 - N - \epsilon)$ or $o(z^3 - N)$ and it is equal to the sum of the residues of $T(s)z^s$ at the poles to the right of the straight part of the contour. By Lemma 5 the sum of these residues is

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m-n+1} G(n)}{(m-n)! n!} z^m + \sum_{m=0}^N A_n z^{3-m}.$$

On inverting the order of summation in the double series and summing for m the theorem follows.

THEOREM 2. If $\kappa = 1$ and $(2\nu - \frac{3}{2})\pi < \arg z < (2\nu + \frac{1}{2})\pi$, where ν is any integer, then

$$\begin{aligned} {}_v\mathcal{F}_q\left\{\frac{z}{k}\right\} &= \frac{e^z z^3 e^{-2\nu\beta\pi i}}{g} \left\{1 + \frac{A_1}{z} + \frac{A_2}{z^2} + \dots + \frac{A_N}{z^N} + o(z^{-N})\right\} \\ &\quad + \sum_{r=1}^p \sum_{\lambda=0}^{\lambda_r} \frac{K_{r,\lambda}}{\{ze^{-(2\nu-1)\pi i}\}^{(\alpha_r+\lambda)/\beta_r}} + o(z^{3-N}), \end{aligned}$$

where N is any positive integer greater than $R(\mathfrak{S})$, λ_r is the greatest positive integer such that $R\{\mathfrak{S}+(a_r+\lambda_r)/\beta_r\} < N$ and the $K_{r,\lambda}$ are in general constants, but in special cases certain of them may be polynomials in $\log z$.

If $(2\nu-\frac{1}{2})\pi < \arg z < (2\nu+\frac{3}{2})\pi$, then the same result is true with the same values of the coefficients, provided that in the double series $ze^{-(2\nu-1)\pi i}$ is replaced by $ze^{-(2\nu+1)\pi i}$.

Consider the integral

(2) $\frac{1}{2\pi i} \int_C \{ \Gamma(-s) G(s) - e^{\mp i\mathfrak{S}\pi} \sum_{m=0}^N (-1)^m A_m \Gamma(-s+\mathfrak{S}-m) \} z^s e^{\pm i s \pi} ds,$

the upper or lower signs being taken together. The contour C is the same as in the proof of Theorem 1, except that the radius of the arc is such that the poles of the integrand of (2) avoid it by an amount greater than δ ($\delta > 0$).

Now apply Lemma 3 to the integrand of (2). The lemma can be applied to the whole of C , with the possible exception of a finite portion. We then see, after some reduction, that, except along this finite portion of C , the integral (2) is equal to

$$\frac{\sin \mathfrak{S} \pi}{2i} \sum_{m=0}^N A_m \int \frac{z^s ds}{\sin s \pi \sin (s-\mathfrak{S}) \pi \Gamma(s-\mathfrak{S}+m+1)} - \frac{1}{2i} \int \frac{e^{\pm s \pi i} z^s O(s^{\mathfrak{S}-N+1})}{\sin s \pi \Gamma(1+s)} ds.$$

Writing $s = \rho e^{i\phi}$ and using Lemma 1, the modulus of the last integral with the upper sign is less than

$$K \int \rho^{R(\mathfrak{S})-N-\frac{3}{2}} \exp [\rho \{ \cos \phi - \cos \phi \log (\rho/|z|) + \sin \phi (\phi - \pi \mp \pi - \arg z) \}] |ds|,$$

the upper or lower sign being taken according as ϕ is positive or negative.

On that part of C for which $\phi < -\frac{1}{2}\pi$ we have

$$-\rho \cos \phi \leq N + \epsilon - R(\mathfrak{S}) \quad \text{and} \quad \cos \phi + \phi \sin \phi \leq -\frac{1}{2}\pi \sin \phi.$$

The modulus of the integral is less than

$$K' \int \rho^{\epsilon-\frac{3}{2}} e^{-\rho \sin \phi (\frac{1}{2}\pi + \arg z)} |ds|.$$

This tends to zero as $\rho \rightarrow \infty$ if $\arg z \leq -\frac{1}{2}\pi$ and $\epsilon < \frac{1}{2}$.

Similar arguments show that the integrals along those parts of C for which $\rho \rightarrow \infty$ as the radius of the arc of C tends to infinity all tend to zero provided that $-\frac{3}{2}\pi < \arg z \leq -\frac{1}{2}\pi$ with the upper signs, or $\frac{1}{2}\pi \leq \arg z < \frac{3}{2}\pi$ with the lower signs. With these conditions the sum

of the residues of the integrand of (2) to the right of the straight part of C is

$$O(z^{3-N-\epsilon}) = o(z^{3-N}).$$

If $K_{r,\lambda}$ is the residue of $\Gamma(-s) G(\delta)/g$ at the pole $s = -(\alpha_r + \lambda)/\beta_r$, the residue of the integrand is in general $gK_{r,\lambda}(ze^{\pm\pi i})^{-(\alpha_r + \lambda)/\beta_r}$, but if poles of $\nu+1$ different Gamma functions in the numerator of $G(s)$ coincide at a point within C , then the corresponding $K_{r,\lambda}$ must be replaced by a polynomial of degree ν in $\log z$, of which the constant term is $K_{r,\lambda}$.

On evaluating the residues at the remaining poles of the integrand of (2), the theorem is proved for $\nu = 0$ for the respective ranges

$$-\frac{3}{2}\pi < \arg z \leq \frac{1}{2}\pi \quad \text{and} \quad \frac{1}{2}\pi \leq \arg z < \frac{3}{2}\pi.$$

These results may now be consolidated with Theorem 1, for if $|\arg z| < \frac{1}{2}\pi$ the terms in $K_{r,\lambda}$ are included in the error term of that theorem. This completes the proof of the theorem for the case $\nu = 0$. The more general form follows on noting that ${}_p\mathcal{F}_q\{z\}$ is unchanged by an increase of $2\nu\pi$ in the argument of z .

THEOREM 3. *If $\kappa = m/n$, where m and n are positive integers other than zero, then*

$$\begin{aligned} {}_p\mathcal{F}_q\left\{\frac{z}{k\kappa}\right\} &= \frac{z^{3/\kappa}}{g\kappa^{3+\frac{1}{2}}} \sum_{s=0}^{m-1} \left\{ B_0(s) + \frac{B_1(s)}{z^{1/\kappa}} + \dots + \frac{B_N(s)}{z^{N/\kappa}} + o(z^{-N/\kappa}) \right\} \\ &\quad \times \exp \left\{ z^{1/\kappa} e^{2(s-\nu)\pi i/m} + 2(s-\nu) \Im \pi i/m \right\} \\ &\quad + \sum_{r=1}^p \sum_{\lambda=0}^{\lambda_r, l} \frac{1}{z^{(\alpha_r + \lambda)/\beta_r}} \sum_{l=0}^{n-1} \frac{K_{r,\lambda,l}}{k^l \kappa^{l\kappa}} \left\{ \frac{m^m}{e^{-(2\nu \pm 1)\pi i}} \right\}^{(\alpha_r + l\beta_r + \lambda)/\beta_r} + o(z^{3-N/\kappa}), \end{aligned}$$

where the upper sign is taken if

$$\frac{2\nu\pi}{n} - \frac{1}{2}\kappa\pi < \arg z < \frac{2(\nu+1)\pi}{n} - \frac{1}{2}\kappa\pi,$$

and the lower sign if

$$\frac{(2\nu-1)\pi}{n} - \frac{1}{2}\kappa\pi < \arg z < \frac{(2\nu+1)\pi}{n} - \frac{1}{2}\kappa\pi,$$

ν being any integer. Also

$$\begin{aligned} B_0(s) &= 1 \text{ if } s-\nu \text{ is a multiple of } n, \text{ including the case } s = \nu, \\ &= 0 \text{ otherwise.} \end{aligned}$$

$$B_\lambda(s) = \frac{1}{n} e^{-2\lambda(s-\nu)\pi i/m} \sum_{l=0}^{n-1} A_\lambda(l) e^{-2(s-\nu)l\pi i/n},$$

where $A_\lambda(l)$ is given, as in Lemma 3, by the expansion

$$\frac{gn^{\frac{1}{2}-\mathfrak{S}+\kappa l}}{n^{-mt}k^{nt}} \cdot \frac{\Gamma(1+t) \prod_{r=1}^n \Gamma(\alpha_r + \beta_r l + n\beta_r t)}{\Gamma(1+l+nt) \prod_{r=1}^q \Gamma(\rho_r + \sigma_r l + n\sigma_r t)} \\ = \sum_{\lambda=0}^m A_\lambda(l) \frac{\Gamma(t+1)}{\Gamma(t-\mathfrak{S}+\kappa l+\lambda+1)} O(t^{\mathfrak{S}-m-1}).$$

If we denote the left-hand side of this equation by $H(t)$, then $K_{r,\lambda,l} z^{-(\alpha_r + l\beta_r + \lambda)/\beta_r}$ is the residue of $\Gamma(-t)H(t)z^{[n]}/gn^{\frac{1}{2}-\mathfrak{S}+\kappa l}$ at the pole $t = -(\alpha_r + l\beta_r + \lambda)/n\beta_r$. If this is not a simple pole, $K_{r,\lambda,l}$ will be a polynomial in $\log z$. λ_r is the greatest positive integer such that

$$R\{\mathfrak{S} - \kappa l + m(\alpha_r + \beta_r l + \lambda_r)/n\beta_r\} < N.$$

The other constants are as defined at the beginning of the paper.

We shall first deal with the case in which κ is a positive integer m . For this case we have identically

$$(3) \quad {}_p\mathcal{F}_q\left\{\frac{z}{k\kappa}\right\} = \frac{1}{m} \sum_{s=0}^{m-1} {}_{p+1}\mathcal{F}_{q+1}\left\{\frac{1}{1}, \frac{\alpha_1}{\beta_1/m}, \frac{\alpha_2}{\beta_2/m}, \dots, \frac{\alpha_p}{\beta_p/m}; \right. \\ \left. \frac{1}{1/m}, \frac{\rho_1}{\sigma_1/m}, \frac{\rho_2}{\sigma_2/m}, \dots, \frac{\rho_q}{\sigma_q/m}; \frac{(ze^{2s\pi i})^{1/m}}{mk^{1/m}}\right\}.$$

Denoting the various functions of the parameters of the hypergeometric functions on the right by $\mu', \mathfrak{S}', k', g', \kappa'$ and $G'(t)$ we have

$$\mu' = \mu, \quad \mathfrak{S}' = \mathfrak{S}, \quad k' = mk^{1/m}, \quad g' = gm^{\mathfrak{S}-\frac{1}{2}}$$

$$\text{and} \quad \frac{G'(t)}{g} = m^{-t} \frac{\Gamma(1+t)}{\Gamma(1+t/m)} \frac{G(t/m)}{g}.$$

Further $\kappa' = 1$, so that we may apply Theorem 2. If

$$(4) \quad -\frac{1}{2}\pi < \frac{\arg z}{m} < -\frac{1}{2}\pi + \frac{2\pi}{m},$$

we can apply Theorem 2 in its second form to every hypergeometric function on the right-hand side of (3). This gives us

$$(5) \quad {}_p\mathcal{F}_q\left\{\frac{z}{k\kappa}\right\} = \frac{1}{gm^{\mathfrak{S}+\frac{1}{2}}} \sum_{s=0}^{m-1} \left\{1 + \frac{A'_1}{(ze^{2s\pi i})^{1/m}} + \dots + \frac{A'_N}{(ze^{2s\pi i})^{N/m}} + O(z^{-N/m})\right\} \\ \times (ze^{2s\pi i})^{\mathfrak{S}/m} \exp\{z^{1/m} e^{2s\pi i/m}\} \\ + \frac{1}{m} \sum_{s=0}^{m-1} \sum_{r=1}^p \sum_{\lambda=0}^{\lambda_r} \frac{K'_{r,\lambda}}{\{(ze^{2s\pi i})^{1/m} e^{-\pi i}\}^{(\alpha_r + \lambda)m/\beta_r}} \\ + \frac{1}{m} \sum_{s=0}^{m-1} \sum_{\lambda=0}^{\lambda_0} \frac{K'_{0,\lambda}}{(ze^{2s\pi i})^{(\lambda+1)/m}} + O(z^{\mathfrak{S}-N/m}),$$

where the A 's are the constants occurring in the expression of $G'(t)$ in the form of Lemma 3; λ_r is the greatest positive integer such that

$$R\{\vartheta + m(a_r + \lambda_r)/\beta_r\} < N;$$

$K'_{r,\lambda} z^{-(a_r+\lambda)/\beta_r}$ is the residue of $\Gamma(-t) G'(t) z^t/g'$ at the pole

$$t = -m(a_r + \lambda)/\beta_r.$$

If this is not a simple pole $K'_{r,\lambda}$ is a polynomial in $\log z$. The residue of the same function at $t = -(\lambda + 1)$ is $K'_{0,\lambda} z^{-(\lambda+1)}$. The series in $K'_{0,\lambda}$ vanishes on summing for s unless $(\lambda + 1)/m$ is an integer, in which case $K'_{0,\lambda} = 0$. The residue of

$$\frac{m^{-t} \sin(\pi t/m) \Gamma(-t/m) G(t/m)}{g \sin \pi t}$$

at $t/m = -(a_r + \lambda)/\beta_r$, when this is a simple pole, is $K'_{r,\lambda}$. Hence, for this case,

$$K'_{r,\lambda} = m^{1+(a_r+\lambda)m/\beta_r} \frac{\sin\{\pi(a_r+\lambda)/\beta_r\}}{\sin\{m\pi(a_r+\lambda)/\beta_r\}} K_{r,\lambda}$$

and so, on summing for s , the algebraic terms become

$$(6) \quad \sum_{r=1}^p \sum_{\lambda=0}^{\lambda_r} K_{r,\lambda} \left\{ \frac{m^m}{ze^{-\pi i}} \right\}^{(a_r+\lambda)/\beta_r}.$$

If $-\frac{1}{2}\pi - (\pi/m) < (\arg z)/m < -\frac{1}{2}\pi + (\pi/m)$, we must apply Theorem 2 in its first form to the first hypergeometric function on the right of (3), and in its second form to the others. It will then be found that (5) is still true with the same values of the coefficients provided that in the algebraical terms as given by (6) we replace $ze^{-\pi i}$ by $ze^{\pi i}$.

On noting that ${}_p\mathcal{F}_q\{z/k\kappa^\kappa\}$ is unchanged by an increase of $2\nu\pi$ in the argument of z , all these results can be gathered together as follows.

If κ is a positive integer m other than zero, then

$$(7) \quad {}_p\mathcal{F}_q\left\{\frac{z}{k\kappa^\kappa}\right\} = \frac{1}{gm^{\theta+\frac{1}{2}}} \left\{ 1 + \frac{A'_1}{[ze^{2(s-\nu)\pi i}]^{1/m}} + \dots + \frac{A'_N}{[ze^{2(s-\nu)\pi i}]^{N/m}} + O(z^{-N/m}) \right\} \\ \times \{ze^{2(s-\nu)\pi i}\}^{3/m} \exp\{z^{1/m} e^{2(s-\nu)\pi i/m}\} \\ + \sum_{r=1}^p \sum_{\lambda=0}^{\lambda_r} K_{r,\lambda} \left\{ \frac{m^m}{ze^{-(2\nu\pm 1)\pi i}} \right\}^{(a_r+\lambda)/\beta_r} + O(z^{(3-N)/m}),$$

the upper sign being taken if

$$\frac{2\nu\pi}{m} - \frac{1}{2}\pi < \frac{\arg z}{m} < \frac{2\nu\pi}{m} - \frac{1}{2}\pi + \frac{2\pi}{m},$$

and the lower sign if

$$\frac{2\nu\pi}{m} - \frac{1}{2}\pi - \frac{\pi}{m} < \frac{\arg z}{m} < \frac{2\nu\pi}{m} - \frac{1}{2}\pi + \frac{\pi}{m},$$

ν being any integer.

We can now deal with the case $\kappa = m/n$, where m and n are positive integers. We have identically

$$\begin{aligned} (8) \quad {}_p\mathcal{F}_q \left\{ \frac{z}{k\kappa^\kappa} \right\} \\ = \sum_{l=0}^{n-1} \left(\frac{z}{k\kappa^\kappa} \right)^l {}_{p+1}\mathcal{F}_{q+1} \left\{ 1, \frac{a_1+l\beta_1}{n\beta_1}, \frac{a_2+l\beta_2}{n\beta_2}, \dots, \frac{a_p+l\beta_p}{n\beta_p}; \right. \\ \left. \frac{1+l}{n}, \frac{\rho_1+l\sigma_1}{n\sigma_1}, \frac{\rho_2+l\sigma_2}{n\sigma_2}, \dots, \frac{\rho_q+l\sigma_q}{n\sigma_q}; \frac{z^\nu}{k^n \kappa^m} \right\}. \end{aligned}$$

Denote the functions of the parameters of the l -th hypergeometric function on the right-hand side by μ_l , $\mathfrak{S}_l$, g_l , k_l , and κ_l . Then we have

$$\mu_l = \mu, \quad \mathfrak{S}_l = \mathfrak{S} - \kappa l, \quad g_l = g n^{\frac{1}{2}-\mathfrak{S}+\kappa l} k^{-l}, \quad k_l = n^{-m} k^n,$$

and finally $\kappa_l = m$. We may therefore apply (7) to the right-hand side of (8), and the result is found to be Theorem 3.

It is of interest to observe how the asymptotic expansion changes for certain values of κ from exponential to algebraic order as $\arg z$ varies from 0 to 2π .

If $\kappa = m/n$ where m and n are in their lowest terms and $m > 2$, it is evident that for at least one of the values $s = 0, 1, 2, \dots, m-1$,

$$(9) \quad \cos \left[\frac{1}{\kappa} \arg z + \frac{2(s-\nu)\pi}{m} \right]$$

must be positive. Hence in this case and for all values of $\arg z$, Theorem 3 shows that

$$(10) \quad \left| {}_p\mathcal{F}_q \left\{ \frac{z}{k\kappa^\kappa} \right\} \right| = O(|z^\lambda| \exp \{M|z^{1/\kappa}|\}),$$

where M is the maximum value of (9) and λ is of the form $(\mathfrak{S}-r)/\kappa$, [$B_r(s)$ being the first coefficient which does not vanish].

If, as before, m and n are in their lowest terms and $m = 2$, s takes only the two values 0 and 1, and so, along the lines on which $\arg z$ is equal to an odd multiple of $\frac{1}{2}\pi\kappa$,

$$\cos \left[\frac{1}{\kappa} \arg z + \frac{2(s-\nu)\pi}{m} \right] = 0$$

for both $s = 0$ and $s = 1$. Hence along these lines ${}_p\mathcal{F}_q\{z/k\kappa^\kappa\}$ is of algebraic order, and in the case where none of the coefficients on the right-hand side of Theorem 3 vanishes, this order is z^λ , where λ is the greatest of $\mathfrak{S}/\kappa$ and $-\alpha_r/\beta_r$ ($r = 1, 2, \dots, p$).

If $\arg z$ is such that z lies between these lines, it is evident that for one of the values $s = 0$ or 1 ,

$$\cos \left[\frac{1}{\kappa} \arg z + \frac{2(s-\nu)\pi}{m} \right]$$

is positive, and so, for such values of $\arg z$, the order of ${}_p\mathcal{F}_q\{z/k\kappa^\kappa\}$ is given by (10).

If $m = 1$, then s in Theorem 3 has only one value, namely $s = 0$, and we have to consider the sign of

$$\cos \left(\frac{1}{\kappa} \arg z \right).$$

Along the lines on which $\arg z$ is equal to an odd multiple of $\frac{1}{2}\pi\kappa$, ${}_p\mathcal{F}_q\{z/k\kappa^\kappa\}$ is of algebraic order just as in the last case. When z lies between these lines

$$\cos \left(\frac{1}{\kappa} \arg z \right)$$

is positive and negative in alternate sectors. In those sectors for which it is positive the order of ${}_p\mathcal{F}_q\{z/k\kappa^\kappa\}$ is given by (10). In the remaining sectors it is evident that the exponential terms are

$$O(|z^\lambda| \exp \{-M|z^{1/\kappa}|\}),$$

where M is positive, and so

$${}_p\mathcal{F}_q\left\{\frac{z}{k\kappa^\kappa}\right\} = O(z^\lambda),$$

where λ is the greatest of $-\alpha_r/\beta_r$ ($r = 1, 2, \dots, p$).

In conclusion, I wish to thank Prof. G. B. Jeffery for valuable help during the course of the paper.

NOTES ON MODULI AND MEAN VALUES

By G. H. HARDY, A. E. INGHAM, and G. PÓLYA.

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1. The two notes which follow have very little connexion with one another, but, since both have their origin in a memoir written by the three authors in collaboration†, they are most conveniently presented together.

A. Limiting values of the modulus of an analytic function.

2. Let $f(z) = f(x+iy)$ be an analytic function regular and bounded in the strip $\alpha < x < \beta$, $y > y_0$. We denote this strip by $(\alpha, \beta; y_0)$ or by S , and any interior strip $(\alpha+\delta, \beta-\delta; y_0)$ by S' .

If $f(z)$ tends to a limit (when $y \rightarrow \infty$) on one interior line $x = a$ of S , then, by a well known theorem of Montel, it tends to the same limit on every interior line, and uniformly in any S' .

There is no corresponding theorem for $|f(z)|$ except when the limit is zero, a case which we exclude as being already settled by Montel's theorem. For example, the function $e^{\sinh z}$ is regular and bounded in any S , and its modulus $e^{\sinh x \cosh y}$ tends to a limit on one line only, namely $x = 0$.

3. In certain circumstances, however, though (as will appear in the sequel) not in general, the existence of $\lim |f|$ on every interior line of S can be inferred from its existence on *two* such lines $x = a$ and $x = b$. This is certainly so if f satisfies inequalities of the type

$$(1) \quad 0 < K_1 < |f| < K_2$$

throughout S . To prove this we have only to apply to the function $\log |f|$, which is harmonic and bounded in S , a known theorem of Carle-

† Hardy, Ingham, and Pólya, 5.

man and Hardy†. The limit exists uniformly in any S' , and its value on the line x is

$$X = A^{(b-x)/(b-a)} B^{(x-a)/(b-a)},$$

A and B being its values on a and b .

A less obvious remark is that condition (1) may be replaced by the wider condition

$$(2) \quad 0 < |f| < K.$$

THEOREM 1. *Suppose that f is regular and bounded and has no zeros in S , and that $|f|$ tends to positive limits A and B on two interior lines $x=a$ and $x=b$ of S . Then $|f|$ tends to the limit X on every interior line, and uniformly in any S' .*

Suppose that $a < b$, and choose δ so that

$$a < a + \delta \leq a < b \leq b - \delta < b.$$

Since $\log |f(a+iy)|$ is continuous for $y > y_0$ and tends to a finite limit $\log A$ when $y \rightarrow \infty$, we have

$$|\log |f|| < H \quad (x = a, y \geq y_0 + \delta),$$

H being a constant. Further, $\log |f|$ is bounded above, say

$$\log |f| < k,$$

in the whole of S . Applying Carathéodory's theorem‡ to pairs of concentric circles of radii δ and $\frac{1}{2}\delta$ with their centres on $x=a$, we deduce

$$|\log |f|| < H_1 \quad (|x-a| \leq \frac{1}{2}\delta, y \geq y_0 + \delta).$$

H_1 depending only on H and k . We can now repeat the argument, starting with one of the lines

$$x = \text{Max}(a - \frac{1}{2}\delta, a + \delta), \quad x = \text{Min}(a + \frac{1}{2}\delta, b - \delta).$$

† Carleman and Hardy, 2, 132; Hardy, 4.

‡ If $g(z)$ is regular for $|z-\zeta| = r < R$, $\Re g(\zeta) = \gamma$, and $\Re g(z) \leq A$ for $|z-\zeta| < R$, then

$$|\Re g(z)| \leq |\gamma| \frac{R+\rho}{R-\rho} + 2A \frac{\rho}{R-\rho}$$

for $r \leq \rho < R$. See for example Landau, 6, 300. Here

$$g = \log f, \quad \Re(g) = \log |f|, \quad \zeta = a + iy, \quad R = \delta, \quad \rho = \frac{1}{2}\delta,$$

and the inequality gives

$$|\log |f|| \leq 3|\gamma| + 2A \leq 3H + 2k = H_1.$$

We conclude, after a finite number of steps, that $|\log|f||$ is bounded in the strip $(a+\frac{1}{2}\delta, \beta-\frac{1}{2}\delta; y_0+\delta)$, to which we now apply the theorem of Carleman and Hardy.

4. There is a more interesting theorem in which no explicit restriction (beyond the general conditions laid down at the beginning) is imposed on f , but the lines a and b are subject to a certain limitation.

THEOREM 2. *Suppose that f is regular and bounded in S , and that $|f|$ tends to positive limits A and B on two lines $x = a$ and $x = b$, where $a < a < b < \beta$ and*

$$(3) \quad b-a < \frac{1}{2}(\beta-a).$$

Then $|f|$ tends to X on every interior line of S , and uniformly in any S' .

We may suppose $A = B = 1$, the general case reducing to this by consideration of the function

$$A^{-(b-z)/(b-a)} B^{-(z-a)/(b-a)} f(z).$$

We suppose first, in addition, that

$$(4) \quad a-a \leq b-a, \quad \beta-b \leq b-a.$$

The function

$$(5) \quad F(z) = F_a(z) = f(z)\bar{f}(2a-z),$$

where $\bar{f}$ is the function conjugate to f , is regular and bounded in $(a, 2a-a; y_0)$, since $a-a < \beta-a$. Further

$$F(a+iy) = |f(a+iy)|^2 \rightarrow 1$$

when $y \rightarrow \infty$. It follows, from the theorem of Montel already referred to, that $F(z) \rightarrow 1$ uniformly in any strip interior to $(a, 2a-a; y_0)$, so that, if δ is any (sufficiently small) positive number, there is a $y_1 = y_1(\delta)$ such that $|F(z)| > \frac{1}{2}$ in $S_a = (a+\delta, 2a-a-\delta; y_1)$. Since $|f| < K$ throughout S , we conclude from (5) that

$$(6) \quad |f(z)| > \frac{1}{2K}$$

in S_a .

By a similar argument we obtain the same inequality in

$$S_b = (2b-\beta+\delta, \beta-\delta; y_2).$$

Now, if δ is small enough, S_a and S_b overlap, in virtue of (3), so that (6) holds in a strip $(a+\delta, \beta-\delta; y_3)$. To this strip we may therefore apply the Carleman-Hardy theorem†, and Theorem 2 is proved under the hypothesis (4).

This restriction is easily removed. We first choose (as is clearly possible) a strip $S_0 = (a_0, \beta_0; y_0)$, contained in S and containing the lines a and b , to which the restricted theorem is applicable. Then, starting with a suitable pair of lines of S_0 , we can apply the restricted theorem to the strip $S_1 = (\alpha_1, \beta_1; y_0)$, where

$$\alpha_1 = \text{Max}(\alpha, a_0 - d_0), \quad \beta_1 = \text{Min}(\beta, \beta_0 + d_0), \quad d_0 = \frac{1}{2}(\beta_0 - a_0).$$

Continuing in this way, we shall eventually reach an S_n which coincides with S . The conclusion of Theorem 2 holds in each S_n and therefore in S .

5. An immediate consequence of Theorem 2 is that the existence of $\lim |f|$ on *three* interior lines of S implies its existence on all. There is thus a marked contrast between this problem and the corresponding problem for mean values

$$\lim_{Y \rightarrow \infty} \frac{1}{2Y} \int_{-Y}^Y |f(x+iy)|^2 dy.$$

It is possible (for an f regular and bounded in $a < x < \beta$) for the mean value to exist on any given finite number of interior lines of the strip (a, β) , or indeed on an infinite set of such lines having no limit line *inside* the strip, but on no other line‡.

6. Theorem 2 is a best possible theorem in the sense that the factor $\frac{1}{2}$ which occurs in (3) cannot be replaced by any larger number. For con-

† Alternatively we may argue as follows, using a classical theorem of Phragmén and Lindelöf. The inequality

$$\overline{\lim} |f| \leq 1$$

holds on the lines $x = a$ and $x = b$, and therefore, since f is bounded, uniformly for $a \leq x \leq b$. By (6) the same argument may be applied to $1/f$, and we conclude that $|f| \rightarrow 1$ uniformly in $a \leq x \leq b$.

Next suppose $a + \delta \leq x \leq a$. We have seen that $F_n(z) \rightarrow 1$ uniformly over this range. Hence

$$|f(x+iy)| |f(2a-x+iy)| = |F_n(x+iy)| \rightarrow 1$$

uniformly. Now $a \leq 2a-x < 2a-a \leq b$, by (4), so that the second factor on the left tends uniformly to 1, by what we have just proved. The same must therefore be true of the first factor. Similarly we can show that $|f| \rightarrow 1$ uniformly in $b \leq x \leq \beta - \delta$.

‡ See Hardy, Ingham, and Pólya, **5**, 565-569; Besicovitch, **1**.

sider the function†
$$f(z) = \prod_1^{\infty} \left(\frac{\rho_n - e^{-i(z-\omega)}}{\rho_n + e^{-i(z+\omega)}} \right),$$

where $-\frac{1}{2}\pi < \omega < \frac{1}{2}\pi, \quad \rho_1 > 0, \quad \rho_{n+1}/\rho_n \geqslant c > 1.$

Then $f(z)$ is regular and bounded in the strip

(7)
$$-\pi - \omega + \delta < x < \pi - \omega - \delta,$$

where we suppose $0 < \delta < \frac{1}{2}\pi - |\omega|$. On the lines $x = \pm \frac{1}{2}\pi$, interior to the strip, $|f(z)| = 1$, so that $\lim |f|$ certainly exists. On the other hand we have, writing $\eta_n = \log \rho_n$ and $\eta'_n = \frac{1}{2}(\eta_n + \eta_{n+1})$,

$$|f(\omega + i\eta_n)| = 0, \quad |f(\omega + i\eta'_n)| \geqslant C(c) > 0,$$

so that the limit does not exist on the line $x = \omega$, nor therefore (by Theorem 2) on *any* interior line of (7) other than $x = \pm \frac{1}{2}\pi$. Finally, in the notation of Theorem 2,

$$\frac{b-a}{\beta-a} = \frac{\pi}{2(\pi-\delta)},$$

and this ratio can, by choice of δ , be made to exceed $\frac{1}{2}$ by as little as we please. The situation of the lines a and b in relation to the strip (α, β) can be varied by varying ω .

We have used here substantially the same function as in our previous paper. In this case we can avoid calculations by using the example

$$f(z) = \mathfrak{Z}(z-a)/\mathfrak{Z}(z),$$

where $\mathfrak{Z}(z)$ is (in the notation of Tannery and Molk) $\mathfrak{Z}_1(z|\tau)$, and $\tau = i$, $0 < a < 1$. It is easily verified that $f(z)$ is regular in $0 < x < 1$ and bounded in any $\delta < x < 1-\delta$, that $|f| = 1$ on the lines $x = \frac{1}{2}a$ and $x = \frac{1}{2} + \frac{1}{2}a$, and that $|f|$ does not tend to a limit on $x = a$.

B. Mean values of subharmonic functions.

7. Suppose that $\psi(z)$ is a function‡ of the complex variable z , continuous and real in a closed domain D . Following F. Riesz§, we say that $\psi(z)$ is *subharmonic* in D if to every z interior to D corresponds a

† Cf. Hardy, Ingham, and Pólya, 5, 565-569, where the calculations are carried out in detail. There we confine ourselves to the case $\omega = 0$, but the work is essentially the same in the general case.

‡ Naturally not analytic.

§ F. Riesz, 9. See also Littlewood, 7.

$$\rho = \rho(z) \text{ such that } \psi(z) \leq \frac{1}{2\pi} \int_0^{2\pi} \psi(z + re^{i\theta}) d\theta$$

for $0 < r < \rho$. It results from the definition that (i) a finite sum of subharmonic functions is subharmonic, (ii) a positive constant multiple of a subharmonic function is subharmonic, and (iii) $\psi(az+b)$, where $a \neq 0$, is subharmonic in the region $D'(z)$ transformed into $D(z')$ by $z' = az+b$. It is also easily proved that (iv) $\psi(z)$ assumes its maximum in D on the boundary of D .

Riesz has proved the following theorem, the analogue for subharmonic functions of one proved by Hardy† for analytic functions.

THEOREM 3. *If $\psi(z)$ is subharmonic for $|z| \leq R$, then the mean value*

$$\mu(r) = \frac{1}{2\pi} \int_0^{2\pi} \psi(re^{i\theta}) d\theta$$

is an increasing and convex function of $\log r$ for $r \leq R$.

This theorem may be proved very simply as follows‡. If $\psi(z)$ is subharmonic in the annulus A defined by $r_1 \leq z \leq r_2$, then, by (i)–(iii),

$$\mu_n(z) = \frac{1}{n} \{ \psi(z) + \psi(z\omega) + \dots + \psi(z\omega^{n-1}) \},$$

where $\omega = e^{2\pi i/n}$, is subharmonic in A , and therefore, by (iv), assumes its maximum on one of the boundary circles of A . If this property is possessed by each of a uniformly convergent sequence of functions, it is possessed by their limit-function; and therefore

$$\mu(r) = \mu(|z|) = \frac{1}{2\pi} \int_0^{2\pi} \psi(|z|e^{i\theta}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \psi(ze^{i\theta}) d\theta = \lim_{n \rightarrow \infty} \mu_n(z)$$

attains its maximum for $r = r_1$ or $r = r_2$.

Applying this result to $\psi(z) + c \log |z|$, where c is an appropriate constant, we obtain the second part of the theorem. The first part now

† Hardy, 3; F. Riesz, 9. The first part of Theorem 3 is a direct generalization of the corresponding part of Hardy's theorem, which is the special case $\psi(z) = |f(z)|^p$. But in the second part, relating to convexity, Hardy's theorem asserts, for the special class of subharmonic functions $|f(z)|^p$, more than Theorem 3, namely that $\log \mu(r)$ is a convex function of $\log r$. To obtain Hardy's result we must extend Theorem 3 (in so far as it asserts convexity) to an annulus, and apply it to $|z|^\beta |f(z)|^p$, with an appropriate β .

‡ Cf. the proof of Hardy's theorem given by Pólya and Szegő, 8, 144, 329–330.

follows from the fact that, since $\psi(z)$ is subharmonic at $z = 0$, we have

$$\mu(0) = \psi(0) \leq \mu(r)$$

for all sufficiently small r .

8. In our former paper† we proved a theorem analogous to the theorem of Hardy just referred to, but involving integrals along parallel lines. This theorem also has an analogue for subharmonic functions.

THEOREM 4. *If $\psi(z) = \psi(x+iy)$ is subharmonic for $a \leq x \leq \beta$, and*

$$0 \leq \psi(z) \leq e^{k|y|} \quad \left(0 < k < \frac{\pi}{\beta-a}\right),$$

and

$$J(x) = \int_{-\infty}^{\infty} \psi(x+iy) dy$$

is convergent for $x = a$ and $x = \beta$, then $J(x)$ is convergent for $a \leq x \leq \beta$, and is a convex function of x .

We may suppose, without real loss of generality, that $a = -\frac{1}{2}\pi + \delta$, $\beta = \frac{1}{2}\pi - \delta$, where $0 < \delta < \frac{1}{2}\pi$, and (by an appropriate choice of δ) that $k < 1$. If now l and ϵ are positive and c real, the function

$$\psi^*(z) = \psi(z) - \frac{c}{2l} \Re(z) - \epsilon \Re(\cos z) = \psi(x+iy) - \frac{cx}{2l} - \epsilon \cos x \cosh y$$

is subharmonic, and tends uniformly to $-\infty$ when $|y| \rightarrow \infty$. Let

$$\Psi(z) = \int_{-l}^l \psi^*(z+it) dt.$$

Then plainly $\Psi(z) \rightarrow -\infty$ when $|y| \rightarrow \infty$. Also, by Theorem 3,

$$\psi^*(z+it) \leq \frac{1}{2\pi} \int_0^{2\pi} \psi^*(z+it+re^{i\theta}) d\theta$$

if $a < x-r < x+r < \beta$, and therefore

$$\Psi(z) \leq \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_{-l}^l \psi^*(z+it+re^{i\theta}) dt = \frac{1}{2\pi} \int_0^{2\pi} \Psi(z+re^{i\theta}) d\theta,$$

so that $\Psi(z)$ is subharmonic.

† Hardy, Ingham, and Pólya, 5, 554.

It follows that $\Psi(z)$ attains its maximum, for the strip (α, β) , at a finite point of the boundary, say at z_0 . Hence

$$\begin{aligned} \int_{-l}^l \psi(x+it) dt - cx - \epsilon \cos x \int_{-l}^l \cosh t dt &= \int_{-l}^l \psi^*(x+it) dt \\ &= \Psi(x) \leq \Psi(z_0) = \int_{-l}^l \psi^*(z_0+it) dt \\ &\leq \int_{-l}^l \psi(z_0+it) dt - cx_0 \leq \text{Max} (J(\alpha) - c\alpha, J(\beta) - c\beta). \end{aligned}$$

Choosing c so that $J(\alpha) - c\alpha = J(\beta) - c\beta$, and then making $\epsilon \rightarrow 0$ and $l \rightarrow \infty$, in that order, we obtain

$$J(x) \leq \frac{\beta-x}{\beta-\alpha} J(\alpha) + \frac{x-\alpha}{\beta-\alpha} J(\beta),$$

the result of the theorem.

9. We can define subharmonic functions $\psi(x, y, z, \dots)$ in space of any number of dimensions in an analogous manner. In one dimension the idea reduces to that of convexity. There is (for example) a three-dimensional theorem concerning

$$J(x, y) = \int_{-\infty}^{\infty} \psi(x, y, z) dz$$

similar to Theorem 4. Here ψ is supposed to be a positive subharmonic function of x, y, z not exceeding $e^{k|z|}$ for an appropriate k , and the integral is supposed to converge along every generator of a cylinder; and the conclusion is that the integral converges inside the cylinder and is a subharmonic function of x, y .

[*Added December 1927.* We observe while correcting the proofs that (contrary to our original expectation) *Theorem 2 is still true in the limiting case*

$$b-a = \frac{1}{2}(\beta-\alpha).$$

Suppose that

$$0 < \delta < l = \text{Min}(\alpha-a, \beta-b), \quad \gamma = 2a-a = 2b-\beta,$$

and let S'_a, S'_b be the closed strips

$$(\alpha+\delta, \gamma-\delta; y_3), \quad (\gamma+\delta, \beta-\delta; y_3).$$

The argument of § 4 shows that there is a $y_3 = y_3(\delta)$ such that (6) holds in S'_a and S'_b . We may suppose that $\delta < l/4K^2$ and $y_3 > l$.

On the other hand, the formula

$$f'(z) = \frac{1}{2\pi i} \int \frac{f(u)}{(u-z)^2} du,$$

applied to the circle $|u-z| = l$, shows that

$$|f'(z)| \leq K/l$$

in $(a, b; y_0+l)$, and in particular in the strip $S_\gamma = (\gamma-\delta, \gamma+\delta; y_3)$ between S'_a and S'_b . Since any point of S_γ is within a distance δ of a point of S' or S'_b , we have

$$|f(z)| > \frac{1}{2K} - \frac{K}{l} \delta > \frac{1}{4K}$$

at all points of S_γ , and so throughout $(a+\delta, \beta-\delta; y_3)$. The proof may then be completed as before.]

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THE BEHAVIOUR OF CERTAIN SERIES ASSOCIATED WITH LIMITING CASES OF ELLIPTIC THETA-FUNCTIONS

By R. COOPER.

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1. The series

$$2 \sum_1^{\infty} e^{\pi i (n-\frac{1}{2})^2 \tau}, \quad 1 + 2 \sum_1^{\infty} e^{\pi i n^2 \tau}, \quad 1 + 2 \sum_1^{\infty} (-1)^n e^{\pi i n^2 \tau}$$

are convergent when $\Im(\tau)$, the imaginary part of τ , is greater than zero. They represent the functions $\mathfrak{S}_2(0, \tau)$, $\mathfrak{S}_3(0, \tau)$, $\mathfrak{S}_4(0, \tau)$. The behaviour of these series when τ is real, and also when $\Im(\tau) \rightarrow 0$, is known*. We discuss here the properties of associated series of the types

$$\sum_1^{\infty} e^{\pi i n^2 \theta} n^{-s} \quad \text{and} \quad \sum_1^{\infty} e^{\pi i n^2 \theta} z^n,$$

where θ is real. We show that the functions defined by these series have properties similar to those which have been found† for the series

$$\sum_1^{\infty} (n\theta) n^{-s} \quad \text{and} \quad \sum_1^{\infty} (n\theta) z^n,$$

where $(n\theta)$ denotes the fractional part of $n\theta$.

* G. H. Hardy and J. E. Littlewood, "Some problems of Diophantine approximation, II", *Acta Math.*, 37 (1914), 193-239; this paper will be referred to later as HL.

† E. Hecke, "Über analytische Funktionen und die Verteilung von Zahlen mod. eins", *Hamburg Math. Abh.*, 1 (1921), 54-76; G. H. Hardy and J. E. Littlewood, "Some problems of Diophantine approximation: the analytic character of the sum of a Dirichlet's series considered by Hecke", *Hamburg Math. Abh.*, 3 (1923), 57-68; G. H. Hardy and J. E. Littlewood, "Some problems of Diophantine approximation: the analytic properties of certain Dirichlet's series associated with the distribution of numbers to modulus unity", *Trans. Camb. Phil. Soc.*, 22 (1923), 519-534.

2. Let θ satisfy $0 < \theta < 1$ and, as usual, let $s = \sigma + it$. We consider the set of six functions defined (at least for $\sigma > 1$) by the series

$$\begin{aligned} F_1(s, \theta) &= \sum_1^{\infty} e^{\pi i n^2 \theta} n^{-s}; & F_2(s, \theta) &= \sum_1^{\infty} (-1)^n e^{\pi i n^2 \theta} n^{-s}; \\ F_3(s, \theta) &= \sum_1^{\infty} e^{-\pi i n^2 \theta} n^{-s}; & F_4(s, \theta) &= \sum_1^{\infty} (-1)^n e^{-\pi i n^2 \theta} n^{-s}; \\ F_5(s, \theta) &= \sum_0^{\infty} e^{\pi i (n+\frac{1}{2})^2 \theta} (n+\frac{1}{2})^{-s}; & F_6(s, \theta) &= \sum_0^{\infty} e^{-\pi i (n+\frac{1}{2})^2 \theta} (n+\frac{1}{2})^{-s}. \end{aligned}$$

We first establish transformation formulae connecting $F_a(s, \theta)$ with $F_{\beta}(s, \theta_1)$, where θ_1 is the fractional part of $1/\theta$.

We shall use $L(a)$ and $L'(a)$ to denote straight lines in a complex plane making angles $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$ respectively with the positive real axis and passing through the point a . The positive direction (of integration) along $L(a)$, $L'(a)$ is that in which the imaginary part of the affix of the point increases.

$F_a(s, \theta, n)$ will denote the sum of the first n terms of the series for $F_a(s, \theta)$. $I(s)$ or $I(s, \theta)$ will denote an integral function of s , and $R(s, x)$ a function of s regular in the half-plane $\sigma > x$; these will not always be the same functions, but may be different each time they occur.

2. 1. The transformation formula for $F_1(s, \theta)$.

Take for contour of integration in a z -plane the parallelogram having for vertices the points

$$\begin{aligned} A &= n + \frac{1}{2} - (1+i)R, & B &= n + \frac{1}{2} + (1+i)S, \\ C &= \frac{1}{2} + (1+i)S, & D &= \frac{1}{2} - (1+i)R, \end{aligned}$$

where R, S are real and positive, and are eventually to tend to infinity. For integrand take the function

$$\frac{e^{\pi i z^2 \theta} z^{-s}}{e^{2\pi i z} - 1},$$

where $z^{-s} = \exp(-s \log z)$ and $\log z$ has the determination which is real for $z = \frac{1}{2}$ and varies continuously on the contour $ABCD$. Then

$$F_1(s, \theta, n) = \int_{ABCD.A} \frac{e^{\pi i z^2 \theta} z^{-s} dz}{e^{2\pi i z} - 1}.$$

The integrand tends to zero uniformly on BC, DA as $R, S \rightarrow \infty$ respectively, and this holds uniformly in any bounded set of values of s . This

also shows that the absolute convergence at both ends of the integrals along $L(n+\frac{1}{2})$ and $L(\frac{1}{2})$ is uniform in any bounded region of s . Hence

$$F_1(s, \theta, n) = \int_{L(n+\frac{1}{2})} - \int_{L(\frac{1}{2})} \frac{e^{\pi i z^2 \theta} z^{-s}}{e^{2\pi i z} - 1} dz.$$

The integral along $L(\frac{1}{2})$ is independent of n and represents an integral function of s . In the other integral make the substitution

$$\frac{1}{e^{2\pi i z} - 1} = \sum_{\nu=1}^M e^{-2\nu\pi i z} + \frac{e^{-2M\pi i z}}{e^{2\pi i z} - 1} \quad \text{where } M = [(n+\frac{1}{2})\theta]^*.$$

$$\begin{aligned} \text{Then } F_1(s, \theta, n) &= I(s, \theta) + \sum_{\nu=1}^M \int_{L(n+\frac{1}{2})} z^{-s} \exp \{ \pi i z^2 \theta - 2\nu\pi i z \} dz \\ &\quad + \int_{L(n+\frac{1}{2})} \frac{z^{-s} \exp \{ \pi i z^2 \theta - 2M\pi i z \}}{e^{2\pi i z} - 1} dz \\ (2.11) \quad &= I(s, \theta) + \sum_{\nu=1}^M f_\nu(s) + R_M(s), \quad \text{say.} \end{aligned}$$

Since each function $z^{-s} \exp \{ \pi i z^2 \theta - 2\nu\pi i z \}$ tends to zero as $z \rightarrow \infty$ along $L(a)$ uniformly in any finite interval of real values of a , the functions $f_\nu(s)$ are independent of n . We have

$$f_\nu(s) = \int_{L(a)} z^{-s} \exp \{ \pi i z^2 \theta - 2\nu\pi i z \} dz,$$

where a is any real number greater than zero.

In $R_M(s)$ make the substitution $z = n + \frac{1}{2} + \zeta$ and write $\mu = (n + \frac{1}{2})\theta - M$ ($0 \leq \mu < 1$). Then

$$\begin{aligned} |R_M(s)| &= \left| \int_{\zeta \text{ on } L(0)} \frac{(n + \frac{1}{2} + \zeta)^{-s} \exp [\pi i \{ \theta \zeta^2 + 2(n + \frac{1}{2})\theta \zeta - 2M\zeta \}]}{e^{2\pi i \zeta} + 1} d\zeta \right| \\ &= \left| \int \frac{(n + \frac{1}{2} + \zeta)^{-s} \exp \{ \pi i (\theta \zeta^2 + 2\mu \zeta) \}}{e^{2\pi i \zeta} + 1} d\zeta \right|. \end{aligned}$$

Now, on the path of integration, if $\sigma > 0$,

$$|(n + \frac{1}{2} + \zeta)^{-s}| \leq \left(\frac{n + \frac{1}{2}}{\sqrt{2}} \right)^{-\sigma} e^{\frac{3}{4}\pi |\zeta|},$$

and

$$\frac{e^{2\pi i \mu \zeta}}{e^{2\pi i \zeta} + 1}$$

* $[x]$ denotes the greatest integer which does not exceed x .

is bounded. Therefore, if $\zeta = (1+i)\xi$ (where ξ is real on the path of integration),

$$(2.12) \quad |R_M(s)| \leq C \left(\frac{n+\frac{1}{2}}{\sqrt{2}} \right)^{-\sigma} e^{\frac{3}{2}\pi|t|} 2^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-2\pi\theta\xi^2} d\xi \rightarrow 0,$$

as $n \rightarrow \infty$, uniformly in any closed bounded region of s in the half-plane $\sigma > 0$, C being a constant.

Consider next $f_\nu(s)$. We have

$$\begin{aligned} f_\nu(s) &= e^{-\pi i \nu^2 / \theta} \int_{L(\nu/\theta)} \exp \left\{ \pi i \theta \left(z - \frac{\nu}{\theta} \right)^2 \right\} \cdot z^{-s} dz \\ &= e^{-\pi i \nu^2 / \theta} \int_{L(0)} e^{\pi i z^2 \theta} \left(z + \frac{\nu}{\theta} \right)^{-s} dz \\ (2.13) \quad &= \sqrt{i} \cdot e^{-\pi i \nu^2 / \theta} \int_{-\infty}^{\infty} e^{-\pi \theta y^2} \left(\frac{\nu}{\theta} + \sqrt{i} \cdot y \right)^{-s} dy^*. \end{aligned}$$

* Now, for $|y| \leq \frac{1}{2}\nu/\theta$,

$$(2.131) \quad \left(1 + \frac{\sqrt{i} \cdot \theta y}{\nu} \right)^{-s} = 1 + \binom{-s}{1} \frac{\sqrt{i} \cdot \theta y}{\nu} + \binom{-s}{2} \left(\frac{\sqrt{i} \cdot \theta y}{\nu} \right)^2 + \binom{-s}{3} \left(\frac{\sqrt{i} \cdot \theta y}{\nu} \right)^3 + O\left(\frac{\theta^4 y^4}{\nu^4} \right)$$

uniformly in any bounded region of s . Applying (2.131) to the integral in (2.13) we find

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{e^{-\pi \theta y^2}}{(1 + \sqrt{i} \cdot \theta y / \nu)^s} dy \\ &= \int_{-\frac{1}{2}\nu/\theta}^{\frac{1}{2}\nu/\theta} + \int_{|y| \geq \frac{1}{2}\nu/\theta} \frac{e^{-\pi \theta y^2}}{(1 + \sqrt{i} \cdot \theta y / \nu)^s} dy \\ &= \left(\int_{-\infty}^{\infty} - \int_{|y| \geq \frac{1}{2}\nu/\theta} \right) e^{-\pi \theta y^2} \left\{ 1 + \binom{-s}{1} \frac{\sqrt{i} \cdot \theta y}{\nu} + \binom{-s}{2} \left(\frac{\sqrt{i} \cdot \theta y}{\nu} \right)^2 + \binom{-s}{3} \left(\frac{\sqrt{i} \cdot \theta y}{\nu} \right)^3 \right\} dy \\ &\quad + \int_{-\frac{1}{2}\nu/\theta}^{\frac{1}{2}\nu/\theta} e^{-\pi \theta y^2} O\left(\frac{\theta^4 y^4}{\nu^4} \right) dy + \int_{|y| \geq \frac{1}{2}\nu/\theta} e^{-\pi \theta y^2} \left(1 + \frac{\sqrt{i} \cdot \theta y}{\nu} \right)^{-s} dy \\ (2.132) \quad &= T_1 + T_2 + T_3 + T_4, \quad \text{say}^\dagger. \end{aligned}$$

* Here and elsewhere in this paper $\sqrt{i}$ denotes $e^{i\pi/4}$.

† The four T 's correspond respectively to the four integral signs.

Now

$$(2.133) \quad T_1 = \frac{1}{\sqrt{\theta}} + \frac{is(s+1)}{4\pi\nu^2} \sqrt{\theta}^*,$$

the terms involving y , y^3 giving zero on integration.

$$(2.134) \quad \begin{aligned} |T_2| &\leq \int_{|y| \geq \frac{1}{2}\nu/\theta} e^{-\pi\theta y^2} dy + \left| \binom{-s}{2} \frac{\theta^2}{\nu^2} \right| \int_{|y| \geq \frac{1}{2}\nu/\theta} e^{-\pi\theta y^2} y^2 dy \\ &\leq e^{-\frac{1}{8}\pi\nu^2/\theta} \int_{|y| \geq \frac{1}{2}\nu/\theta} e^{-\frac{1}{2}\pi\theta y^2} dy + \left| \binom{-s}{2} \frac{\theta^2}{\nu^2} \right| e^{-\frac{1}{8}\pi\nu^2/\theta} \int_{|y| \geq \frac{1}{2}\nu/\theta} e^{-\frac{1}{2}\pi\theta y^2} y^2 dy \\ &< e^{-\frac{1}{8}\pi\nu^2/\theta} \left\{ \sqrt{\frac{2}{\theta}} + \left| \binom{-s}{2} \frac{\theta^2}{\nu^2} \right| \frac{1}{2\pi} \left(\frac{2}{\theta} \right)^{\frac{3}{2}} \right\}, \end{aligned}$$

since the values of the integrals are increased by extending the ranges of integration to the whole of $(-\infty, \infty)$.

$$(2.135) \quad |T_3| \leq A(s) \frac{\theta^4}{\nu^4} \int_{-\frac{1}{2}\nu/\theta}^{\frac{1}{2}\nu/\theta} e^{-\pi\theta y^2} y^4 dy < \frac{3A(s)}{4\pi} \frac{\theta^{\frac{5}{2}}}{\nu^4},$$

where $A(s)$ is a positive constant depending on the bounded region in which s varies.

$$(2.136) \quad \begin{aligned} |T_4| &\leq e^{-\frac{1}{8}\pi\nu^2/\theta} \int_{|y| \geq \frac{1}{2}\nu/\theta} e^{-\frac{1}{2}\pi\theta y^2} \left| \left(1 + \frac{\sqrt{i} \cdot \theta y}{\nu} \right)^{-s} \right| dy \\ &\leq e^{-\frac{1}{8}\pi\nu^2/\theta} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\pi\theta y^2} \left\{ e^{-\frac{1}{2}\pi\theta y^2} \left| \left(1 + \frac{\sqrt{i} \cdot \theta y}{\nu} \right)^{-s} \right| \right\} dy \\ &\leq A(s) e^{-\frac{1}{8}\pi\nu^2/\theta} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\pi\theta y^2} dy = A(s) e^{-\frac{1}{8}\pi\nu^2/\theta} 2\theta^{-\frac{1}{2}}, \end{aligned}$$

where $A(s)$ is a positive constant depending on the bounded region in which s varies.

Collecting the results (2.11, 2.12, 2.13, 2.132, ..., 2.136), observing that the integral function in (2.11) is independent of n , and making $n \rightarrow \infty$, we find that

$$\begin{aligned} \sum_1^{\infty} e^{\pi i n^2 \theta} n^{-s} &= I(s, \theta) + \sqrt{i} \theta^{s-\frac{1}{2}} \sum_1^{\infty} e^{-\pi i \nu^2/\theta} \nu^{-s} \\ &\quad + \frac{i \sqrt{i} s(s+1)}{4\pi} \theta^{s+\frac{1}{2}} \sum_1^{\infty} e^{-\pi i \nu^2/\theta} \nu^{-(s+2)} + \theta^{s+\frac{3}{2}} R(s, -3) \end{aligned}$$

in that portion of the half-plane $\sigma > 0$ in which the series are all convergent; the last term arises from T_3 , the terms T_2 and T_4 contributing

* $\sqrt{\theta}$ is to mean the positive root.

to $I(s, \theta)$. If, therefore, $a_1 = [1/\theta]$ and $\theta(a_1 + \theta_1) = 1$, we have a formula connecting $F_1(s, \theta)$ with $F_3(s, \theta_1)$, or $F_4(s, \theta_1)$, according as a_1 is even or odd, and this must hold for all values of s for which the functions concerned are all regular. We might have taken more terms in the expansion (2.131). If we do this we obtain the following theorem*.

THEOREM 1.

$$\begin{aligned} F_1(s, \theta) = I(s, \theta) + \sqrt{i} \bigg\{ & \theta^{s-\frac{1}{2}} F_a(s, \theta_1) + \frac{is(s+1)}{4\pi} \theta^{s+\frac{1}{2}} F_a(s+2, \theta_1) + \dots \\ & + i^n \binom{-s}{2n} \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{(2\pi)^n} \theta^{s+n-\frac{1}{2}} F_a(s+2n, \theta_1) \\ & + \theta^{s+n+\frac{1}{2}} R(s, -2n-1) \bigg\}, \end{aligned}$$

where $a = 3$ or 4 according as a_1 is even or odd.

2.2. The transformation formula for $F_2(s, \theta)$.

With the same contour as in § 2.1 we have

$$F_2(s, \theta, n) = \int_{ABCD A} \frac{e^{\pi iz^2 \theta + \pi iz} z^{-s}}{e^{2\pi iz} - 1} dz,$$

and then

$$\begin{aligned} F_2(s, \theta, n) &= \sum_{\nu=1}^M \int_{L(n+\frac{1}{2})} e^{\pi iz^2 \theta - (2\nu-1)\pi iz} z^{-s} dz \\ &\quad + \int_{L(n+\frac{1}{2})} \frac{e^{\pi iz^2 \theta - (2M-1)\pi iz} z^{-s}}{e^{2\pi iz} - 1} dz - \int_{L(\frac{1}{2})} \frac{e^{\pi iz^2 \theta + \pi iz} z^{-s}}{e^{2\pi iz} - 1} dz \\ &= \sum_{\nu=1}^M f_\nu(s) + R_M(s) + I(s, \theta), \quad \text{say.} \end{aligned}$$

This time we take $M = [(n+\frac{1}{2})\theta + \frac{1}{2}]$ and then $R_M(s) \rightarrow 0$ uniformly in any closed bounded region in $\sigma > 0$.

For $f_\nu(s)$ we find

$$\begin{aligned} f_\nu(s) &= \int_{L(n+\frac{1}{2})} \exp \left\{ \pi i \theta \left(z - \frac{\nu-\frac{1}{2}}{\theta} \right)^2 - \frac{\pi i}{\theta} (\nu-\tfrac{1}{2})^2 \right\} \cdot z^{-s} dz \\ &= \sqrt{i} \exp \left\{ -\frac{\pi i}{\theta} (\nu-\tfrac{1}{2})^2 \right\} \int_{-\infty}^{\infty} e^{-\pi \theta y^2} \left(\frac{\nu-\frac{1}{2}}{\theta} + y\sqrt{i} \right)^{-s} dy. \end{aligned}$$

Expanding this as the expression in (2.13) was expanded, and proceeding as before, we get the formula required.

* Since $\int_{-\infty}^{\infty} y^{2n} e^{-\pi \theta y^2} dy = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{(2\pi)^n} \theta^{-(n+\frac{1}{2})}$,

THEOREM 2.

$$\begin{aligned}
 F_2(s, \theta) = I(s, \theta) + \sqrt{i} \exp(-\tfrac{1}{4}a_1\pi i) \\
 \times \left\{ \theta^{s-\frac{1}{2}} F_6(s, \theta_1) + \frac{is(s+1)}{4\pi} \theta^{s+\frac{1}{2}} F_6(s+2, \theta_1) + \dots \right. \\
 \left. + i^n \binom{-s}{2n} \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{(2\pi)^n} \theta^{s+n-\frac{1}{2}} F_6(s+2n, \theta_1) \right. \\
 \left. + \theta^{s+n+\frac{1}{2}} R(s, -2n-1) \right\}.
 \end{aligned}$$

2.3. The transformation formulae for $F_3(s, \theta)$ and $F_4(s, \theta)$.

For contour of integration in the z -plane take the parallelogram with vertices

$$A' = n + \tfrac{1}{2} - (i-1)R; \quad B' = n + \tfrac{1}{2} + (i-1)S;$$

$$C' = \tfrac{1}{2} + (i-1)S; \quad D' = \tfrac{1}{2} - (i-1)R;$$

where R, S are real and positive. Then

$$F_3(s, \theta, n) = \int_{A'B'C'D'A'} \frac{e^{-\pi i z^2 \theta} z^{-s}}{e^{2\pi i z} - 1} dz$$

with a similar expression for F_4 . After making R and S tend to infinity, in both cases make the substitution

$$\frac{1}{e^{2\pi i z} - 1} = -1 - \sum_{r=1}^{M-1} e^{2r\pi i z} + \frac{e^{2M\pi i z}}{e^{2\pi i z} - 1}$$

in the integrals along $L'(n+\frac{1}{2})$. Choosing $M = 1 + [(n+\frac{1}{2})\theta]$ for F_3 , and $M = 1 + [(n+\frac{1}{2})\theta + \frac{1}{2}]$ for F_4 , and proceeding as above we obtain the following theorem.

THEOREM 3.

$$\begin{aligned}
 F_3(s, \theta) = I(s, \theta) - i\sqrt{i} \left\{ \theta^{s-\frac{1}{2}} F_a(s, \theta_1) - \frac{is(s+1)}{4\pi} \theta^{s+\frac{1}{2}} F_a(s+2, \theta_1) + \dots \right. \\
 \left. + (-i)^n \binom{-s}{2n} \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{(2\pi)^n} \theta^{s+n-\frac{1}{2}} F_a(s+2n, \theta_1) \right. \\
 \left. + \theta^{s+n+\frac{1}{2}} R(s, -2n-1) \right\},
 \end{aligned}$$

where a is 1 or 2 according as a_1 is even or odd; and

$$\begin{aligned} F_4(s, \theta) = I(s, \theta) &- i\sqrt{i} \exp\left(\tfrac{1}{4}a_1\pi i\right) \\ &\times \left\{ \theta^{s-\frac{1}{2}} F_5(s, \theta_1) - \frac{is(s+1)}{4\pi} \theta^{s+\frac{1}{2}} F_5(s+2, \theta_1) + \dots \right. \\ &\quad + (-i)^n \binom{-s}{2n} \frac{1 \cdot 3 \dots (2n-1)}{(2\pi)^n} \theta^{s+n-\frac{1}{2}} F_5(s+2n, \theta_1) \\ &\quad \left. + \theta^{s+n+\frac{1}{2}} R(s, -2n-1) \right\}. \end{aligned}$$

2.4. The transformation formulae for $F_5(s, \theta)$ and $F_6(s, \theta)$.

For F_5 we use the parallelogram $ABC''D''$ where A, B are as in § 2.1 and

$$C'' = -\tfrac{1}{4} + (1+i)S, \quad D'' = -\tfrac{1}{4} - (1+i)R;$$

and for F_6 we take $A'B'C'''D'''$ where A', B' are as in § 2.3 and

$$C''' = -\tfrac{1}{4} + (i-1)S, \quad D''' = -\tfrac{1}{4} - (i-1)R,$$

and proceed as before.

THEOREM 4.

$$\begin{aligned} F_5(s, \theta) = I(s, \theta) + \sqrt{i} \left\{ \theta^{s-\frac{1}{2}} F_a(s, \theta_1) + \frac{is(s+1)}{4\pi} \theta^{s+\frac{1}{2}} F_a(s+2, \theta_1) + \dots \right. \\ \quad + i^n \binom{-s}{2n} \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{(2\pi)^n} \theta^{s+n-\frac{1}{2}} F_a(s+2n, \theta_1) \\ \quad \left. + \theta^{s+n+\frac{1}{2}} R(s, -2n-1) \right\}, \end{aligned}$$

and

$$\begin{aligned} F_6(s, \theta) = I(s, \theta) - i\sqrt{i} \left\{ \theta^{s-\frac{1}{2}} F_\beta(s, \theta_1) - \frac{is(s+1)}{4\pi} \theta^{s+\frac{1}{2}} F_\beta(s+2, \theta_1) + \dots \right. \\ \quad + (-i)^n \binom{-s}{2n} \frac{1 \cdot 3 \dots 2n-1}{(2\pi)^n} \theta^{s+n-\frac{1}{2}} F_\beta(s+2n, \theta_1) \\ \quad \left. + \theta^{s+n+\frac{1}{2}} R(s, -2n-1) \right\}, \end{aligned}$$

where a is 4 or 3, and β is 2 or 1, according as a_1 is even or odd.

The case of quadratic irrationals.

3. We now consider the special case in which θ is a quadratic irrational, and show that then the six functions $F_a(s, \theta)$ are meromorphic functions of s having at most simple poles at a half lattice-work of points

$s = s_0 - am + ibn$, where $m = 0, 1, 2, 3, \dots$, $n = 0, \pm 1, \pm 2, \dots$, and a, b are real and positive; s_0 and b are not necessarily the same for all the six functions associated with a given θ . We first prove the following theorem.

THEOREM 5. *If the partial quotients a_n in the expression of θ as a simple continued fraction*

$$\theta = \frac{1}{a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_n + \dots}}}$$

are bounded, and, in particular, if θ is a quadratic irrational, the six functions $F_a(s, \theta)$ are all regular in the half-plane $\sigma > \frac{1}{2}$.

Let s_n denote the sum of the first n coefficients (of n^{-s}) in the series defining $F_a(s, \theta)$. Then* $|s_n| < K\sqrt{n}$, where K is a constant (depending on θ).

$$\begin{aligned} F_a(\sigma, \theta, n) &= \sum_{\nu=1}^n s_\nu \{ \nu^{-\sigma} - (\nu+1)^{-\sigma} \} + O(1) \\ &= \sum_{\nu=1}^n \sigma s_\nu \{ \nu^{-\sigma-1} + o(\nu^{-\sigma-1}) \} + O(1). \end{aligned}$$

The series for $F_a(\sigma, \theta)$ is therefore convergent for $\sigma > \frac{1}{2}$, and hence the series for $F_a(s, \theta)$ is also convergent for $\sigma > \frac{1}{2}$, and so represents a function regular in that half-plane.

COROLLARY. *For almost all irrationals θ , $F_a(s, \theta)$ is regular in $\sigma > \frac{1}{2}$.*

This follows from the result (2.112) of the paper last referred to, $s_n = o(n^{\frac{1}{2}+\epsilon})$ for almost all θ , in a way similar to the proof of Theorem 5.

THEOREM 6. *Let θ be a quadratic irrational, whose expression as a simple continued fraction is*

$$\theta = \frac{1}{a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_n + \dots}}},$$

and write

$$\theta = \frac{1}{a_1 + \theta_1}, \quad \theta_1 = \frac{1}{a_2 + \theta_2}, \quad \dots$$

Then $F_a(s, \theta)$ is a meromorphic function having for singularities at most simple poles at the points

$$s = \frac{1}{2} - 2q + \frac{(8\mu - \gamma)}{4\rho \log \Theta} \pi i$$

$$(q = 0, 1, 2, \dots; \mu = 0, \pm 1, \pm 2, \dots),$$

* HL, Theorem 2.141.

where* Θ is the product of the numbers θ_k corresponding to the periodic part of the continued fraction, ρ is one of the numbers 1, 2, ..., 6, and γ is one of 0, 1, 2, ..., 7.

We shall use α, β with and without suffixes to denote one of the numbers 1, 2, ..., 6, and δ to denote one of 0, 1, 2, ..., 7. δ will not necessarily always denote the same number; it may represent different numbers even in the same equation, and, when it occurs inside a summation sign, it need not have the same value in the various terms of the summation.

Let m be the length of the periodic part in the continued fraction for θ , so that $\alpha_{k+m} = \alpha_k$ when k is sufficiently large, say when $k \geq h$. Then $\Theta = \theta_{h+1} \theta_{h+2} \dots \theta_{h+m}$. Now apply repeatedly the transformation formulae of Theorems 1, 2, 3, 4, to $F_\alpha(s, \theta)$, taking $n = 1$ and absorbing the integral functions and the terms involving the functions F with argument $s+2$ in the remainder term. We find by Theorem 5

(3.1)
$$F_\alpha(s, \theta) = e^{\frac{1}{2}\delta\pi i}(\theta\theta_1\theta_2\dots\theta_{k-1})^{s-\frac{1}{2}}F_{\alpha_k}(s, \theta_k)+R(s, -\frac{3}{2}).$$

We can find integers $g \geq h$ and $\rho = 1, 2, 3, 4, 5$, or 6 such that, taking (3.1) as defining α_k for all integers k (positive), we have†

$$\alpha_{g+\rho m} = \alpha_g,$$

and so, of course, if r is any positive integer,

$$\alpha_{g+r\rho m} = \alpha_g.$$

Write $\beta = \alpha_g$. By the transformation formulae we have

(3.2)
$$F_\beta(s, \theta_g) = e^{\frac{1}{2}\delta\pi i}(\theta_g\theta_{g+1}\dots\theta_{g+m-1})^{\rho(s-\frac{1}{2})}F_\beta(s, \theta_g)+R(s, -\frac{3}{2}).$$

Use γ to denote the value of δ in (3.2). (3.2) shows that the function

$$[1 - e^{\frac{1}{2}\gamma\pi i} \Theta^{\rho(s-\frac{1}{2})}] F_\beta(s, \theta_g)$$

is regular for $\sigma > -\frac{3}{2}$ and so the only possible singularities of $F_\beta(s, \theta_g)$ in $\sigma > -\frac{3}{2}$ are poles at the zeros of

$$1 - e^{\frac{1}{2}\gamma\pi i} \Theta^{\rho(s-\frac{1}{2})}.$$

We see, therefore, that $F_\beta(s, \theta_g)$ is regular in the half-plane $\sigma > -\frac{3}{2}$ save possibly for simple poles at the points

$$s = \frac{1}{2} + \frac{(8\mu - \gamma)\pi i}{4\rho \log \Theta} \quad (\mu = 0, \pm 1, \pm 2, \dots).$$

* The real determination of $\log \Theta$ is to be taken here and elsewhere.

† If m is odd, ρ must be even.

Equation (3.1) now shows that the same is true first for $F_a(s, \theta)$ and then generally for $F_{a_k}(s, \theta_k)$.

To get the other lines of simple poles we must take more terms in the transformation formulae. We have

$$\begin{aligned}
 (3.3) \quad F_a(s, \theta) &= e^{\frac{1}{2}\delta\pi i}(\theta\theta_1\ldots\theta_{k-1})^{s-\frac{1}{2}}F_{a_k}(s, \theta_k) \\
 &\quad + \frac{1}{2\pi} \binom{-s}{2} \sum_{j=0}^{k-1} e^{\frac{1}{2}\delta\pi i}(\theta\theta_1\theta_2\ldots\theta_j)^{s-\frac{1}{2}}\theta_j F_{a_{j+1}}(s+2, \theta_{j+1}) + \ldots \\
 &\quad + \frac{(2n)!}{n!(4\pi)^n} \binom{-s}{2n} \sum_{j=0}^{k-1} e^{\frac{1}{2}\delta\pi i}(\theta\theta_1\ldots\theta_j)^{s-\frac{1}{2}}\theta_j^n F_{a_{j+1}}(s+2n, \theta_{j+1}) \\
 &\quad + R(s, -2n-\frac{3}{2}),
 \end{aligned}$$

where $\theta_0 = \theta$. Also

$$\begin{aligned}
 (3.4) \quad \{1 - e^{\frac{1}{2}\gamma\pi i} \Theta^{\rho(s-\frac{1}{2})}\} F_\beta(s, \theta_g) \\
 &= \frac{1}{2\pi} \binom{-s}{2} \sum_{j=0}^{\rho m-1} e^{\frac{1}{2}\delta\pi i}(\theta_g\theta_{g+1}\ldots\theta_{g+j})^{s-\frac{1}{2}}\theta_{g+j} F_{a_{g+j}}(s+2, \theta_{g+j}) + \ldots \\
 &\quad + \frac{(2n)!}{n!(4\pi)^n} \binom{-s}{2n} \sum_{j=0}^{\rho m-1} e^{\frac{1}{2}\delta\pi i}(\theta_g\theta_{g+1}\ldots\theta_{g+j})^{s-\frac{1}{2}}\theta_{g+j}^n F_{a_{g+j}}(s+2n, \theta_{g+j}) \\
 &\quad + R(s, -2n-\frac{3}{2}).
 \end{aligned}$$

Write for short
$$\frac{(8\mu-\gamma)\pi i}{4\rho \log \Theta} = i(\mu a + b).$$

Since we know that the functions $F_{a_k}(s, \theta_k)$ are all regular in $\sigma > -\frac{3}{2}$, save for possible simple poles at the points $s = \frac{1}{2} + i(\mu a + b)$, it follows that the right-hand side of (3.4), for $n \geq 1$, is regular in $\sigma > -\frac{7}{2}$, save possibly for simple poles at the points $s = i(\mu a + b) - \frac{3}{2}$. Therefore $F_\beta(s, \theta_g)$ is regular in $\sigma > -\frac{7}{2}$, save for possible simple poles at the points

$$(3.5) \quad s = -\frac{3}{2} + i(\mu a + b) \quad \text{and} \quad s = \frac{1}{2} + i(\mu a + b) \quad (\mu = 0, \pm 1, \pm 2, \ldots).$$

Taking $k = g$, all the terms in (3.3) after the first on the right-hand side are regular in $\sigma > -\frac{7}{2}$, save for possible simple poles at

$$s = i(\mu a + b) - \frac{3}{2},$$

and so $F_a(s, \theta)$ has simple poles possibly at the points (3.5) and is otherwise regular in $\sigma > -\frac{7}{2}$. Another application of (3.3) then shows that the same is true for $F_{a_k}(s, \theta_k)$.

Using this last result in (3.4) and (3.3) we find the next line of poles along $\sigma = -\frac{7}{2}$. Proceeding in this way, we deduce that $F_a(s, \theta)$ and

all the functions $F_a(s, \theta_k)$ derived from it are regular in the whole s -plane save for possible simple poles at the points

$$s = \frac{1}{2} - 2q + \frac{(8\mu - \gamma)\pi i}{4\rho \log \Theta} \quad (q = 0, 1, 2, \dots, \mu = 0, \pm 1, \pm 2, \dots).$$

It should be observed that ρ and γ depend on the non-periodic part of the continued fraction for θ as well as on the periodic part. They also depend on the particular function F_a which is considered. It is only for special values of θ that all six functions have the same pair of values for ρ, γ . To see this it is sufficient to take for θ a continued fraction with period $m = 2$, with no non-recurring elements, and with a_{2r+1} even and a_{2r} odd. Then an examination of the transformation formulae shows at once that $\rho = 2$ for F_1, F_2, F_3, F_4 , and $\rho = 1$ for F_5, F_6 . For a similar fraction with a_{2r+1} odd and a_{2r} even we see that $\rho = 2$ for F_1, F_3, F_5, F_6 , and $\rho = 1$ for F_2, F_4 . It is obvious after this that ρ (and so also γ) must depend on the non-periodic part of θ , for it is this which determines β .

In this respect the series considered behave differently from $\sum_1^\infty (n\theta)n^{-s}$, for which the positions of the possible simple poles are the points

$$s = -2q + \frac{2\mu\pi i}{\log \Theta},$$

and so depend only on the periodic part of the continued fraction.

The power series.

4. We consider next the behaviour of the power series $\sum_1^\infty e^{\pi i n^2 \theta} z^n$, etc.

It is convenient to change the variable by the transformation $z = e^{-x}$ and so get series which are convergent in the half-plane $\Re x > 0$ instead of in the unit circle $|z| < 1$. We establish first transformation formulae for the six functions

$$\begin{aligned} f_1(x, \theta) &= \sum_1^\infty e^{\pi i n^2 \theta - nx}; & f_2(x, \theta) &= \sum_1^\infty (-1)^n e^{\pi i n^2 \theta - nx}; \\ f_3(x, \theta) &= \sum_1^\infty e^{-\pi i n^2 \theta - nx}; & f_4(x, \theta) &= \sum_1^\infty (-1)^n e^{-\pi i n^2 \theta - nx}; \\ f_5(x, \theta) &= \sum_1^\infty e^{\pi i (n - \frac{1}{2})^2 \theta - nx}; & f_6(x, \theta) &= \sum_1^\infty e^{-\pi i (n - \frac{1}{2})^2 \theta - nx}; \end{aligned}$$

where $0 < \theta < 1$. The functions are all regular in $\Re x > 0$, and, being power series in e^{-x} with $\Re x = 0$ as line of convergence, must have some

singularities on the imaginary axis. It may be observed that, if θ is replaced by a complex number τ and the summation is extended from $-\infty$ to ∞ , the series for f_1, f_2, f_5 when $\Re \tau > 0$, and f_3, f_4, f_6 when $\Re \tau < 0$, become elliptic $\mathfrak{S}$ -series, the other three series in each case becoming divergent.

4.1. The transformation formulae.

Let $L(a), L'(a)$ have the same meanings as in § 2. As in § 2 we find that

$$\begin{aligned} \sum_1^n e^{\pi i n^2 \theta - n x} &= \int_{L(n+\frac{1}{2})} - \int_{L(\frac{1}{2})} \frac{e^{\pi i z^2 x - z x}}{e^{2\pi i z} - 1} dz \\ &= \sum_{\nu=1}^M \int_{L(n+\frac{1}{2})} \exp \{ \pi i z^2 \theta - 2\nu \pi i z - z x \} dz \\ &\quad + \int_{L(n+\frac{1}{2})} \frac{\exp \{ \pi i z^2 \theta - 2M \pi i z - z x \}}{e^{2\pi i z} - 1} dz + I(x, \theta) \\ &= \sum_{\nu=1}^M g_\nu(x) + R_M(x) + I(x, \theta) \quad \text{say,} \end{aligned}$$

where $I(x, \theta)$ is an integral function of x .

In the integral for $R_M(x)$ write $z = n + \frac{1}{2} + \zeta$, and then take

$$M = [(n + \frac{1}{2}) \theta], \quad \mu = (n + \frac{1}{2}) \theta - M$$

so that $0 \leq \mu < 1$. Then

$$|R_M(x)| = \left| \int_{\zeta \text{ on } L(\nu)} \frac{\exp [\pi i \{ \theta \zeta^2 + 2(n + \frac{1}{2}) \theta \zeta - 2M \zeta \}] \cdot e^{-\nu(n + \frac{1}{2}) - \zeta x}}{e^{2\pi i \zeta} + 1} d\zeta \right|.$$

Now on the path of integration

$$\left| \frac{e^{2\mu \pi i \zeta}}{e^{2\pi i \zeta} + 1} \right| \leq C,$$

where C is some constant, and so

$$\begin{aligned} |R_M(x)| &\leq C \exp \{ -\Re(x)(n + \frac{1}{2}) \} \int |e^{\pi i \theta \zeta^2 - \zeta x}| |d\zeta| \\ &= C \exp \{ -\Re(x)(n + \frac{1}{2}) \} \int_{-\infty}^{\infty} \exp \left[-\pi \theta y^2 + \frac{y}{\sqrt{2}} \{ \Im(x) - \Re(x) \} \right] dy \\ &= \frac{C}{\sqrt{\theta}} \exp \left[-(n + \frac{1}{2}) \Re(x) + \frac{1}{8\pi\theta} \{ \Im(x) - \Re(x) \}^2 \right] \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{when } \Re(x) > 0. \end{aligned}$$

Next

$$\begin{aligned}
 g_\nu(x) &= \int_{L(n+\frac{1}{2})} \exp \left[\pi i \theta \left\{ z^2 - 2z \left(\frac{\nu}{\theta} - \frac{ix}{2\pi\theta} \right) \right\} \right] dz \\
 &= \exp \left\{ -\frac{\pi i}{\theta} \left(\nu - \frac{ix}{2\pi} \right)^2 \right\} \int_{L[(\nu/\theta) - ix/(2\pi\theta)]} \exp \left[\pi i \theta \left\{ z - \left(\frac{\nu}{\theta} - \frac{ix}{2\pi\theta} \right) \right\}^2 \right] dz \\
 &= \exp \left\{ -\frac{\pi i}{\theta} \left(\nu - \frac{ix}{2\pi} \right)^2 \right\} \int_{L(\epsilon)} e^{\pi \theta z^2} dz \\
 &= \frac{\sqrt{i}}{\sqrt{\theta}} \exp \left\{ -\frac{\pi i}{\theta} \left(\nu - \frac{ix}{2\pi} \right)^2 \right\}.
 \end{aligned}$$

Summing and making $n \rightarrow \infty$ we find, when $\Re(x) > 0$,

$$f_1(x, \theta) = \frac{\sqrt{i}}{\sqrt{\theta}} \exp \left(\frac{ix^2}{4\pi\theta} \right) f_1 \left(\frac{x}{\theta}, -\frac{1}{\theta} \right) + I(x, \theta).$$

Precisely similar arguments give the corresponding results for the other five functions. Those for f_5 , f_6 can alternatively be obtained from those for f_1 , f_3 since

$$f_5(x, \theta) = e^{2\pi i \theta} f_1(x + \pi i \theta, \theta); \quad f_6(x, \theta) = e^{-\frac{1}{2}\pi i \theta} f_3(x - \pi i \theta, \theta).$$

In all we have the following theorem.

THEOREM 7.

$$\begin{aligned}
 f_1(x, \theta) &= \frac{\sqrt{i}}{\sqrt{\theta}} \exp \left(\frac{ix^2}{4\pi\theta} \right) f_1 \left(\frac{x}{\theta}, -\frac{1}{\theta} \right) + I(x, \theta), \\
 f_2(x, \theta) &= \frac{\sqrt{i}}{\sqrt{\theta}} \exp \left(\frac{ix^2}{4\pi\theta} + \frac{x}{\theta} - \frac{a_1 \pi i}{4} \right) f_6 \left(\frac{x}{\theta}, \theta_1 \right) + I(x, \theta), \\
 f_3(x, \theta) &= -\frac{\sqrt{i}}{\sqrt{\theta}} \exp \left(-\frac{ix^2}{4\pi\theta} \right) f_3 \left(\frac{x}{\theta}, -\frac{1}{\theta} \right) + I(x, \theta), \\
 f_4(x, \theta) &= -\frac{\sqrt{i}}{\sqrt{\theta}} \exp \left(-\frac{ix^2}{4\pi\theta} + \frac{x}{\theta} + \frac{a_1 \pi i}{4} \right) f_5 \left(\frac{x}{\theta}, \theta_1 \right) + I(x, \theta), \\
 f_5(x, \theta) &= \frac{\sqrt{i}}{\sqrt{\theta}} \exp \left(\frac{ix^2}{4\pi\theta} - \frac{x}{\theta} \right) f_2 \left(\frac{x}{\theta}, -\frac{1}{\theta} \right) + I(x, \theta), \\
 f_6(x, \theta) &= -\frac{\sqrt{i}}{\sqrt{\theta}} \exp \left(-\frac{ix^2}{4\pi\theta} - \frac{x}{\theta} \right) f_4 \left(\frac{x}{\theta}, -\frac{1}{\theta} \right) + I(x, \theta),
 \end{aligned}$$

where $a_1 = [1/\theta]$ and $\theta(a_1 + \theta_1) = 1$.

5. Write $\phi_a(z, \theta)$ for the function defined by the series obtained from that for $f_a(x, \theta)$ by replacing e^{-x} by z . We prove that, when θ is irrational, the unit circle is a singular line for $\phi_a(z, \theta)$. We require the following lemma.

LEMMA. Let $f(x) = a_0 + a_1x + \dots + a_nx^n + \dots$ be regular in the unit circle and also at each point of the arc $\phi_1 \leq \phi \leq \phi_2$, where $x = e^{i\phi}$. Then if the coefficients a_n are bounded, the sums $\sum_{v=1}^n a_v x^v$ are uniformly bounded on the arc $\phi_1 \leq \phi \leq \phi_2$.

Write $x_1 = e^{i\phi_1}$, $x_2 = e^{i\phi_2}$, and suppose $|a_n| < K$. The proof is almost the same as, though simpler than, that given for a theorem of Fatou and M. Riesz in Landau's book* when the ϵ 's there appearing are all replaced by K . For then the argument shows (using Landau's notation) that $g_n(x)$ is bounded on the boundary of the sector Oz_1z_2 , and therefore on the arc $\phi_1 \leq \phi \leq \phi_2$. Then, on this arc,

$$|f(x) - (a_0 + \dots + a_nx^n)| \leq \frac{|g_n(x)|}{L^2} < A,$$

a constant, and so, if M is the upper bound of $|f(z)|$ in Oz_1z_2 , then

$$|a_0 + a_1x + \dots + a_nx^n| < A + M.$$

THEOREM 8. If θ is irrational each of the functions $\phi_a(z, \theta)$ has the circle $|z| = 1$ for a singular line.

Write s_n for the sum of the first n coefficients of the power series for $\phi_a(z, \theta)$. Since every coefficient has modulus unity, the lemma shows that if the function were regular at $z = 1$, we should have $s = O(1)$. But this is impossible, since we know that $s = \Omega(\sqrt{n})^\dagger$. It follows that $f_a(x, \theta)$ has singularities at all the points

$$x = 2r\pi i \quad (r = 0, \pm 1, \pm 2, \dots).$$

Repeated applications of the transformation formulae of Theorem 7 give

$$(5.1) \quad f_a(x, \theta) = \frac{\omega}{\sqrt{(\theta\theta_1 \dots \theta_n)}} e^{g(x)} f_\beta\left(\frac{x}{\theta\theta_1 \dots \theta_n}, \theta_{n+1}\right) + I(x, \theta),$$

* E. Landau, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie* (1916), 66-69.

† HL, Theorem 2.22. The same argument is not available for the points of the unit circle other than $z = 1$.

where ω is an eighth root of unity, $g(x)$ is of the form iAx^2+Bx , where A, B are real, and $\theta_1, \theta_2, \dots, \theta_n, \dots$ are defined by

$$\theta = \frac{1}{a_1 + a_2 + \dots}; \quad \theta = \frac{1}{a_1 + \theta_1}, \quad \theta_1 = \frac{1}{a_2 + \theta_2}, \quad \dots$$

Therefore $f_a(x, \theta)$ has singularities at all the points

$$x = 2r\pi i \theta \theta_1 \dots \theta_n + 2m\pi i \\ (r = 0, \pm 1, \pm 2, \dots; \quad m = 0, \pm 1, \pm 2, \dots).$$

These points form a set everywhere dense on the imaginary axis. Suppose, if possible, that there is a length δ free from these points. It is impossible for $\theta \theta_1 \dots \theta_n$ to tend to a non-zero limit as $x \rightarrow \infty$; for that would necessitate the convergence of $\Sigma(1-\theta_n)$. Now

$$1 - \theta_{n-1} = 1 - \frac{1}{a_n + \theta_n} > 1 - \frac{1}{a_n} \geq \frac{1}{2} \quad \text{unless } a_n = 1.$$

This shows that the series is divergent, and therefore $\theta \theta_1 \dots \theta_n \rightarrow 0$, except possibly when $a_n = 1$ for all sufficiently large n . If $a_n = 1$ for $n \geq n_0$, $\theta_n = \frac{1}{2}(\sqrt{5}-1)$ for $n > n_0$, and so the product still tends to zero.

We can therefore choose n so large that $2r\pi\theta\theta_1 \dots \theta_n < \frac{1}{2}\delta$, and then the points $2r\pi i \theta \theta_1 \dots \theta_n$ are at distances less than $\frac{1}{2}\delta$ apart. There cannot be an interval of length δ free from the singular points. The imaginary axis is therefore a singular line for $f_a(x, \theta)$, and so the unit circle is a singular line for $\phi_a(x, \theta)$.

The rational cases.

6. When θ is rational, the behaviour of both sets of functions is quite different from their behaviour in the cases considered above. The transformation formulae, of course, still hold. In the case of the Dirichlet's series, since the continued function for θ terminates, repeated applications of Theorems 1, 2, 3, 4 express $F_a(s, \theta)$ in the form

$$(6.1) \quad F_a(s, \theta) = I(s, \theta) + I(s, \theta) \xi(s) + I(s, \theta) \xi(s+2) + \dots \\ + I(s, \theta) \xi(s+2n) + R(s, -2n-1),$$

where $\xi(s)$ is one of the six functions $F_\beta(s, 0)$ and the functions $I(s, \theta)$ are integral functions, not all the same. $\xi(s)$ is the function defined (at least for $\sigma > 1$) by one or other of the series

$$(i) \quad \sum_1^\infty n^{-s}, \quad (ii) \quad \sum_1^\infty (-1)^n n^{-s}, \quad (iii) \quad \sum_0^\infty (n + \frac{1}{2})^{-s}.$$

In case (i) $\xi(s) = \zeta(s)$, the Riemann ζ -function, and it is therefore regular in the whole plane save for a simple pole at $s = 1$.

In case (ii) $\xi(s) = (1-2^{1-s})\zeta(s)$ and it is an integral function.

$$\begin{aligned}\text{In case (iii)} \quad \xi(s) &= 2^s \sum_0^{\infty} (2n+1)^{-s} \\ &= 2^s \sum_1^{\infty} n^{-s} - 2^s \sum_1^{\infty} (2n)^{-s} \\ &= (2^s - 1) \zeta(s),\end{aligned}$$

which is regular in the whole plane save for a simple pole at $s = 1$.

Since $F_a(s, \theta)$ is independent of n , the number of terms to which the expansion (6.1) is carried, it is either an integral function or a meromorphic function whose only singularities are simple poles at the points

$$s = 1, -1, -3, \dots, -(2n+1), \dots;$$

in the latter case $s = 1$ must always be a pole. Whether we get an integral function or a meromorphic function depends on the parity of the partial quotients in the continued fraction for θ .

7. Let n be the greatest integer for which $\theta_n \neq 0$. Equation (5.1), which does not depend on θ being irrational, becomes

$$f_a(x) = \frac{\omega}{\sqrt{(\theta\theta_1 \dots \theta_n)}} e^{g(x)} f_\beta\left(\frac{x}{\theta\theta_1 \dots \theta_n}, 0\right) + I(x, \theta).$$

The series for $f_\beta(x, 0)$ is a geometric series. Summing it by the ordinary rule we see that $f_\beta(x, 0)$ has simple poles at the points $2r\pi i$ or $(2r+1)\pi i$ ($r = 0, \pm 1, \pm 2, \dots$), according as $\beta = 1, 3, 5, 6$ or $\beta = 2, 4$, and it has no other singularities.

Hence $f_a(x, \theta)$ has simple poles at each point of *one* of the sets

$$\begin{aligned}2r\pi i \theta \theta_1 \dots \theta_n + 2m\pi i, \quad (2r+1)\pi i \theta \theta_1 \dots \theta_n + 2m\pi i \\ (r, m = 0, \pm 1, \pm 2, \dots).\end{aligned}$$

Finally $\phi_a(z, \theta)$ has simple poles at each point of one of the sets

$$\exp(2r\pi i \theta \theta_1 \dots \theta_n), \quad \exp\{(2r+1)\pi i \theta \theta_1 \dots \theta_n\} \quad (r = 0, \pm 1, \pm 2, \dots),$$

and no other singularities. These poles lie on the unit circle, and there are only a finite number of them, since $\theta\theta_1 \dots \theta_n$ is rational.

ON THE REAL FOCI AND DIRECTRICES OF A CLASS CUBIC
AND OTHER PLANE ALGEBRAIC CURVES

By HAROLD HILTON and SYBIL D. JERVIS.

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1. A curve of class m ("class m -ic") has, in general, m real foci, *i.e.* intersections of tangents to the curve from the circular points at infinity I and J . If O is such a focus, the line joining the points of contact of OI and OJ is the *directrix* corresponding to O . The *foot of the directrix* is its intersection with the perpendicular from O .

Foci were first defined by Plücker*, who proved that, if the sum of the angles made by the tangents from P to a class m -ic with a given line is zero or $\frac{1}{2}\pi$, the locus of P is an m -ic (curve of degree m) through the foci. Fifteen years later Kummer† proved that, if a singly infinite family of curves is such that through each point of the plane two curves pass which cut orthogonally, then the curves are confocal. In the years 1864 and 1865 the subject was taken up again by Siébeck‡, Cremona§, and especially by Laguerre||, who proved, among other results, that, if a curve has four concyclic foci, it has sixteen foci lying by fours on four orthogonal circles, that the sum of the angles which the tangents to a curve from a point P make with any direction is equal to the similar sum for the lines joining P to the real foci, and that the centroid of the foci of a curve coincides with the centroid of the points of contact of parallel tangents. This work was partly extended and partly rediscovered by

* *Journal für Math.*, 10 (1832), 84.

† *Journal für Math.*, 35 (1847), 5.

‡ *Journal für Math.*, 64 (1864), 175.

§ *Nouv. Annales Math.* (2), 3 (1864), 21.

|| *Nouv. Annales Math.* (2), 3 (1864), 141; *Œuvres*, 2 (1898–1905), 18, 23, 537.

other authors; for instance, Clifford*, Holst†, Humbert‡, Franklin§, Balitrand||, Roberts¶, and Juhel-Renoy**. Zimmermann †† is one of the very few writers who has considered the properties of directrices; he gives a method for the construction of the directrices and foci.

We give here some theorems concerning foci and directrices of a class m -ic which we believe to be new, considering especially the case $m = 3$.

2. In rectangular Cartesian coordinates the general tangential equation (condition that it should touch $\lambda x + \mu y + \nu = 0$) of a class m -ic with real foci (x_1, y_1) , (x_2, y_2) , ..., (x_m, y_m) is

$$2(\lambda x_1 + \mu y_1 + \nu)(\lambda x_2 + \mu y_2 + \nu) \dots (\lambda x_m + \mu y_m + \nu) + \phi(\lambda, \mu, \nu)(\lambda^2 + \mu^2) = 0, \quad (i)$$

where ϕ is a homogeneous polynomial in λ, μ, ν of degree $m-2$.

The equation (i) is satisfied by any values of λ, μ, ν such that $\phi = 0$, $\lambda x_1 + \mu y_1 + \nu = 0$. Hence

(1) "The non-circular tangents from the real foci of a class m -ic all touch a class $(m-2)$ -ic".

If (ξ_1, η_1) , (ξ_2, η_2) , ..., (ξ_{m-2}, η_{m-2}) are the real foci of $\phi = 0$, we may put (i) into the form

$$2(\lambda x_1 + \mu y_1 + \nu) \dots (\lambda x_m + \mu y_m + \nu) + k(\lambda \xi_1 + \mu \eta_1 + \nu) \dots (\lambda \xi_{m-2} + \mu \eta_{m-2} + \nu) \\ \times (\lambda^2 + \mu^2) + \psi(\lambda, \mu, \nu)(\lambda^2 + \mu^2)^2 = 0, \quad (ii)$$

where ψ is a homogeneous polynomial in λ, μ, ν of degree $m-4$, and $\psi = 0$ is the class $(m-4)$ -ic touched by the non-circular tangents from the foci of $\phi = 0$.

To get the directrix corresponding to a focus O of a class m -ic $f(\lambda, \mu, \nu) = 0$, we obtain λ, μ, ν for a circular line through O , and then get its point of contact $(\partial f / \partial \lambda, \partial f / \partial \mu, \partial f / \partial \nu)$. This is one point on the directrix. Another is obtained by changing $i(\equiv \sqrt{-1})$ into $-i$, and the line joining them is the required directrix.

* *Math. Papers* (1882), 130.

† *Math. Annalen*, 11 (1877), 341, 575.

‡ *Amer. J. of Math.*, 10 (1888), 258.

§ *Amer. J. of Math.*, 12 (1890), 161.

|| *Nouv. Annales Math.* (3), 12 (1893), 256.

¶ *Bull. Amer. Math. Soc.*, 2 (1896), 98.

** *Comptes rendus*, 142 (1906), 700.

†† *Journal für Math.*, 126 (1903), 171.

For the curve (i) or (ii) write

$$x_t - x_s = r_{st} \cos \alpha_{st}, \quad y_t - y_s = r_{st} \sin \alpha_{st},$$

$$\xi_t - x_s = \rho_{st} \cos \epsilon_{st}, \quad \eta_t - y_s = \rho_{st} \sin \epsilon_{st},$$

and
$$a_1 = a_{12} + a_{13} + \dots + a_{1m}, \quad \epsilon_1 = \epsilon_{11} + \epsilon_{12} + \dots + \epsilon_{1m}, \quad \text{etc.}$$

Here r_{st} is the distance from the focus (x_s, y_s) to the focus (x_t, y_t) , and α_{st} is the angle which this line makes with $y = 0$. Also ρ_{st} is the distance from the focus (x_s, y_s) to the focus (ξ_t, η_t) of $\phi = 0$, and ϵ_{st} is the angle which this line makes with $y = 0$.

We find readily that the directrix of (i) or (ii) corresponding to the focus (x_1, y_1) is

$$(x - x_1) \cos (\alpha_1 - \epsilon_1) + (y - y_1) \sin (\alpha_1 - \epsilon_1) = (k \rho_{11} \rho_{12} \dots \rho_{1m-2}) / (r_{12} r_{13} \dots r_{1m}).$$

It follows that :

(2) "The distance of each real focus of a class m -ic Σ_m from the corresponding directrix is proportional to the product of its distances from the real foci of the class $(m-2)$ -ic Σ_{m-2} , which touches the non-circular tangents from the real foci of Σ_m , divided by the product of its distances from the other real foci of Σ_m ".

(3) "The lines joining any real focus O of Σ_m to the other real foci make angles with the perpendicular from O to the corresponding directrix, whose sum differs by a multiple of π from the sum of the angles made with this perpendicular by the lines joining O to the foci of Σ_{m-2} ".

It is well known that the sum is also equal to the sum of the angles made with the perpendicular by the non-circular tangents from O to Σ_{m-2} .

(4) "If the real foci of a class m -ic and $\frac{1}{2}(m-2)(m+1)$ real tangents are given, there are $\frac{1}{2}(m-1)(m-2)$ other fixed tangents. The class $(m-2)$ -ics touched by the non-circular tangents from the foci form a tangential pencil. Each directrix envelops a parabola whose focus is the corresponding focus of the class m -ic.

"The locus of the points of contact of the non-circular tangents from one of the foci O is a curve of degree $2m-3$ with a multiple point of order $2m-4$ at O . Two of the tangents at O are circular lines".

(5) "If PX is the tangent at P to a class m -ic, while $A_1, A_2, \dots, A_m$ are the real foci, and O is the polar point of PX with respect to the class

$(m-2)$ -ic touched by the non-circular tangents from these foci, then

$$(m-2) \cot OPX = \cot A_1 PX + \cot A_2 PX + \dots + \cot A_m PX".*$$

This may be proved by taking P as the origin and PX as the axis of x in (i).

If in (i) $(x_{m-1}, y_{m-1}), (x_m, y_m)$ are the same point, the curve goes through the circular points, and that point is a "singular focus". It has, of course, no directrix, but the above properties of the directrices corresponding to the other foci still hold good with slight and obvious modifications.

3. We now discuss in more detail the case $m = 3$. The results of § 2 (1) to (4) give us :

(1) "If a class cubic has three real foci A, B, C , the non-circular tangents from these foci meet in a point O called the *pole*. [It is the point O of § 2 (5).] The product of the perpendiculars from A, B, C on any tangent is proportional to the perpendicular from O on the tangent".

(2) "The perpendiculars from A, B, C on the corresponding directrices meet at a point O' . The points O and O' are foci of a conic touching the sides of the triangle ABC .

"The lengths of these three perpendiculars are in the ratios

$$OA.BC : OB.CA : OC.AB.$$

"The directrices are parallel to the lines joining the feet of the perpendiculars from O to the sides of the triangle ABC .

"If A, B, C are collinear, O and O' are reflexions of each other in ABC ".

We shall speak of all class cubics with the same real foci and pole as a *confocal and copolar family*.

We find without difficulty, using the equation of § 2 (i) with $m = 3$, and the origin at a focus or at the pole :

(3) "Given the real foci of a class cubic and the ratio of the distances of two directrices from the corresponding foci, the locus of the pole is a circle".

(4) "Four members of a given confocal and copolar family go through a given point".

* This theorem is given by Salmon, *Higher plane curves*, 3rd edition (1879), 126, for the case $m = 3$.

(5) "The normals at the points of contact of the real tangents from the foci of a given confocal and copolar family are concurrent for *two* members of the family in general; or for *one* member, if the foci and pole are concyclic".

(6) "Given the foci A, B, C and the directrix corresponding to C , the envelope of the directrix corresponding to A is a parabola with focus A and latus rectum along AB ".

(7) "If P is any point on the pedal of a class cubic with respect to the point V , the product of the tangents from P to the circles on VA, VB, VC as diameters varies as the product of VP and the tangent from P to the circle on VO as diameter".

(8) "Given two tangents, the focus, and the directrix of a circular class cubic, the locus of the singular focus is a conic".

4. If we take the foci A, B, C in § 3 as vertices of the triangle of reference of trilinear coordinates, and suppose that (p, q, r) is the pole, the tangential equation of the class cubic is

$$2\lambda\mu\nu + (p\lambda + q\mu + r\nu)(\lambda^2 + \mu^2 + \nu^2 - 2\mu\nu \cos A - 2\nu\lambda \cos B - 2\lambda\mu \cos C) = 0.$$

The directrix corresponding to $C(0, 0, 1)$ is found by the method of § 2 to be

$$\begin{aligned} \sin C(p^2 + 2pq \cos C + q^2)(\sin A\alpha + \sin B\beta + \sin C\gamma) \\ = (p \cos C + q)\alpha + (p + q \cos C)\beta, \end{aligned}$$

and so for the foci A and B .

If instead we take trilinear coordinates with $(1, 0, 0)$ and $(0, 1, 0)$ as two foci, $(0, 0, 1)$ as pole, and (p, q, r) as the third focus, the equation of the class cubic is

$$2\lambda\mu(p\lambda + q\mu + r\nu) + \nu(\lambda^2 + \mu^2 + \nu^2 - 2\mu\nu \cos A - 2\nu\lambda \cos B - 2\lambda\mu \cos C) = 0,$$

and the equation of the directrix of $(1, 0, 0)$ is

$$\beta(q \cos A + r) + \gamma(q \cos 2A + r \cos A) = \sin A(\sin A\alpha + \sin B\beta + \sin C\gamma),$$

and so for $(0, 1, 0)$.

Some of the more interesting of the many results obtainable from these trilinear equations are the following:—

(1) "Given the foci of a class cubic, a tangent, and a point on one directrix, the locus of the pole is a conic through the corresponding focus".

(2) "Given the three foci and a point on each of two directrices, the locus of the pole is a circular cubic through the two corresponding foci".

(3) "Given the three foci, one tangent, and the distance of one focus from the corresponding directrix, the locus of the pole is a conic with that focus and the given tangent as focus and corresponding directrix".

(4) "In a confocal and copolar family of class cubics there is *one* member whose points of contact A' , B' , C' with the real tangents from the foci A , B , C are collinear".

(5) "Given the foci, and the fact that A' , B' , C' are collinear with a fixed point, the locus of the pole is a circular cubic touching a conic at the foci".

(6) "In a confocal and copolar family of class cubics there are in general *two* members whose directrices have collinear feet. But, if the foci and pole are concyclic, the directrices are parallel and there is only *one* member".

(7) "Given one tangent and the foci of a class cubic whose directrices have collinear feet, the locus of the pole is a conic through the intersections of the given tangent with the circle through the foci".

(8) "Given the pole O and two foci A , B of a class cubic, and also a point on each of the corresponding directrices, the locus of the third focus is a straight line.

"The locus of the point of contact of the third tangent from O is a cubic with node at O and OA , OB as nodal tangents".

(9) "Given two foci and the points of contact of the real tangents from those foci, the corresponding directrices go through fixed points, and the third focus lies on a fixed line through the pole".

(10) "If two foci and the pole are given, while the points of contact of the real tangents from the foci lie on a line through a fixed point, the locus of the third focus is a conic through the pole".

5. Properties of the foci of a class cubic may also be obtained by reciprocation. If we reciprocate with respect to a focus O we obtain a circular cubic. The directrix of O reciprocates into the singular focus, and the point of contact of the real tangent from O becomes the real asymptote. As illustrations we enunciate the theorems:—

(1) "Any point R is taken on the third tangent from the point of contact of the real tangent to a class cubic from its focus O . If the remain-

ing tangents from R meet the directrix corresponding to O in L and M , then RL and RM subtend equal angles at O'' .

(2) "Perpendiculars OA , OP , OQ , OR are drawn from the focus O of a class cubic to the corresponding directrix and to the three tangents drawn from any point of that directrix. Then

$$\frac{OQ \cdot OR}{AQ \cdot AR} + \frac{OR \cdot OP}{AR \cdot AP} + \frac{OP \cdot OQ}{AP \cdot AQ}$$

is constant".

6. The most interesting of the specialized class cubics is that which has the line at infinity as inflexional tangent. It has only one focus O and corresponding directrix. Let I be the infinite inflexion, A the foot of the directrix, E the point of contact of the real tangent from O , and F the point of contact of the remaining tangent from E . The properties of the curve may be obtained by reciprocating it with respect to O into a cuspidal cubic. As an alternative we may use its tangential equation taking O as origin and either OA or OI as axis of x ; or, again, we may take the tangent and normal at any point as axes of reference. The following are some of the more interesting properties of the curve obtained by these methods:

(1) "The line OI bisects the angle EOA , and the third tangent from E is parallel to OA ".

(2) "Given the focus O and the directrix, the locus of E is a parabola with focus O , axis OA , and latus rectum equal to $2OA$. The locus of F is a parabola with OA as axis and latus rectum $\frac{2}{3}OA$ ".

(3) "If PT is the tangent at any point P meeting OE and OI in H and K , then

$$\cot OHT = \cot OPT + 2 \cot OKT.$$

The ratio of the distances of O from the directrix and tangent is

$$\sin^2 OKT \cdot \operatorname{cosec} OHT''.$$

(4) "If OM is the perpendicular from O to EF , the distance from O of the tangent parallel to the directrix is $EM \cdot OA^2/OM^2$ ".

(5) "If the perpendicular through O to OI meets the directrix in L , OL is twice the distance of E from OI ".

(6) "The circumcircle of the triangle formed by these tangents, which make with OI angles whose sines have a constant product, cuts orthogonally a fixed circle with centre O ".

(7) "If three tangents are drawn from a point P , the locus of P and the locus of the centroid of the triangle formed by the points of contact are of the same degree".

7. Another special type of interest is the class cubic for which the line at infinity is a bitangent. Let O be the focus, H and K the points of contact of the infinite bitangent, A the foot of the directrix, and E the point of contact of the real tangent from O .

Taking the bisectors of the angle HOK as axes of reference, or using reciprocation, we have :

(1) "The line-pairs OA , OE and OH , OK have common bisectors".

(2) "The tangents from any point P make with either of these bisectors angles whose sum differs by a multiple of π from the angle made with this bisector by OP ".

(3) "If the tangents parallel to the bisectors of the angle HOK meet OE in Q and R . $1/OQ \sim 1/OR = 2/OA$ ".

(4) "The third tangent from E is parallel to OA ".

ON CERTAIN FUNCTIONS REPRESENTED BY DIRICHLET SERIES

By T. ESTERMANN.

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Introduction.

Let $F(h)$ be a polynomial such that $F(h)$ is an integer for every integral value of h (*ganzwertiges Polynom*), and that

$$(i.1) \quad F(0) = 1;$$

let

$$(i.2) \quad g(n) = F(h_1) F(h_2) \dots F(h_m),$$

where

$$n = p_1^{h_1} p_2^{h_2} \dots p_m^{h_m}$$

is the decomposition of n into prime factors, and let

$$(i.3) \quad f(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}.$$

Taking, in particular,

$$(i.4) \quad F_{l,m}(h) = \left[\frac{(h+m-1)!}{h! (m-1)!} \right]^l,$$

we have

$$g_{l,m}(n) = [d_m(n)]^l,$$

where $d_m(n)$ is the number of the decompositions of n into m factors, and

$$(i.5) \quad f_{l,m}(s) = \sum_{n=1}^{\infty} \frac{[d_m(n)]^l}{n^s}.$$

Our main results are the following :

The functions (i. 3) can be divided into two classes. The first consists of meromorphic functions, the second of functions whose natural boundary is the imaginary axis*, and which have no singularities other than poles in the half-plane $\sigma > 0$. Our results concerning the functions of the first class are an immediate consequence of the formula (4. 3). If we confined our attention to the functions of the first class, we could have avoided the considerations of §§ 2-4 and proved (4. 3) by means of the two classical theorems stated in the appendix. We need §§ 2-4, however, for the discussion of the functions of the second class, and we obtain a new proof of the said theorems as an accessory result.

Some of the functions (i. 5) are trivial :

$$f_{l,1}(s) = \xi(s) \quad (l = 1, 2, \dots),$$

$$f_{1,m}(s) = [\xi(s)]^m \quad (m = 1, 2, \dots).$$

For $m = 2, l = 2$ we have†

$$f_{2,2}(s) = \frac{[\xi(s)]^4}{\xi'(2s)}.$$

All the other functions (i. 5) belong to the second class. It was pointed out to me by Mr. Titchmarsh‡ that, if the Riemann hypothesis is true, then $f_{3,2}(s)$ has the said properties of the functions of the second class; but the general result which I have obtained is independent of any hypothesis.

1. Let k be the degree of the polynomial $F(h)$. Then we can write $F(h)$ in the form

$$(1.1) \quad F(h) = a_0 + \sum_{m=1}^k a_m (h+1)(h+2)\dots(h+m),$$

and we have (for $|x| < 1$)

$$(1.2) \quad \sum_{h=0}^{\infty} F(h) x^h = \sum_{m=0}^k \frac{a_m m!}{(1-x)^{m+1}} = \frac{\sum_{m=0}^k a_m m! (1-x)^{k-m}}{(1-x)^{k+1}} = \frac{\sum_{m=0}^k b_m x^m}{(1-x)^{k+1}}.$$

* A similar result concerning Σp^{-s} is given by E. Landau and A. Walfisz, *Rend. di Palermo*, 44 (1919), 82-86.

† S. Ramanujan, *Messenger of Math.*, 45 (1916), 81-84; B. M. Wilson, *Proc. London Math. Soc.* (2), 21 (1922), 235-236.

‡ I take this opportunity to express my thanks to Mr. Titchmarsh for having drawn my attention to the subject and for his kind permission to avail myself of a manuscript in which he proved Theorem 3 for the particular function $f_{3,2}(s)$.

From the identity

$$\sum_{m=0}^k b_m x^m = \sum_{h=0}^{\infty} F(h) x^h \cdot (1-x)^{k+1} = \sum_{h=0}^{\infty} F(h) x^h \cdot \sum_{n=0}^{k+1} \binom{k+1}{n} (-1)^n x^n$$

we obtain

$$(1.3) \quad b_m = \sum_{h=0}^m F(h) \binom{k+1}{m-h} (-1)^{m-h}.$$

Hence $b_0, b_1, \dots, b_k$ are integers, and, in particular, by (i. 1),

$$(1.4) \quad b_0 = 1,$$

$$(1.5) \quad b_1 = -(k+1) + F(1).$$

Let b_r be the last of the numbers $b_0, b_1, \dots$, which is not zero, and let us exclude the trivial case $r = 0$, so that

$$(1.6) \quad 1 \leq r \leq k.$$

Taking $x = p^{-s}$ in (1.2), we have

$$(1.7) \quad \sum_{h=0}^{\infty} \frac{F(h)}{p^{hs}} = \frac{\sum_{m=0}^r b_m p^{-sm}}{(1-p^{-s})^{k+1}} \quad (s = \sigma + it, \sigma > 1).$$

$$\text{Now} \quad \sum_{n=1}^{\infty} \frac{g(n)}{n^s} = \prod_p \sum_{h=0}^{\infty} \frac{F(h)}{p^{hs}} \quad (\sigma > 1),$$

this being a trivial generalization of the known formula

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \sum_{h=0}^{\infty} \frac{1}{p^{hs}}.$$

Hence by (1.7) and (i. 3)

$$(1.8) \quad f(s) = [\xi(s)]^{k+1} \prod_p \sum_{m=0}^r b_m p^{-sm} \quad (\sigma > 1).$$

2. We have

$$(2.1) \quad \sum_{m=0}^r b_m x^m = \prod_{m=1}^r (1 - a_m x),$$

where $a_1, a_2, \dots, a_r$ are the roots of the equation

$$(2.2) \quad y^r + b_1 y^{r-1} + \dots + b_r = 0.$$

Let
$$\alpha = \max(|a_1|, |a_2|, \dots, |a_r|),$$

then, since $|a_1| |a_2| \dots |a_r| = |b_r| \geq 1,$

$$(2.3) \quad \alpha \geq 1.$$

Let

$$(2.4) \quad \lambda_{m,n} = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) a_m^d,$$

then

$$(2.5) \quad |\lambda_{m,n}| \leq \alpha^n$$

and

$$(2.6) \quad \sum_{n|\nu} n \lambda_{m,n} = a_m^\nu.$$

It follows from (2.3) and (2.5) that

$$\sum_{n=1}^{\infty} \left[|\lambda_{m,n}| \sum_{\nu=1}^{\infty} \frac{|x|^{n\nu}}{\nu} \right]$$

is convergent for $|x| < 1/\alpha$. Hence

$$\begin{aligned} \sum_{n=1}^{\infty} \lambda_{m,n} \log(1-x^n) &= - \sum_{n=1}^{\infty} \left[\lambda_{m,n} \sum_{\nu=1}^{\infty} \frac{x^{n\nu}}{\nu} \right] = - \sum_{h=1}^{\infty} \left[x^h \sum_{n|h} \lambda_{m,n} \cdot \frac{n}{h} \right] \\ &= - \sum_{h=1}^{\infty} \left[\frac{x^h}{h} \sum_{n|h} n \lambda_{m,n} \right] = - \sum_{h=1}^{\infty} \frac{x^h a_m^h}{h} = \log(1 - a_m x) \quad \left(|x| < \frac{1}{\alpha} \right) \end{aligned}$$

and

$$(2.7) \quad 1 - a_m x = \prod_{n=1}^{\infty} (1 - x^n)^{\lambda_{m,n}} \quad \left(|x| < \frac{1}{\alpha} \right).$$

Hence by (2.1)

$$(2.8) \quad \sum_{m=0}^r b_m x^m = \prod_{n=1}^{\infty} (1 - x^n)^{\kappa_n} \quad \left(|x| < \frac{1}{\alpha} \right),$$

where

$$(2.9) \quad \kappa_n = \sum_{m=1}^r \lambda_{m,n} = \frac{1}{n} \sum_{d|n} \left[\mu\left(\frac{n}{d}\right) \sum_{m=1}^r a_m^d \right].$$

3. LEMMA. Let $a_1, a_2, \dots, a_r$ be any numbers whose elementary symmetrical functions are integers. Then, for every prime p and every

natural number h ,

$$(3.1) \quad \sum_{m=1}^r a_m^{ph} \equiv \sum_{m=1}^r a_m^{h-1} \pmod{p^h}.$$

We prove this by induction.

(a) *Proof** for $h = 1$. By the multinomial theorem

$$(3.2) \quad \left(\sum_{m=1}^r a_m \right)^p = \sum_{m=1}^r a_m^p + \sum' \left[\frac{p!}{n_1! n_2! \dots n_r!} \sum_i \beta_i(n_1, n_2, \dots, n_r) \right],$$

where the sum Σ' is taken over all the sets $n_1, n_2, \dots, n_r$ for which $n_1 + n_2 + \dots + n_r = p$ and $0 \leq n_1 \leq n_2 \leq \dots \leq n_r < p$, and where $\beta_1(n_1, n_2, \dots, n_r), \beta_2(n_1, n_2, \dots, n_r), \dots$ are all the distinct power-products which can be obtained from

$$a_1^{n_1} a_2^{n_2} \dots a_r^{n_r}$$

by permutations of the indices.

Now

$$(3.3) \quad \frac{p!}{n_1! n_2! \dots n_r!} \equiv 0 \pmod{p} \quad (n_1 \leq \dots \leq n_r < p),$$

and $\sum_i \beta_i(n_1, n_2, \dots, n_r)$ is an integer. Hence, by (3.2),

$$\left(\sum_{m=1}^r a_m \right)^p \equiv \sum_{m=1}^r a_m^p \pmod{p},$$

and since

$$\left(\sum_{m=1}^r a_m \right)^p \equiv \sum_{m=1}^r a_m \pmod{p}$$

by Fermat's theorem,

$$\sum_{m=1}^r a_m^p \equiv \sum_{m=1}^r a_m \pmod{p}.$$

(b) *Proof for $h = H+1$, assuming the lemma for $h = H$.* As we assume the lemma for $h = H$, we have

$$(3.4) \quad \sum_{m=1}^r a_m^{pH} \equiv \sum_{m=1}^r a_m^{pH-1} \pmod{p^H}$$

and

$$(3.5) \quad \sum_i [\beta_i(n_1, n_2, \dots, n_r)]^{p^H} \equiv \sum_i [\beta_i(n_1, n_2, \dots, n_r)]^{p^{H-1}} \pmod{p^H}.$$

* See R. Fricke, *Lehrbuch der Algebra*, 405 (7).

Now, if $x \equiv y \pmod{p^H}$, then $x^p \equiv y^p \pmod{p^{H+1}}$, for then there is an integer q such that $x = y + qp^H$, and

$$x^p = y^p + py^{p-1}qp^H + \dots = y^p + q_1p^{H+1},$$

where q_1 is an integer. Taking

$$x = \sum_{m=1}^r \alpha_m^{p^H}, \quad y = \sum_{m=1}^r \alpha_m^{p^{H-1}},$$

we have by (3.4)

$$(3.6) \quad \left(\sum_{m=1}^r \alpha_m^{p^H} \right)^p \equiv \left(\sum_{m=1}^r \alpha_m^{p^{H-1}} \right)^p \pmod{p^{H+1}}.$$

Now

$$\left(\sum_{m=1}^r \alpha_m^{p^H} \right)^p - \sum_{m=1}^r \alpha_m^{p^{H+1}} = \sum_n' \left\{ \frac{p!}{n_1! n_2! \dots n_r!} \sum_i [\beta_i(n_1, n_2, \dots, n_r)]^{p^H} \right\}$$

and

$$\left(\sum_{m=1}^r \alpha_m^{p^{H-1}} \right)^p - \sum_{m=1}^r \alpha_m^{p^H} = \sum_n' \left\{ \frac{p!}{n_1! n_2! \dots n_r!} \sum_i [\beta_i(n_1, n_2, \dots, n_r)]^{p^{H-1}} \right\},$$

hence, by (3.3) and (2.5),

$$\left(\sum_{m=1}^r \alpha_m^{p^H} \right)^p - \sum_{m=1}^r \alpha_m^{p^{H+1}} \equiv \left(\sum_{m=1}^r \alpha_m^{p^{H-1}} \right)^p - \sum_{m=1}^r \alpha_m^{p^H} \pmod{p^{H+1}}.$$

Hence, by (3.6),

$$\sum_{m=1}^r \alpha_m^{p^{H+1}} \equiv \sum_{m=1}^r \alpha_m^{p^H} \pmod{p^{H+1}},$$

which proves the lemma.

4. THEOREM 1*. Let $\alpha_1, \alpha_2, \dots, \alpha_r$ be any numbers whose elementary symmetrical functions are integers. Then the numbers

$$\kappa_n = \frac{1}{n} \sum_{d|n} \left[\mu \left(\frac{n}{d} \right) \sum_{m=1}^r \alpha_m^d \right]$$

are also integers.

Proof. Let p be a prime divisor of n , so that

$$n = p^l \nu, \quad (p, \nu) = 1.$$

Then for every divisor d of n we have $d = p^l \delta$, where $0 \leq l \leq h$ and

* See supplement.

$\delta | \nu$. Hence

$$(4.1) \quad \sum_{d|n} \left[\mu \left(\frac{n}{d} \right) \sum_{m=1}^r \alpha_m^d \right] = \sum_{l=0}^h \sum_{\delta | \nu} \left[\mu \left(\frac{p^h \nu}{p^l \delta} \right) \sum_{m=1}^r \alpha_m^{p^l \delta} \right] \\ = \sum_{\delta | \nu} \sum_{l=0}^h \left[\mu \left(p^{h-l} \cdot \frac{\nu}{\delta} \right) \sum_{m=1}^r \alpha_m^{p^l \delta} \right].$$

Now

$$\mu \left(p^{h-l} \cdot \frac{\nu}{\delta} \right) = \begin{cases} 0 & (l < h-1), \\ -\mu(\nu/\delta) & (l = h-1), \\ \mu(\nu/\delta) & (l = h). \end{cases}$$

Hence

$$\sum_{l=0}^h \left[\mu \left(p^{h-l} \cdot \frac{\nu}{\delta} \right) \sum_{m=1}^r \alpha_m^{p^l \delta} \right] = \mu \left(\frac{\nu}{\delta} \right) \left[\sum_{m=1}^r (\alpha_m^{\delta})^{p^h} - \sum_{m=1}^r (\alpha_m^{\delta})^{p^{h-1}} \right],$$

hence by the lemma and (4.1)

$$\sum_{d|n} \left[\mu \left(\frac{n}{d} \right) \sum_{m=1}^r \alpha_m^d \right] \equiv 0 \pmod{p^h},$$

Since this is true for every prime divisor p of n , we have

$$\sum_{d|n} \left[\mu \left(\frac{n}{d} \right) \sum_{m=1}^r \alpha_m^d \right] \equiv 0 \pmod{n},$$

which proves the theorem.

In the case

$$\alpha = 1$$

$$(2.9) \quad |\kappa_n| \leq \frac{1}{n} \sum_{d|n} r = \frac{rd(n)}{n},$$

where $d(n)$ is the number of the divisors of n . Hence $\kappa_n \rightarrow 0$, and by Theorem 1

$$(4.2) \quad \kappa_n = 0 \quad (n > N),$$

and we can write instead of (2.8)

$$(4.3) \quad \sum_{m=0}^r b_m x^m = \prod_{n=1}^N (1-x^n)^{\kappa_n}.$$

On this formula a new proof of two known algebraic theorems can be based, as will be shown in the appendix.

5. We return to the function $f(s)$ defined by (i.3). By (2.3) we have either $\alpha = 1$ or $\alpha > 1$. In the first case we shall call $f(s)$ a *function of the first class*, in the second a *function of the second class*.

THEOREM 2. *The functions of the first class are meromorphic.*

Proof. Putting $x = p^{-s}$ in (4.3), we have

$$\sum_{m=0}^r b_m p^{-sm} = \prod_{n=1}^N (1 - p^{-sn})^{\kappa_n}.$$

Hence by (1.8)

$$f(s) = [\zeta(s)]^{k+1} \prod_{n=1}^N \prod_p (1 - p^{-sn})^{\kappa_n} = [\zeta(s)]^{k+1} \prod_{n=1}^N [\zeta(ns)]^{-\kappa_n},$$

which proves the theorem.

6. THEOREM 3. *The functions of the second class have in the half-plane $\sigma > 0$ no singularities other than poles.*

Proof. Let δ be any positive number less than 1, and take the integer

$$(6.1) \quad M > a^{1/\delta}.$$

By (2.9)

$$(6.2) \quad |\kappa_n| \leq r a^n;$$

hence, for $p > M$ and $\sigma > \delta$,

$$\left| \frac{\kappa_n}{p^{ns}} \right| \leq r \left(\frac{a}{p^\sigma} \right)^n \quad \text{and} \quad \frac{a}{p^\sigma} < \frac{a}{M^\delta} < 1,$$

so that

$$(6.3) \quad \sum_{n=1}^{\infty} \left| \frac{\kappa_n}{p^{ns}} \right| \leq r \frac{a}{p^\sigma} \frac{1}{1 - ap^{-\sigma}} < \frac{ra}{1 - aM^{-\delta}} \frac{1}{p^\sigma}.$$

Now $\Sigma(1/p^\sigma)$ is convergent for $\sigma > 1$; hence, by (6.3),

$$\sum_{p > M} \sum_{n=1}^{\infty} \left| \frac{\kappa_n}{p^{ns}} \right|$$

is convergent for $\sigma > 1$. Hence

$$(6.4) \quad \prod_{p > M} \prod_{n=1}^{\infty} (1 - p^{-ns})^{\kappa_n} = \prod_{n=1}^{\infty} \left[\prod_{p > M} (1 - p^{-ns}) \right]^{\kappa_n} \quad (\sigma > 1).$$

Putting $x = p^{-s}$ in (2.8), we have

$$\sum_{m=0}^r b_m p^{-ms} = \prod_{n=1}^{\infty} (1 - p^{-ns})^{\kappa_n} \quad (p > M, \sigma \geq \delta).$$

Hence, by (6.4),

$$(6.5) \quad \prod_{p > M} \sum_{m=0}^{\prime} b_m p^{-ms} = \prod_{n=1}^{\infty} [\xi_M(ns)]^{-\kappa_n} \quad (\sigma > 1),$$

where

$$(6.6) \quad \xi_M(ns) = \frac{1}{\prod_{p > M} (1-p^{-ns})} = \prod_{p \leq M} (1-p^{-ns}) \cdot \xi(ns).$$

Hence, by (1.8),

$$(6.7) \quad f(s) = [\xi(s)]^{k+1} \prod_{p \leq M} \sum_{m=0}^{\prime} b_m p^{-ms} \prod_{n=1}^{\infty} [\xi_M(ns)]^{-\kappa_n} = f_1(s) f_2(s) f_3(s),$$

where

$$(6.8) \quad \begin{cases} f_1(s) = \prod_{p \leq M} \sum_{m=0}^{\prime} b_m p^{-ms} \\ f_2(s) = [\xi(s)]^{k+1} \prod_{n=1}^{[1/\delta]} [\xi_M(ns)]^{-\kappa_n} \\ f_3(s) = \prod_{n=[1/\delta]+1}^{\infty} [\xi_M(ns)]^{-\kappa_n} \end{cases}$$

Here $f_1(s)$ is an integral function, $f_2(s)$ is a meromorphic function, and the product for $f_3(s)$ converges, as we shall show, uniformly for $\sigma \geq \delta$. In fact, for $n \geq [1/\delta] + 1 > 1/\delta$ and $\sigma \geq \delta$, we have

$$(6.9) \quad |\xi_M(ns) - 1| \leq \sum_{m=M+1}^{\infty} m^{-n\delta} < \int_M^{\infty} x^{-n\delta} dx = \frac{1}{n\delta-1} \frac{1}{M^{n\delta-1}},$$

and, by (6.2) and (6.1),

$$\frac{|\kappa_n|}{M^{n\delta}} \leq r \left(\frac{\alpha}{M^\delta} \right)^n, \quad \frac{\alpha}{M^\delta} < 1,$$

so that

$$\sum_{n=[1/\delta]+1}^{\infty} \frac{|\kappa_n|}{M^{n\delta}} \frac{M}{n\delta-1}$$

converges. Hence, by (6.9),

$$\sum_{n=[1/\delta]+1}^{\infty} |\kappa_n| |\xi_M(ns) - 1|$$

converges uniformly, and, therefore, so does

$$\prod_{n=[1/\delta]+1}^{\infty} [\xi_M(ns)]^{-\kappa_n}.$$

Hence $f_3(s)$ is regular in the half-plane $\sigma > \delta$, and $f(s)$ has there (and, in consequence, in the half-plane $\sigma > 0$) no singularities other than poles.

7. THEOREM 4. *The imaginary axis is the natural boundary of every function $f(s)$ of the second class.*

Proof. It is sufficient to show that $f(s)$ has a zero in every neighbourhood of every point iu (u real), or in every square

$$(7.1) \quad 0 < \sigma < \epsilon, \quad u < t < u + \epsilon.$$

Now at least one of the numbers $|a_1|, |a_2|, \dots, |a_r|$ —say $|a_1|$ —is equal to α , and $\alpha > 1$. So we have

$$(7.2) \quad a_1 = e^{\beta + i\gamma}, \quad \alpha = |a_1| = e^\beta, \quad \beta > 0.$$

Take, for instance, $u > 0$. Then, if we choose the integer ν large enough, we have

$$(7.3) \quad \frac{1}{\nu} < \epsilon,$$

$$(7.4) \quad \frac{2\pi}{\nu\beta} < \epsilon,$$

$$(7.5) \quad \varpi(e^{2\nu\beta}) - \varpi(e^{\nu\beta}) > e^{\nu\beta},$$

where $\varpi(x)$ is the number of the primes less than or equal to x ; and

$$(7.6) \quad e^{\nu\beta} > 2\nu N[2\nu(u + \epsilon)],$$

where $N(x)$ is the number of the zeros of $\xi(s)$ for which $0 < t < x$. The inequalities (7.5) and (7.6) follow from the known results $\varpi(x) \sim x/\log x$ and $N(x) = O(x \log x)^*$.

By (7.3) the rectangle

$$(7.7) \quad \frac{1}{2\nu} \leq \sigma < \frac{1}{\nu}, \quad u < t < u + \epsilon$$

is contained in the square (7.1).

Now take

$$(7.8) \quad \delta = \frac{1}{2\nu}, \quad M = [\alpha^{1/\delta}] + 1 = [e^{2\nu\beta}] + 1.$$

Then $f_3(s)$ is regular in the rectangle (7.7). Let Z denote the number of distinct zeros of $f_1(s)$, P the number of distinct poles of $f_2(s)$ in this rectangle. Then it is sufficient to show that

$$(7.9) \quad Z > P.$$

For, if this is true, then there must be a zero of $f_1(s)$ in this rectangle

* It is sufficient to know that $\{\varpi(x)/\sqrt{x}\} \rightarrow \infty$ and $N(x) = O(e^{\epsilon x})$ for every positive ϵ .

[and *a fortiori* in the square (7.1)], which is not a pole of $f_2(s)$, and which is, therefore, by (6.7), a zero of $f(s)$.

Now, by (2.1),

$$\sum_{m=0}^{\infty} b_m p^{-ms} = 0$$

if $p^s = a_1$, which is the case for

$$s = \frac{\beta + i\gamma}{\log p} + \frac{2\pi im}{\log p}$$

or

$$(7.10) \quad \sigma = \frac{\beta}{\log p}, \quad t = \frac{\gamma}{\log p} + \frac{2\pi m}{\log p},$$

where m is any integer. Hence, by (6.8), all those points (7.10) for which $p \leq M$ are zeros of $f_1(s)$. If, moreover,

$$(7.11) \quad e^{\nu\beta} < p \leq e^{2\nu\beta},$$

we have, by (7.4),

$$\frac{1}{2\nu} \leq \frac{\beta}{\log p} < \frac{1}{\nu}, \quad \frac{2\pi}{\log p} < \epsilon,$$

which mean that, for each prime p which satisfies (7.11), at least one of the numbers (7.10) is in the rectangle (7.7). Now the number of these primes p is $\varpi(e^{2\nu\beta}) - \varpi(e^{\nu\beta})$; hence

$$(7.12) \quad Z \geq \varpi(e^{2\nu\beta}) - \varpi(e^{\nu\beta}).$$

On the other hand, each singularity of $f_2(s)$ must be a zero of one of the functions $\zeta_M(s)$, $\zeta_M(2s)$, ..., $\zeta_M(2\nu s)$, which have the same zeros in the half-plane $\sigma > 0$ as $\zeta(s)$, $\zeta(2s)$, ..., $\zeta(2\nu s)$. Hence, since $2\nu = 1/\delta$,

$$(7.13) \quad P \leq \sum_{n=1}^{1/\delta} P_n,$$

where P_n is the number of the zeros of $\zeta(ns)$ in the rectangle (7.7), or the number of the zeros of $\zeta(s)$ in the rectangle

$$\frac{n}{2\nu} \leq \sigma < \frac{n}{\nu}, \quad un < t < (u + \epsilon)n.$$

Hence, obviously,

$$P_n \leq N[(u + \epsilon)n] \leq N[(u + \epsilon)/\delta],$$

and, by (7.13) and (7.8),

$$(7.14) \quad P \leq 2\nu N[2\nu(u+\epsilon)].$$

From (7.12), (7.5), (7.6), and (7.14) we obtain (7.9), which proves the theorem.

8. THEOREM 5. *If $l \geq 2$ and $m \geq 2$, while l and m are not both equal to 2, then $f_{l,m}(s)$ is a function of the second class.*

Proof. In this case, obviously,

$$2l-1 < m^{l-1} + m^{l-2} + \dots + m + 1;$$

hence

$$(m-1)(2l-1) < (m-1)(m^{l-1} + m^{l-2} + \dots + m + 1) = m^l - 1$$

and

$$(8.1) \quad (m-1)(l-1) < m^l - (m-1)l - 1.$$

Now by (i.4), k being the degree of the polynomial $F_{l,m}(h)$,

$$(8.2) \quad F_{l,m}(1) = m^l, \quad k = (m-1)l.$$

Hence, by (1.5),

$$(8.3) \quad b_1 = m^l - (m-1)l - 1.$$

By (i.4), $(h+1)(h+2) \dots (h+m-1)$ is a divisor of $F_{l,m}(h)$. Hence, by (1.1),

$$a_\nu = 0 \quad (\nu < m-1),$$

and, by (1.2) and (8.2),

$$\sum_{\nu=0}^k b_\nu x^\nu = \sum_{\nu=0}^k a_\nu \nu! (1-x)^{k-\nu}$$

is a polynomial of degree

$$(8.4) \quad r \leq k - (m-1) = (m-1)(l-1).$$

$$\text{Now, by (2.1),} \quad |b_1| = \left| \sum_{\nu=1}^r a_\nu \right| \leq ra;$$

hence, by (8.3), (8.1), and (8.4),

$$ra > r, \quad a > 1,$$

which was to be proved.

Appendix.

New proof of two algebraic theorems.

THEOREM 6*. If, for all the roots $\alpha_1, \alpha_2, \dots, \alpha_r$ of an equation

$$\sum_{m=0}^r b_m y^{r-m} = 0 \quad (b_0 = 1, b_1, b_2, \dots, b_r \text{ integers}),$$

we have $|\alpha_v| = 1$, then $\alpha_1, \alpha_2, \dots, \alpha_r$ are roots of unity.

Proof. By (4.3) the zeros of $\sum_{m=0}^r b_m x^m$ are roots of unity. Now these zeros are $1/\alpha_1, 1/\alpha_2, \dots, 1/\alpha_r$, and so $\alpha_1, \alpha_2, \dots, \alpha_r$ are also roots of unity.

THEOREM 7†. The equation

$$P(y) = y^{\phi(n)} + \alpha_1 y^{\phi(n)-1} + \dots + \alpha_{\phi(n)} = 0,$$

whose roots are all the primitive n -th roots of unity, is irreducible.

Proof. Let
$$P_1(y) = \sum_{m=0}^r b_m y^{r-m} \quad (b_0 = 1, r \geq 1)$$

be a divisor of $P(y)$, and let

$$P_2(x) = x^r P_1\left(\frac{1}{x}\right) = \sum_{m=0}^r b_m x^m.$$

Then, by (4.3),

$$(a.1) \quad P_2(x) = \prod_{m=1}^N (1 - x^m)^{\kappa_m}.$$

Now consider $P_2(\omega + \delta)$, where ω is a primitive n -th root of unity, for $\delta \rightarrow 0$. We have

$$1 - (\omega + \delta)^m \rightarrow c_m \neq 0 \quad [m \not\equiv 0 \pmod{n}],$$

$$1 - (\omega + \delta)^m \sim c_m \delta, \quad c_m \neq 0 \quad [m \equiv 0 \pmod{n}];$$

hence, by (a.1),

$$(a.2) \quad P_2(\omega + \delta) \sim c \delta^\kappa,$$

where

$$c = \prod_{m=1}^N c_m^{\kappa_m} \neq 0, \quad \kappa = \sum_{\nu=1}^{[N/n]} \kappa_{\nu n}.$$

(a.2) shows that $P_2(\omega) = 0$ if $\kappa > 0$, and *vice versa*. Since κ depends only on n (and not on the particular choice of ω), either all or none of the

* Kronecker, *Journal für Math.*, 53 (1857), 173.

† Dedekind, *Journal für Math.*, 54 (1857), 27-30.

primitive n -th roots of unity are zeros of $P_2(x)$. The latter is impossible, since it follows easily from the definition of $P_2(x)$ that all its zeros are primitive n -th roots of unity. So all these are zeros of $P_2(x)$; and this means that r cannot be less than $\phi(n)$. We could have $r < \phi(n)$, however, if $P(y)$ were reducible.

SUPPLEMENT.

(Added 14 January, 1928.)

Since this paper was written I have discovered that Theorem 1 has been previously proved (in a different way) by Jänichen [*Berliner Sitzungsberichte*, 20 (1921), 23-29]. Combining the results of § 2 with an idea taken from Jänichen's memoir, I find the following short proof, which makes § 3 unnecessary.

It is obvious that κ_1 is an integer. Suppose that $\kappa_1, \kappa_2, \dots, \kappa_n$ are integers (for a particular value of n). To show that κ_{n+1} is then also an integer, we expand as power series both sides of the equation

$$\left(\sum_{m=1}^r b_m x^m \right) \prod_{h=1}^n (1-x^h)^{-\kappa_h} = \prod_{h=n+1}^{\infty} (1-x^h)^{\kappa_h},$$

which follows from (2.8). The expression on the left gives a power series with integral coefficients, and, in the expansion of the product on the right, the coefficient of x^{n+1} is $-\kappa_{n+1}$. Hence, since both sides give the same power series, κ_{n+1} is an integer.

ON THE REMAINDER IN THE FORMULA FOR $N(T)$, THE
NUMBER OF ZEROS OF $\zeta(s)$ IN THE STRIP $0 < t < T$.

By E. C. TITCHMARSH.

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1. We denote, as usual, by $N(T)$ the number of zeros $\beta + i\gamma$ of $\zeta(s)$ for which $0 < \gamma < T$, if T is not equal to any γ ; and otherwise

$$N(T) = \lim_{\epsilon \rightarrow 0} \frac{1}{2} \{N(T+\epsilon) + N(T-\epsilon)\}.$$

It is well known that

$$(1.1) \quad N(T) = \frac{1}{2\pi} T \log T - \frac{1 + \log 2\pi}{2\pi} T + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right),$$

where $\pi S(T) = \arg \zeta(\frac{1}{2} + iT)$, the value of the argument being found by continuous variation along the straight line joining 2 , $2 + iT$, $\frac{1}{2} + iT$, beginning with the value zero. The function $S(t)$ is of an oscillatory character, and our knowledge of its behaviour is far from complete. However, if the Riemann hypothesis is true, we have the following results, all due to Littlewood*. We write

$$S_1(T) = \int_0^T S(t) dt + c_1,$$

and generally
$$S_n(T) = \int_0^T S_{n-1}(t) dt + c_n,$$

where the numbers c_n are constants, the value of which does not concern us here. Then

$$(1.2) \quad S(t) = O(\log t / \log \log t),$$

$$(1.3) \quad S_n(t) = O\{\log t / (\log \log t)^{n+1}\},$$

* J. E. Littlewood, *Proc. Camb. Phil. Soc.*, 22 (1924), 295–318; and *Proc. London Math. Soc.* (2), 24 (1925), 175–201. We shall refer to these papers as L (1) and L (2).

and if $T^a \leq H \leq T$, $0 < a \leq 1$,

$$(1.4) \quad \frac{1}{H} \int_T^{T+H} |S(t)| dt = O(\log \log T),$$

$$(1.41) \quad \frac{1}{T} \int_0^T |S(t)| dt = O(\log \log T),$$

$$(1.5) \quad \frac{1}{H} \int_T^{T+H} |S_n(t)|^2 dt = O(1) \quad (n > 0),$$

$$(1.51) \quad \frac{1}{T} \int_0^T |S_n(t)|^2 dt = O(1) \quad (n > 0).$$

Throughout the present paper the truth of the Riemann hypothesis is assumed. Our first result is

THEOREM I. *There is a positive constant A such that*

$$(1.6) \quad \frac{1}{T} \int_0^T |S(t)|^2 dt > A \log \log T \quad (T > T_0).$$

This is proved by an argument like that which is used to prove Bessel's inequality in the theory of Fourier series. It does not fit in exactly with any of Littlewood's results, since he does not consider $|S(t)|^2$. All that seems to be known about this mean-value on the other side is that it is $O(\log T)$; this obviously follows from (1.2) and (1.41). We may add to (1.41) the inequality

$$(1.7) \quad \frac{1}{T} \int_0^T |S(t)| dt > A.$$

The proof of this is trivial, but I have failed to prove anything more.

Let us write, as usual,

$$-\frac{\xi'(s)}{\xi(s)} = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s}$$

and $\Lambda_r(n) = \Lambda(n)/\log^r n$; and let

$$C = \frac{1}{2\pi^2} \sum_{n=2}^{\infty} \frac{\Lambda_2^2(n)}{n}.$$

Then we have

THEOREM II. *There are constants A , B , such that*

$$(1.81), (1.82) \quad C - \frac{A}{\log^2 T} < \frac{1}{T} \int_0^T |S_1(t)|^2 dt < C + \frac{B}{\log \log T} \quad (T > T_0).$$

This is a more precise form of (1.51) for $n=1$. The left-hand inequality (1.81) follows from the "Bessel's inequality" method, while (1.82) is obtained by combining (1.2) with the method of L(2). Similar results might be obtained for $n>1$; but in this case there is nothing new to prove, since Littlewood's work really leads to a precise limiting value.

Both theorems may be extended to integrals over $(T, T+H)$.

2. We begin by proving some lemmas.

2.1. LEMMA α . If $\phi(s) = \sum a_n n^{-s}$ ($\sigma > 1$), and $a_n = O(1)$, then

$$\int_0^T \phi(2+it) n^{2+it} dt = a_n T + O(n^2),$$

$$\int_0^T \phi(2+it) n^{-2-it} dt = O(n^{-2}).$$

$$\begin{aligned} \text{We have } \int_0^T \phi(2+it) n^{2+it} dt &= \int_0^T \sum_{m=1}^{\infty} \frac{a_m}{m^{2+it}} n^{2+it} dt \\ &= a_n T + n^2 \sum' \frac{a_m}{m^2} \frac{(n/m)^{iT} - 1}{i \log n/m} \\ &= a_n T + n^2 O \left\{ \sum' \frac{1}{m^2 |\log n/m|} \right\}. \end{aligned}$$

The last sum is bounded*. This gives the first result. Also

$$\int_0^T \phi(2+it) n^{-2-it} dt = \frac{1}{n^2} \sum_{m=1}^{\infty} \frac{a_m}{m^2} \frac{(mn)^{-iT} - 1}{-i \log mn},$$

whence the second result is obvious.

2.2. LEMMA β . As $t \rightarrow \infty$

$$\int_{\frac{1}{2}}^2 |\log \zeta(\sigma+it)| d\sigma = O(\log t).$$

This is true without any hypothesis, for

$$\begin{aligned} \int_{\frac{1}{2}}^2 |\log \zeta(\sigma+it)| d\sigma &= \int_{\frac{1}{2}}^2 \left| \log |\zeta(\sigma+it)| - \log t + \log t + i \arg \zeta(\sigma+it) \right| d\sigma \\ &\leq \int_{\frac{1}{2}}^2 \{ \log t - \log |\zeta(\sigma+it)| \} d\sigma + \frac{3}{2} \log t + \int_{\frac{1}{2}}^2 |\arg \zeta(\sigma+it)| d\sigma, \end{aligned}$$

* E. Landau, *Handbuch*, 806.

since $\log |\zeta(\sigma + it)| < \log t$, for $\frac{1}{2} \leq \sigma \leq 2$ and t large enough. Now it is known that $\arg \zeta(s) = O(\log t)$ for $\sigma \geq \frac{1}{2}$, and*

$$\int_{\frac{1}{2}}^2 \log |\zeta(\sigma + it)| d\sigma = O(\log t).$$

Hence the result.

2. 3. LEMMA γ .

$$\int_0^T S(t) \sin(t \log n) dt = -\frac{1}{2\pi} \frac{\Lambda_1(n)}{\sqrt{n}} T + O(n^{\frac{1}{2}} \log T).$$

Consider the integral $\int \log \zeta(s) n^s ds$

taken round the rectangle $\frac{1}{2}, 2, 2+iT, \frac{1}{2}+iT$, with suitable indentations to exclude the singularities of $\log \zeta(s)$. The integrand is regular inside the contour, and the radii of the indentations may be made to tend to zero. Hence

$$\begin{aligned} i \int_0^T \log \zeta(\tfrac{1}{2} + it) n^{\frac{1}{2} + it} dt \\ = \int_{\frac{1}{2}}^2 \log \zeta(\sigma) n^{\sigma} d\sigma + i \int_0^T \log \zeta(2 + it) n^{2 + it} dt - \int_{\frac{1}{2}}^2 \log \zeta(\sigma + iT) n^{\sigma + iT} d\sigma \\ = I_1 + I_2 + I_3. \end{aligned}$$

Obviously $I_1 = O(n^2)$.

By Lemma α , $I_2 = i\Lambda_1(n) T + O(n^2)$.

By Lemma β , $I_3 = O(n^2 \log T)$.

Hence $\int_0^T \log \zeta(\tfrac{1}{2} + it) n^{\frac{1}{2} + it} dt = \Lambda_1(n) T + O(n^2 \log T)$.

Similarly, taking $\int \log \zeta(s) n^{-s} ds$

round the same contour, we obtain

$$\int_0^T \log \zeta(\tfrac{1}{2} + it) n^{-\frac{1}{2} - it} dt = O(n^{-\frac{1}{2}} \log T).$$

Hence $\int_0^T \log \zeta(\tfrac{1}{2} + it) \sin(t \log n) dt = \frac{\Lambda_1(n)}{2i\sqrt{n}} T + O(n^{\frac{1}{2}} \log T)$,

and the imaginary part of this gives the result stated.

* L (1), § 12.

2.4. *Proof of Theorem I.* We have

$$\begin{aligned} 0 &\leq \int_0^T \left\{ S(t) + \frac{1}{\pi} \sum_{n=2}^N \frac{\Lambda_1(n)}{\sqrt{n}} \sin(t \log n) \right\}^2 dt \\ &= \int_0^T \{S(t)\}^2 dt + \frac{2}{\pi} \sum_{n=2}^N \frac{\Lambda_1(n)}{\sqrt{n}} \int_0^T S(t) \sin(t \log n) dt \\ &\quad + \frac{1}{\pi^2} \sum_{n=2}^N \frac{\Lambda_1^2(n)}{n} \int_0^T \sin^2(t \log n) dt \\ &\quad + \frac{1}{\pi^2} \sum_{m \neq n} \sum \frac{\Lambda_1(m) \Lambda_1(n)}{\sqrt{mn}} \int_0^T \sin(t \log m) \sin(t \log n) dt. \end{aligned}$$

$$\text{Now} \quad \int_0^T \sin^2(t \log n) dt = \frac{1}{2} T - \frac{\sin(2T \log n)}{4 \log n} = \frac{1}{2} T + O(1),$$

$$\begin{aligned} \text{and} \quad \int_0^T \sin(t \log m) \sin(t \log n) dt &= \frac{1}{2} \left(\frac{\sin(T \log m/n)}{\log m/n} - \frac{\sin(T \log mn)}{\log mn} \right) \\ &= O(|\log m/n|^{-1}). \end{aligned}$$

Hence, using Lemma γ , and the fact that $\Lambda_1(n) = O(1)$,

$$\begin{aligned} 0 &\leq \int_0^T \{S(t)\}^2 dt - \frac{T}{\pi^2} \sum_2^N \frac{\Lambda_1^2(n)}{n} + \sum_2^N \Lambda_1(n) O(n \log T) \\ &\quad + \frac{T}{2\pi^2} \sum_2^N \frac{\Lambda_1^2(n)}{n} + \sum_2^N O\left(\frac{1}{n}\right) + \sum_{m \neq n} O\left(\frac{1}{\sqrt{mn} |\log m/n|}\right). \end{aligned}$$

The third term is $O(N^2 \log T)$, the fifth term is $O(\log N)$, and, as in L(2) Lemma 7, the last term is $O(N \log N)$. Hence

$$\begin{aligned} \int_0^T \{S(t)\}^2 dt &> \frac{T}{2\pi^2} \sum_2^N \frac{\Lambda_1^2(n)}{n} + O(N^2 \log T) \\ &> \frac{T}{2\pi^2} \sum_{p < N} \frac{1}{p} + O(N^2 \log T) \\ &> AT \log \log N + O(N^2 \log T)^*. \end{aligned}$$

Taking $N = [T^{\frac{1}{2}}]$ the result follows.

2.5. Since

$$\int_0^T |S(t)| dt > \left| \int_0^T S(t) \sin(t \log 2) dt \right|,$$

(1.7) also follows from Lemma γ .

* E. Landau, *Handbuch*, 197.

3. *Proof of (1.81).* We have

$$\pi S_1(t) = R \{F(\tfrac{1}{2} + it)\},$$

$$\text{where} \quad F(s) = \int_{\sigma}^{\infty} \log \xi(\sigma_1 + it) d\sigma_1 = - \sum_{n=2}^{\infty} \frac{\Lambda_2(n)}{n^s}.$$

We now consider the integral

$$\int_0^T \left\{ S_1(t) - \frac{1}{\pi} \sum_{n=2}^N \frac{\Lambda_2(n)}{n} \cos(t \log n) \right\}^2 dt,$$

as in § 2.4. In this case the result is obviously

$$\begin{aligned} \int_0^T \{S_1(t)\}^2 dt &> \frac{T}{2\pi^2} \sum_2^N \frac{\Lambda_2^2(n)}{n} + O(N^2 \log T) \\ &= CT - \frac{T}{2\pi^2} \sum_{N+1}^{\infty} \frac{\Lambda_2^2(n)}{n} + O(N^2 \log T). \end{aligned}$$

$$\text{Now} \quad \sum_{N+1}^{\infty} \frac{\Lambda_2^2(n)}{n} = \sum_{m=1}^{\infty} \frac{1}{n^2} \sum_{p^m > N} \frac{1}{p^m \log^2 p^m},$$

$$\begin{aligned} \text{and} \quad \sum_{p > N} \frac{1}{p \log^2 p} &= \sum_{r=\pi(N)+1}^{\infty} \frac{1}{p_r \log^2 p_r} = O \left\{ \sum_{(\pi(N))}^{\infty} \frac{1}{r \log r \cdot \log^2(r \log r)} \right\} \\ &= O \left\{ \sum_{(\pi(N))}^{\infty} \frac{1}{r \log^3 r} \right\} = O \{ \log^{-2} \pi(N) \} = O(\log^{-2} N). \end{aligned}$$

Also for $m > 1$,

$$\sum_{p^m > N} \frac{1}{p^m \log^2 p^m} < \sum_{n^m > N} \frac{1}{n^m} < \int_{N^{1/m}}^{\infty} \frac{dx}{x^m} + \frac{1}{N} = \frac{N^{1/m-1}}{m-1} + \frac{1}{N} < \frac{2}{\sqrt{N}}.$$

$$\text{Hence} \quad \sum_{m=2}^{\infty} \frac{1}{m^2} \sum_{p^m > N} \frac{1}{p^m \log^2 p^m} = O\left(\frac{1}{\sqrt{N}}\right).$$

$$\text{Thus} \quad \int_0^T \{S_1(t)\}^2 dt > CT + O(T \log^{-2} N) + O(N^2 \log T),$$

and, taking $N = [T^{\frac{1}{2}}]$, the result follows.

4. For the proof of (1.82) we require another lemma.

LEMMA δ . *Let y be real, and let*

$$\psi(s) = \sum_{\rho} \int_s^{\frac{3}{2} + it} y^{s_1 - \rho} \Gamma(\rho - s_1) ds_1 \quad \left(\tfrac{1}{2} < \sigma < \tfrac{9}{8}\right),$$

the series, which is absolutely convergent, being taken over all the com-

plex zeros ρ of $\zeta(s)$. Then, if $y = O(t^{-A})$,

$$\int_{\frac{1}{2}}^{\frac{3}{8}} |\psi(\sigma + it)| d\sigma = O\left(\frac{1}{\log \log t}\right).$$

We have, if $s_1 = \sigma_1 + it$, $\frac{1}{2} < \sigma_1 < \frac{9}{8}$, $\rho = \frac{1}{2} + i\gamma$,

$$|\Gamma(\rho - s_1)| = \left| \frac{\Gamma(\rho - s_1 + 1)}{\rho - s_1} \right| < A \frac{e^{-A|t-\gamma|}}{\sqrt{\{(\sigma_1 - \frac{1}{2})^2 + (t - \gamma)^2\}}}$$

by the asymptotic formula for the Γ -function, since $\frac{3}{8} < R(\rho - s_1 + 1) < 1$. Hence

$$\left| \sum_{\rho} y^{\rho_1 - \rho} \Gamma(\rho - s_1) \right| < A y^{\sigma_1 - \frac{1}{2}} \sum_{\gamma} \frac{e^{-A|t-\gamma|}}{\sqrt{\{(\sigma_1 - \frac{1}{2})^2 + (t - \gamma)^2\}}}.$$

Let us write $\mu(t) = 1/\log \log t$. Then

$$\begin{aligned} \Sigma &= \sum_{\gamma, |t-\gamma| < \mu(t)} + \sum_{\mu(t) \leq |t-\gamma| < 1} + \sum_{|t-\gamma| \geq 1} \\ &= \Sigma_1 + \Sigma_2 + \Sigma_3. \end{aligned}$$

Now it follows from (1.1) and (1.2) that

$$N\{t + \mu(t)\} - N(t) = O\{\mu(t) \log t\}.$$

$$\text{Hence } \Sigma_1 < \frac{1}{\sigma_1 - \frac{1}{2}} \sum_{|t-\gamma| < \mu(t)} e^{-A|t-\gamma|}$$

$$< \frac{1}{\sigma_1 - \frac{1}{2}} [N\{t + \mu(t)\} - N\{t - \mu(t)\}]$$

$$= O\left(\frac{\mu(t) \log t}{\sigma_1 - \frac{1}{2}}\right),$$

$$\Sigma_2 < \sum_{\mu(t) \leq |t-\gamma| < 1} \frac{e^{-A|t-\gamma|}}{|t-\gamma|}$$

$$= \sum_{m=1}^{\log \log t-1} \sum_{m\mu(t) \leq |t-\gamma| < (m+1)\mu(t)} \frac{e^{-A|t-\gamma|}}{|t-\gamma|}$$

$$< \sum_{m=1}^{\log \log t} [N\{t + (m+1)\mu(t)\} - N\{t + m\mu(t)\}] \frac{e^{-Am\mu(t)}}{m\mu(t)}$$

$$+ \sum_{m=1}^{\log \log t} [N\{t - m\mu(t)\} - N\{t - (m+1)\mu(t)\}] \frac{e^{-Am\mu(t)}}{m\mu(t)}$$

$$< \sum_{m=1}^{\infty} O(\mu(t) \log t) \cdot \frac{e^{-Am\mu(t)}}{m\mu(t)} = O\left\{\log t \int_1^{\infty} e^{-Ax\mu(t)} \frac{dx}{x}\right\}$$

$$= O\left\{\log t \int_{\mu(t)}^{\infty} e^{-Au} \frac{du}{u}\right\} = O(\log t \log \log \log t),$$

$$\begin{aligned} \text{and} \quad \Sigma_8 &< \sum_{|t-\gamma| \geq 1} e^{-A|t-\gamma|} = \sum_{m=1}^{\infty} \sum_{m \leq |t-\gamma| < m+1} e^{-A|t-\gamma|} \\ &< A \sum_{m=1}^{\infty} \log(t+m+1) e^{-Am} = O(\log t). \end{aligned}$$

Hence

$$\sum_{\rho} y^{s_1-\rho} \Gamma(\rho-s_1) = O\left(\mu(t) \log t \frac{y^{\sigma_1-\frac{1}{2}}}{\sigma_1-\frac{1}{2}}\right) + O(\log t \log \log \log t y^{\sigma_1-\frac{1}{2}}).$$

$$\begin{aligned} \text{Thus} \quad \int_{\frac{1}{2}}^{\frac{9}{8}} |\psi(\sigma+it)| d\sigma &= O\left(\mu(t) \log t \int_{\frac{1}{2}}^{\frac{9}{8}} d\sigma \int_{\sigma}^{\frac{9}{8}} \frac{y^{\sigma_1-\frac{1}{2}}}{\sigma_1-\frac{1}{2}} d\sigma_1\right) \\ &\quad + O\left(\log t \log \log \log t \int_{\frac{1}{2}}^{\frac{9}{8}} d\sigma \int_{\sigma}^{\frac{9}{8}} y^{\sigma_1-\frac{1}{2}} d\sigma_1\right). \end{aligned}$$

$$\begin{aligned} \text{Now} \quad \int_{\frac{1}{2}}^{\frac{9}{8}} d\sigma \int_{\sigma}^{\frac{9}{8}} \frac{y^{\sigma_1-\frac{1}{2}}}{\sigma_1-\frac{1}{2}} d\sigma_1 &< \int_{\frac{1}{2}}^{\infty} d\sigma \int_{\sigma}^{\infty} \frac{y^{\sigma_1-\frac{1}{2}}}{\sigma_1-\frac{1}{2}} d\sigma_1 = \int_{\frac{1}{2}}^{\infty} \frac{y^{\sigma_1-\frac{1}{2}}}{\sigma_1-\frac{1}{2}} d\sigma_1 \int_{\frac{1}{2}}^{\sigma_1} d\sigma \\ &= \int_{\frac{1}{2}}^{\infty} y^{\sigma_1-\frac{1}{2}} d\sigma_1 = \left(\log \frac{1}{y}\right)^{-1} = O\left(\frac{1}{\log t}\right), \end{aligned}$$

$$\text{and similarly} \quad \int_{\frac{1}{2}}^{\frac{9}{8}} d\sigma \int_{\sigma}^{\frac{9}{8}} y^{\sigma_1-\frac{1}{2}} d\sigma_1 = O\left(\frac{1}{\log^2 t}\right).$$

This proves the lemma.

5. *Proof of (1.82).* By Theorem I of L(2),

$$-\frac{\xi'(s)}{\xi(s)} = \Sigma \Lambda(n) n^{-s} e^{-ny} + \sum_{\rho} y^{s-\rho} \Gamma(r-s) + O(y^{\sigma-\frac{1}{2}} \log t)$$

uniformly in $\frac{1}{2} \leq \sigma \leq \frac{9}{8}$, $e^{-yt} \leq y \leq 1$. Integrating from s to $\frac{9}{8}+it$,

$$\log \xi(s) - \log \xi\left(\frac{9}{8}+it\right) = \phi(s) - \phi\left(\frac{9}{8}+it\right) + \psi(s) + O\left(\log t \int_{\sigma}^{\frac{9}{8}} y^{\sigma_1-\frac{1}{2}} d\sigma_1\right),$$

where

$$\phi(s) = \Sigma \Lambda_1(n) n^{-s} e^{-ny}.$$

Hence

$$\begin{aligned} \int_{\frac{1}{2}}^{\frac{9}{8}} \log \xi(\sigma+it) d\sigma &= \int_{\frac{1}{2}}^{\frac{9}{8}} \phi(\sigma+it) d\sigma + \frac{5}{8} \left\{ \log \xi\left(\frac{9}{8}+it\right) - \phi\left(\frac{9}{8}+it\right) \right\} \\ &\quad + \int_{\frac{1}{2}}^{\frac{9}{8}} \psi(\sigma+it) d\sigma + O\left(\log t \int_{\frac{1}{2}}^{\frac{9}{8}} d\sigma \int_{\sigma}^{\frac{9}{8}} y^{\sigma_1-\frac{1}{2}} d\sigma_1\right). \end{aligned}$$

Let $y = T^{-\frac{1}{2}}$ and $T \leq t \leq 2T$. Then the last term is

$$O\left(\log T \cdot y^{\frac{1}{2}} \log^{-2} \frac{1}{y}\right) = O(T^{-\frac{1}{2}}),$$

and, by Lemma δ , the term involving ψ is $O\{\mu(T)\}$. Also

$$\begin{aligned} \log \zeta\left(\frac{9}{8} + it\right) - \phi\left(\frac{9}{8} + it\right) &= \sum \Lambda_1(n) n^{-\frac{9}{8} - it} (1 - e^{-ny}) \\ &= O\left\{\sum n^{-\frac{9}{8}} (1 - e^{-ny})\right\} = O\left(\sum_{ny \leq 1} n^{-\frac{9}{8}} y\right) + O\left(\sum_{ny > 1} n^{-\frac{9}{8}}\right) \\ &= O(y^{\frac{1}{2}}) = O(T^{-\frac{1}{2}}). \end{aligned}$$

Hence
$$\int_{\frac{1}{2}}^{\frac{9}{8}} \log \zeta(\sigma + it) d\sigma = \int_{\frac{1}{2}}^{\frac{9}{8}} \phi(\sigma + it) d\sigma + O\{\mu(T)\}.$$

Also
$$\int_{\frac{9}{8}}^{\infty} \log \zeta(\sigma + it) d\sigma = \int_{\frac{9}{8}}^{\infty} \phi(\sigma + it) d\sigma + O(T^{-\frac{1}{2}}).$$

Hence, adding and taking the real part,

$$\pi S_1(t) = -\sum \Lambda_2(n) n^{-\frac{1}{2}} \cos(t \log n) e^{-ny} + O\{\mu(T)\}.$$

Now, as in Lemma 7, of L(2),

$$\begin{aligned} \int_T^{2T} |\sum \Lambda_2(n) n^{-\frac{1}{2}} \cos(t \log n) e^{-ny}|^2 dt &= \frac{1}{2} T \sum \Lambda_2^2(n) n^{-1} e^{-2ny} + O(y^{-1} \log y^{-1}) \\ &= \frac{1}{2} T \sum \Lambda_2^2(n) n^{-1} e^{-2ny} + O(T^{\frac{1}{2}} \log T), \end{aligned}$$

$$\begin{aligned} \text{and } \sum \Lambda_2^2(n) n^{-1} (1 - e^{-2ny}) &= O\left(\sum_{ny \leq 1} y \log^{-2} n\right) + O\left(\sum_{ny > 1} n^{-1} \log^{-2} n\right) \\ &= O(\log^{-2} y^{-1}) + O(\log^{-1} y^{-1}) \\ &= O(\log^{-1} T). \end{aligned}$$

Hence
$$\int_T^{2T} |\sum \Lambda_2(n) n^{-\frac{1}{2}} \cos(t \log n) e^{-ny}|^2 dt = \pi^2 CT + O(T/\log T).$$

$$\begin{aligned} \text{Hence } \int_T^{2T} |S_1(t)|^2 dt &= \frac{1}{\pi^2} \int_T^{2T} |\sum \Lambda_2(n) n^{-\frac{1}{2}} \cos(t \log n) e^{-ny}|^2 dt \\ &\quad + O\left\{\mu(T) \int_T^{2T} |\sum \Lambda_2(n) n^{-\frac{1}{2}} \cos(t \log n) e^{-ny}| dt\right\} + O\{T\mu^2(T)\} \\ &= CT + O(T/\log T) + O\{T\mu(T)\} + O\{T\mu^2(T)\} \\ &= CT + O\{T\mu(T)\}. \end{aligned}$$

Writing $\frac{1}{2}T$, $\frac{1}{4}T$, in succession for T , and adding, we have, if ν is defined by $2^{-\nu-1}T \leq 4 < 2^{-\nu}T$,

$$\begin{aligned} \int_0^T |S_1(t)|^2 dt &= \sum_{n=1}^{\nu} \int_{2^{-n}T}^{2^{-n+1}T} |S_1(t)|^2 dt + O(1) \\ &= \sum_{n=1}^{\nu} C 2^{-n}T + \sum_{n=1}^{\nu} O\{2^{-n}T\mu(2^{-n}T)\} + O(1) \\ &= CT + O\{T\mu(T)\}. \end{aligned}$$

This completes the proof.

ON THE INTEGRATION OF THE EQUATIONS OF THE
ELECTROMAGNETIC THEORY OF LIGHT

By M. KOURENSKY.

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1. In the electromagnetic theory, Maxwell's equations for the ether are

$$\left. \begin{aligned} \operatorname{rot} H &= \frac{1}{c} \frac{\partial E}{\partial t} \\ \operatorname{rot} E &= -\frac{1}{c} \frac{\partial H}{\partial t} \\ \operatorname{div} E &= 0 \\ \operatorname{div} H &= 0 \end{aligned} \right\} \quad (1)$$

Following Silberstein and Bateman, we may write these equations in the form

$$\begin{aligned} \operatorname{rot} M &= \mp \frac{i}{c} \frac{\partial M}{\partial t}, \\ \operatorname{div} M &= 0, \end{aligned}$$

where M is the complex vector defined by the equation

$$M = H \pm iE.$$

Here i denotes $\sqrt{-1}$, c is the velocity of light, and E and H are the vectors of electric and magnetic force.

By means of Bateman's substitution*

$$\left. \begin{aligned} M_x &= f(\alpha, \beta) \frac{\partial(\alpha, \beta)}{\partial(y, z)} \\ M_y &= f(\alpha, \beta) \frac{\partial(\alpha, \beta)}{\partial(z, x)} \\ M_z &= f(\alpha, \beta) \frac{\partial(\alpha, \beta)}{\partial(x, y)} \end{aligned} \right\} \quad (2)$$

* H. Bateman, *The mathematical analysis of electrical and optical wave-motion* Cambridge (1915), 1, 4, 12.

where $f(\alpha, \beta)$ is an arbitrary function of α and β , these equations transform into the system

$$\left. \begin{aligned} c \frac{\partial(\alpha, \beta)}{\partial(y, z)} &= i \frac{\partial(\alpha, \beta)}{\partial(x, t)} \\ c \frac{\partial(\alpha, \beta)}{\partial(z, x)} &= i \frac{\partial(\alpha, \beta)}{\partial(y, t)} \\ c \frac{\partial(\alpha, \beta)}{\partial(x, y)} &= i \frac{\partial(\alpha, \beta)}{\partial(z, t)} \end{aligned} \right\} \quad (3)$$

By algebraic transformations, system (3) may be expressed in alternative interesting forms; for example,

$$\left. \begin{aligned} \left[\left(\frac{\partial \alpha}{\partial x} \right)^2 + \left(\frac{\partial \alpha}{\partial y} \right)^2 + \left(\frac{\partial \alpha}{\partial z} \right)^2 \right] \frac{\partial \beta}{\partial t} &= \left(\frac{\partial \alpha}{\partial x} \frac{\partial \beta}{\partial x} + \frac{\partial \alpha}{\partial y} \frac{\partial \beta}{\partial y} + \frac{\partial \alpha}{\partial z} \frac{\partial \beta}{\partial z} \right) \frac{\partial \alpha}{\partial t} \\ c \left[\left(\frac{\partial \alpha}{\partial x} \right)^2 + \left(\frac{\partial \alpha}{\partial y} \right)^2 \right] \frac{\partial \beta}{\partial z} &= c \left(\frac{\partial \alpha}{\partial x} \frac{\partial \beta}{\partial x} + \frac{\partial \alpha}{\partial y} \frac{\partial \beta}{\partial y} \right) \frac{\partial \alpha}{\partial z} + i \left(\frac{\partial \alpha}{\partial x} \frac{\partial \beta}{\partial y} - \frac{\partial \alpha}{\partial y} \frac{\partial \beta}{\partial x} \right) \frac{\partial \alpha}{\partial t} \\ c \left[\left(\frac{\partial \beta}{\partial x} \right)^2 + \left(\frac{\partial \beta}{\partial y} \right)^2 \right] \frac{\partial \alpha}{\partial z} &= c \left(\frac{\partial \beta}{\partial x} \frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} \frac{\partial \alpha}{\partial y} \right) \frac{\partial \beta}{\partial z} + i \left(\frac{\partial \beta}{\partial x} \frac{\partial \alpha}{\partial y} - \frac{\partial \beta}{\partial y} \frac{\partial \alpha}{\partial x} \right) \frac{\partial \beta}{\partial t} \end{aligned} \right\}, \quad (3')$$

$$\left. \begin{aligned} J_{yt}^2 + J_{zt}^2 &= c^2 J_{yz}^2 \\ J_{xt}^2 + J_{zt}^2 &= c^2 J_{zx}^2 \\ J_{xt}^2 + J_{zx}^2 &= c^2 J_{xy}^2 \end{aligned} \right\}, \quad (3'')$$

$$\left. \begin{aligned} J_{yz}^2 + J_{zx}^2 + J_{xy}^2 &= 0 \\ J_{xt}^2 + J_{yt}^2 + J_{zt}^2 &= 0 \\ \text{one of equations (3'')} & \end{aligned} \right\}, \quad (3''')$$

where $J_{yt}, J_{zt}, \dots$ denote the Jacobians

$$\frac{\partial(\alpha, \beta)}{\partial(y, t)}, \quad \frac{\partial(\alpha, \beta)}{\partial(z, t)}, \quad \dots$$

2. Since the number of equations in systems (3), (3'), ... is greater than the number of unknown functions α and β , it is necessary to examine the question of their compatibility. According to investigations by E. von Weber* and myself†, the conditions of compatibility will

* E. von Weber, *Math. Annalen*, 49 (1897), 548.

† M. Kourensky, "Sur l'intégration des équations différentielles aux dérivées partielles avec plusieurs variables dépendantes", *Mémoires de la Cl. des Sci. Phys. et Math. de l'Académie des Sci. de l'Ukraine*, 5 (Kieff, 1927).

lead to the vanishing of nine determinants of the twelfth order from the elements of a matrix with twelve rows and twenty columns. The elements of this matrix are the coefficients of the derivatives with respect to $\partial a/\partial x, \dots, \partial \beta/\partial t$ in the equations obtained by differentiating any of the systems (3), (3'), ... with respect to $\partial a/\partial x, \dots, \partial \beta/\partial t$.

The theorem of Bourlet and E. von Weber* can be applied to systems (3), (3'), ...; according to this theorem the general integral must contain two arbitrary functions; one of these has three arguments (independent functions of a, β), and the other has two arguments.

Besides particular solutions of equations (1), due to Hertz, Fitzgerald, and Whittaker, there exist also integrals of systems (3), which were given by Bateman† :

$$\left. \begin{aligned} z-ct &= \phi(a, \beta) + (x+iy)\theta(a, \beta) \\ \theta(a, \beta)(z+ct) &= \psi(a, \beta) - (x-iy) \end{aligned} \right\}, \quad (4)$$

where ϕ, θ , and ψ denote arbitrary functions of a and β . These integrals likewise are only particular solutions, because the elimination of arbitrary elements from (4) leads not only to the system of three equations (3), but also to systems of three and four equations which are not equivalent to the equations of system (3). It is easy to see that the integral

$$\left. \begin{aligned} z-ct &= \phi(a, \beta) + \kappa(x+iy) \\ \kappa(z+ct) &= \psi(a, \beta) - (x-iy) \end{aligned} \right\}, \quad (5)$$

where κ is an arbitrary constant, also leads to such special systems.

The integral (4) is no more general than the integral (5), and also these equations (4) cannot be solved for the required functions a, β , which are connected by means of three arbitrary functions ϕ, θ , and ψ .

3. The integrals of equations in the form (3), like the solutions of many other equations of mathematical physics involving the Jacobians of unknown functions $z_1, \dots, z_n$, with respect to various independent variables $x_1, \dots, x_m$, may be found by my method of integration, of which a par-

* C. Bourlet, *Annales Sci. de l'Ecole Normale*, 3, 8 (1891), 43; E. von Weber, *Math. Annalen*, 49 (1897), 551.

† H. Bateman, *Amer. J. of Math.*, 37 (1915), 192-197; *Proc. London Math. Soc.* (2), 21 (1922), 256-270.

ticular case is Hamburger's method* of integrating linear equations either in derivatives or in derivatives and Jacobians.

As an example, we restrict ourselves to the system of two equations of the form

$$\left. \begin{aligned} \sum_i a_i^1 p_i^1 + \sum_i a_i^2 p_i^2 + \sum_{ij} b_{ij} (p_i^1 p_j^2 - p_j^1 p_i^2) + c &= 0 \\ \sum_i a_i^1 p_i^1 + \sum_i a_i^2 p_i^2 + \sum_{ij} \beta_{ij} (p_i^1 p_j^2 - p_j^1 p_i^2) + \gamma &= 0 \\ (i, j = 1, 2, \dots, m; i \neq j) \end{aligned} \right\}, \quad (6)$$

where the coefficients a , b , c and α , β , γ are functions of $x_1, \dots, x_m; z_1, z_2$, and where

$$p_i^k = \frac{\partial z_k}{\partial x_i} \quad (i = 1, 2, \dots, m; k = 1, 2).$$

Multiplying the second equation by an indeterminate multiplier λ , and adding the result to the first, we have

$$\sum_i A_i^1 p_i^1 + \sum_i A_i^2 p_i^2 + \sum_{ij} B_{ij} (p_i^1 p_j^2 - p_j^1 p_i^2) + C = 0 \quad (7)$$

where $a_i^k + \lambda \alpha_i^k = A_i^k$, $b_{ij} + \lambda \beta_{ij} = B_{ij}$, $c + \lambda \gamma = C$.

The solution of the system (6) can be written

$$u_k(x_1, \dots, x_m; z_1, z_2) = 0 \quad (k = 1, 2).$$

Differentiating these equations with respect to $x_1, \dots, x_m$, we get a system of equations from which the derivatives p_i^k may be deduced by differentiating u_1, u_2 with respect to $x_1, \dots, x_m; z_1, z_2$. Substituting the expressions for these in (7), we have

$$\begin{aligned} \sum_i A_i^1 \left(\frac{\partial u_1}{\partial x_i} \frac{\partial u_2}{\partial z_2} - \frac{\partial u_1}{\partial z_2} \frac{\partial u_2}{\partial x_i} \right) + \sum_i A_i^2 \left(\frac{\partial u_1}{\partial z_1} \frac{\partial u_2}{\partial x_i} - \frac{\partial u_1}{\partial x_i} \frac{\partial u_2}{\partial z_1} \right) \\ + \sum_{ij} B_{ij} \left(\frac{\partial u_1}{\partial x_j} \frac{\partial u_2}{\partial x_i} - \frac{\partial u_1}{\partial x_i} \frac{\partial u_2}{\partial x_j} \right) - C \left(\frac{\partial u_1}{\partial z_1} \frac{\partial u_2}{\partial z_2} - \frac{\partial u_1}{\partial z_2} \frac{\partial u_2}{\partial z_1} \right) = 0. \end{aligned}$$

The integration of this equation can be reduced to the integration of a system of equations of the first order in one function u . For this it is sufficient to equate to zero the coefficients of the derivatives of one of the functions u_1, u_2 . Neglecting the index of u , we obtain the system

* M. Hamburger, *Journal für Math.*, 100 (1886), 390; A. R. Forsyth, *Theory of differential equations*, 5 (1906), 428-455.

we have an algebraic equation for λ of degree p , and for each of its roots we obtain a system of equations (8) of rank $2p-1$.

When m is an even number, $m = 2p$, the determinant in (7) of order $2p+2$ is a perfect square, and it has $p+1$ different roots λ to determine the system (8) for finding the particular integrals u .

The rank of the system (8), by reason of condition (9), is decreased by two. If the system (8) is an integrable system, it may give two particular integrals u for each of the roots λ .

From the system of equations (6), when m is an even number, we may obtain a system with an odd number of independent variables*. In the particular case when the $\frac{1}{2}m(m-1)$ algebraical conditions of Hamburger, namely

$$B_{ij}C = A_i^1 A_j^2 - A_j^1 A_i^2 \quad (i, j = 1, 2, \dots, m; i \neq j) \quad (11)$$

for $C \neq 0^\dagger$, are satisfied by the coefficients of equation (7), then all the minors of the third order of the determinant (9) are zero, and therefore Hamburger's conditions lead to the condition that the rank of the system (8) is two. In this case, the system (8) leads to the system of two auxiliary equations of Hamburger,

$$\left. \begin{aligned} \sum_i A_i^1 \frac{\partial u}{\partial x_i} - C \frac{\partial u}{\partial z_1} &= 0 \\ \sum_i A_i^2 \frac{\partial u}{\partial x_i} - C \frac{\partial u}{\partial z_2} &= 0 \end{aligned} \right\} \quad (12)$$

The verification of all the equations (11) is a *necessary and sufficient* condition for the general integral of the proposed equations (6) to be of the form

$$\phi_1(u_{11}, u_{21}, \dots, u_{m1}) = 0, \quad \phi_2(u_{12}, u_{22}, \dots, u_{m2}) = 0,$$

where $u_{11}, u_{21}, \dots, u_{m2}$ represent particular integrals of system (12) for distinct roots λ . The method of integration indicated is useless for equations linear as regards only p^k , but for these equations it reduces to Hamburger's method. If Hamburger's conditions of proportionality of coefficients are satisfied, then system (8) also leads to Hamburger's system

$$\begin{aligned} \sum_i A_i' \frac{\partial u}{\partial x_i} - C \frac{\partial u}{\partial z_1} &= 0 \\ A \frac{\partial u}{\partial z_1} - \frac{\partial u}{\partial z_2} &= 0, \end{aligned}$$

* M. Kourensky, *loc. cit.*, ch. II.

† Conditions for mn other cases, when $A_i^k \neq 0$ ($i = 1, 2, \dots, m; k = 1, 2, \dots, n$), were generalized in my article, *Bull. de l'Acad. des Sci. de l'Ukraine*, II, 1 (1926), 20-31.

where

$$A = \frac{A_i^2}{A_j^1}.$$

Hamburger's systems (12) do not exist for equations of the form (3), but in the method indicated, equation (9) gives the roots $\lambda_1 = i$, $\lambda_2 = -i$ for the first two equations of systems (3), from which we obtain two systems of the form

$$\begin{cases} \frac{\partial u_1}{\partial x} + i \frac{\partial u_1}{\partial y} = 0, \\ c \frac{\partial u_1}{\partial z} + \frac{\partial u_1}{\partial t} = 0, \end{cases} \quad \begin{cases} i \frac{\partial u_2}{\partial x} + \frac{\partial u_2}{\partial y} = 0, \\ c \frac{\partial u_2}{\partial z} - \frac{\partial u_2}{\partial t} = 0. \end{cases}$$

By the integration of these systems particular integrals may be deduced, namely

$$\alpha_1 \equiv C_1, \quad \alpha_2 \equiv z - ct = C_2, \quad \alpha_3 \equiv x + iy = C_3,$$

$$\beta_1 \equiv C_4, \quad \beta_2 \equiv z + ct = C_5, \quad \beta_3 \equiv x - iy = C_6,$$

and particular integrals may be taken to be

$$\alpha = \phi(z - ct, x + iy), \quad \beta = \psi(z + ct, x - iy), \quad (13)$$

where ϕ and ψ are arbitrary functions. The third equation of system (3) will be satisfied when we introduce a new argument v in one of the expressions for ϕ or ψ ; this function v may be determined by substituting the expressions for α and β in system (3). The equations

$$\alpha = \phi(z - ct, x + iy), \quad \beta = \psi(z + ct, x - iy, v)$$

lead to relations

$$\frac{\frac{\partial \phi}{\partial(x+iy)}}{\frac{\partial \phi}{\partial(z-ct)}} = -c^2 \frac{\frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x}}{c \frac{\partial v}{\partial z} - \frac{\partial v}{\partial t}} = \frac{c \frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}}{\frac{\partial v}{\partial y} - i \frac{\partial v}{\partial x}} = -\frac{\frac{\partial \psi}{\partial(z+ct)}}{\frac{\partial \psi}{\partial(x-iy)}},$$

which limit the arbitrariness of the functions ϕ and ψ , and also give Hamilton's equation

$$\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 = \frac{1}{c^2} \left(\frac{\partial v}{\partial t}\right)^2. \quad (14)$$

This equation, according to Bateman*, is satisfied by an arbitrary function of α and β .

* H. Bateman, *Electrical and optical wave motion*, 13.

4. The complete integral of system (3) must contain seven arbitrary constants. It may be written

$$\left. \begin{aligned} \alpha &= K_1 + C_1 x + C_2 y + C_3 z + ic \frac{(C_1 C' + C_2 C'') C_3 - (C_1^2 + C_2^2) C'''}{C_1 C'' - C_2 C'} t \\ \beta &= K' + C' x + C'' y + C''' z + ic \frac{(C_1 C' + C_2 C'') C''' - (C'^2 + C''^2) C_3}{C' C_2 - C_1 C''} t \end{aligned} \right\}; \quad (15)$$

six of the eight constants K and C are restricted by the equation

$$\left. \begin{aligned} &C_1^2(C''^2 + C'''^2) + C_2^2(C'^2 + C'''^2) + C_3^2(C'^2 + C'''^2) \\ &= C_1 C' (C_2 C'' + C_3 C''') + C_2 C'' (C_1 C' + C_3 C''') + C_3 C''' (C_1 C' + C_2 C'') \end{aligned} \right\}. \quad (16)$$

A solution more general than the integral (5) may be taken to be

$$\left. \begin{aligned} \alpha &= \phi(\xi, \eta, \zeta) \\ \beta &= \psi(\xi, \eta, \zeta) \end{aligned} \right\}, \quad (17)$$

where ϕ and ψ are arbitrary functions of the arguments ξ, η, ζ , which are determined by the formulae

$$\left. \begin{aligned} \xi &= k_1 y + k_2 z \pm c \sqrt{(k_1^2 + k_2^2)} t \\ \eta &= \pm i \frac{k_1 k_3}{\sqrt{(k_1^2 + k_2^2)}} x + k_3 z \pm c \frac{k_2 k_3}{\sqrt{(k_1^2 + k_2^2)}} t \\ \zeta &= \mp i \frac{k_2 k_4}{\sqrt{(k_1^2 + k_2^2)}} x + k_4 y \pm c \frac{k_1 k_4}{\sqrt{(k_1^2 + k_2^2)}} t \end{aligned} \right\}, \quad (18)$$

$$\text{and} \quad k_3 k_4 \xi + k_2 k_4 \eta + k_1 k_3 \zeta = 0; \quad (19)$$

where k_1, k_2, k_3, k_4 denote arbitrary constants.

5. The group of conformal transformations of space of four dimensions is connected with the fundamental equations of light, with equation (14), and with*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}. \quad (20)$$

With the solution of equation (14) is connected the transition from components (E_x, E_y, E_z) and (M_x, M_y, M_z) of electric and magnetic force of an electromagnetic field in the system of coordinates x, y, z, t to com-

* H. Bateman, *Proc. London Math. Soc.* (2), 7 (1909), 71; 8 (1910), 223,

ponents (E_x, E_y, E_z) and (M_x, M_y, M_z) in the new system of coordinates* x', y', z', t' .

The general integral of equation (14) must contain an arbitrary function of three arguments. Its complete integral is

$$v = K + C_1x + C_2y + C_3z + c\sqrt{(C_1^2 + C_2^2 + C_3^2)}t. \quad (21)$$

It is easily seen that the functions α and β , which are determined by formulae (15) and condition (16), represent simultaneously the complete integrals of equation (14). Also the expressions for α and β determined by formulae (17), (18), (19) represent integrals of equation (14).

To interpret these results geometrically, the arbitrary constants may be associated with the cosines of the angles which a straight line makes with the axes of coordinates. For example, the complete integral (21) may be written in the form

$$v = k + K(x \cos \alpha + y \cos \beta + z \cos \gamma) + K^2 ct \quad (21')$$

$$(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1),$$

where k and K are arbitrary constants. The integral of the form (17) for equation (14) may be written

$$v = \phi[k(y \cos \alpha + z \sin \alpha \pm ct), \quad K \cos \beta(xi \cos \alpha \pm z + ct \sin \alpha), \\ \mp K \sin \beta(xi \sin \alpha \mp y - ct \cos \alpha)]. \quad (21'')$$

An arbitrary function $\Omega(v)$ of the integral v of Hamilton's equation (14) satisfies equation (20).

According to Bateman†, the expressions for the components of the vector M in formulae (2) are also integrals of the wave-equation (20), and therefore the functions α and β in the equations numbered from (15) to (19) represent integrals both for system (3) and for equations (14), (20). These integrals may be called the wave-functions.

Substituting the constant w for c in all the preceding equations, where w represents the velocity of propagation of electromagnetic waves in a homogeneous and isotropic dielectric in accordance with the formula‡

$$w = \frac{c}{\sqrt{(\epsilon\mu)}},$$

* H. Bateman, *Proc. London Math. Soc.* (2), 10 (1912), 9-11.

† H. Bateman, *Electrical and optical wave motion*, 122.

‡ M. Abraham, *Theorie der Elektrizität* (1), Leipzig (1907), 250, 304.

we arrive at the integrals for the motion of electromagnetic waves in this dielectric.

The theory of conformal transformations of space of four dimensions (and, in particular, the transformations of the theory of relativity) gives another series of integrals of equations (1), (3), (14), (20); in the integrals given already the symbols x, y, z, t are replaced* by functions of x, y, z, t .

6. My method of variation of arbitrary constants of complete integrals of differential equations with partial derivatives of the first order of one or more unknown functions† may be applied to find the general integrals of equations (1), (3), (14), (20).

Consider, for example, the variation of arbitrary constants for equations with one dependent variable z , and several independent variables $x_1, \dots, x_m$. The passage from the complete integral

$$z = F(x_1, \dots, x_m, C_1, \dots, C_{m-1}, C_m) \quad (23)$$

of the equation

$$f\left(x_1, \dots, x_m, z, \frac{\partial z}{\partial x_1}, \dots, \frac{\partial z}{\partial x_m}\right) = 0 \quad (24)$$

to a general integral

$$z = F[x_1, \dots, x_m; C_1, \dots, C_{m-1}, \phi(C_1, \dots, C_{m-1})] \quad (25)$$

will be determined by the system of equations

$$\frac{\partial F}{\partial C_i} + \frac{\partial F}{\partial C_m} \frac{\partial \phi}{\partial C_i} = 0 \quad (i = 1, 2, \dots, m-1) \quad (26)$$

$$C_m = \phi(C_1, \dots, C_{m-1}), \quad (27)$$

where $C_1, \dots, C_m$ denote arbitrary constants in (23), $C_1, \dots, C_{m-1}$ are unknown functions of $x_1, \dots, x_m$ in (25), and ϕ denotes an arbitrary function.

The functions $C_1, \dots, C_{m-1}$ may be found from (26) for a given form of the function ϕ , and the expression for C_m found‡ from (27), but there is no method of finding an expression for $C_1, \dots, C_{m-1}$, if ϕ denotes a *completely arbitrary* function.

* H. Bateman, *Proc. London Math. Soc.* (2), 7 (1909), 74, 76; 8 (1910), 240-243, 469.

† *Loc. cit.*, ch. VI, VII.

‡ G. Imschenetsky, *Sur l'intégration des équations . . . du premier ordre* (Paris, 1869), 16,

To find $C_1, \dots, C_{m-1}$ by means of $x_1, \dots, x_m$, we use the identities

$$\frac{\partial^2 \phi}{\partial x_g \partial x_h} \equiv \frac{\partial^2 \phi}{\partial x_h \partial x_g} \quad (g, h = 1, 2, \dots, m; \quad g \neq h)$$

and the formulae

$$\frac{\partial \phi}{\partial x_g} = \frac{\partial \phi}{\partial C_1} \frac{\partial C_1}{\partial x_g} + \frac{\partial \phi}{\partial C_2} \frac{\partial C_2}{\partial x_g} + \dots + \frac{\partial \phi}{\partial C_{m-1}} \frac{\partial C_{m-1}}{\partial x_g},$$

or

$$\frac{\partial \phi}{\partial x_g} = \frac{\frac{\partial F}{\partial C_1} \frac{\partial C_1}{\partial x_g} + \frac{\partial F}{\partial C_2} \frac{\partial C_2}{\partial x_g} + \dots + \frac{\partial F}{\partial C_{m-1}} \frac{\partial C_{m-1}}{\partial x_g}}{\frac{\partial F}{\partial C_m}},$$

in consequence of (26) and (27).

These equations lead to the system

$$\sum_{i=1}^{m-1} \left[\left(\frac{\partial^2 F}{\partial C_i \partial x_h} \frac{\partial F}{\partial C_m} - \frac{\partial^2 F}{\partial C_m \partial x_h} \frac{\partial F}{\partial C_i} \right) \frac{\partial C_i}{\partial x_g} + \left(\frac{\partial^2 F}{\partial C_m \partial x_g} \frac{\partial F}{\partial C_i} - \frac{\partial^2 F}{\partial C_i \partial x_g} \frac{\partial F}{\partial C_m} \right) \frac{\partial C_i}{\partial x_h} \right] = 0, \quad (28)$$

which determines the functions $C_1, \dots, C_{m-1}$.

This system may be integrated (i) if it does not contain an arbitrary function ϕ in the coefficients of $\partial C_i / \partial x_g$, $\partial C_i / \partial x_h$, and (ii) if the condition of compatibility are satisfied for the system. The system (28) contains no arbitrary function ϕ and it is of the form

$$\sum_{i=1}^{m-1} \left(\frac{\partial^2 F}{\partial C_i \partial x_h} \frac{\partial C_i}{\partial x_g} - \frac{\partial^2 F}{\partial C_i \partial x_g} \frac{\partial C_i}{\partial x_h} \right) = 0$$

for complete integrals of the type

$$z = F(x_1, \dots, x_m; \quad C_1, \dots, C_{m-1}) + C_m, \quad (29)$$

which satisfy the equations

$$f(x_1, \dots, x_m; \quad \frac{\partial z}{\partial x_1}, \dots, \frac{\partial z}{\partial x_m}) = 0;$$

and for complete integrals of the type

$$z = F_1(x_1, \dots, x_m; \quad C_1, \dots, C_{m-1}) + C_m F_2(x_1, \dots, x_m),$$

which satisfy special differential equations with an unknown function z , the system (28) is

$$\sum_{i=1}^{m-1} \left[\left(\frac{\partial^2 F_1}{\partial C_i \partial x_h} F_2 - \frac{\partial F_2}{\partial x_h} \frac{\partial F_1}{\partial C_i} \right) \frac{\partial C_1}{\partial x_g} + \left(\frac{\partial F_2}{\partial x_g} \frac{\partial F_1}{\partial C_i} - F_2 \frac{\partial^2 F_1}{\partial C_i \partial x_g} \right) \frac{\partial C_1}{\partial x_h} \right] = 0.$$

These formulae may be applied to the complete integral (21) of equation (14), and we obtain the result :

If K is regarded as an arbitrary function of C_1, C_2, C_3 , and if we put $x = x_1, y = x_2, z = x_3, t = x_4$, the passage from a complete integral to the general integral reduces to the determination of any three particular independent integrals of the system

$$\begin{aligned} \frac{\partial^2 v}{\partial C_1 \partial x_h} \frac{\partial C_1}{\partial x_g} - \frac{\partial^2 v}{\partial C_1 \partial x_g} \frac{\partial C_1}{\partial x_h} + \frac{\partial^2 v}{\partial C_2 \partial x_h} \frac{\partial C_2}{\partial x_g} - \frac{\partial^2 v}{\partial C_2 \partial x_g} \frac{\partial C_2}{\partial x_h} \\ + \frac{\partial^2 v}{\partial C_3 \partial x_h} \frac{\partial C_3}{\partial x_g} - \frac{\partial^2 v}{\partial C_3 \partial x_g} \frac{\partial C_3}{\partial x_h} = 0 \\ (g, h = 1, 2, 3, 4; g \neq h), \end{aligned}$$

or
$$\frac{\partial C_1}{\partial y} = \frac{\partial C_2}{\partial x}, \quad \frac{\partial C_1}{\partial y} = \frac{\partial C_3}{\partial x}, \quad \frac{\partial C_2}{\partial z} = \frac{\partial C_3}{\partial y}$$

$$c \left(C_1 \frac{\partial C_i}{\partial y} + C_2 \frac{\partial C_i}{\partial y} + C_3 \frac{\partial C_i}{\partial z} \right) = \sqrt{(C_1^2 + C_2^2 + C_3^2)} \frac{\partial C_i}{\partial z}$$

$$(i = 1, 2, 3).$$

When particular integrals C_1, C_2, C_3 have been obtained, the general integral of equation (14) is

$$v = xC_1 + yC_2 + zC_3 + ct\sqrt{(C_1^2 + C_2^2 + C_3^2)} + \phi(C_1, C_2, C_3).$$

ON A PROBLEM OF ANALYTIC CONTINUATION

By T. ESTERMANN.

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*Introduction.*If $\varliminf |a_n|^{-1/n} > 0$, the series

$$(1) \quad \sum_{n=1}^{\infty} a_n [\zeta(ns) - 1]$$

is obviously convergent for sufficiently large values of σ ($s = \sigma + it$) and represents an analytic function $f(s)$. It is in most cases still unknown whether this function can be continued beyond the region of convergence of the series (1), and how far it can be continued. The following theorems may throw some light on this problem, though they do not show the way to a general solution:

THEOREM 1. *Let*

$$(2) \quad 0 < r = \varliminf |a_n|^{-1/n} < 1,$$

and let the function

$$(3) \quad g(x) = \sum_{n=1}^{\infty} a_n x^n$$

be singular at every point of the circle $|x| = r$. Then the function

$$(4) \quad f(s) = \sum_{n=1}^{\infty} a_n [\zeta(ns) - 1]$$

is singular at every point of the line $\sigma = -\log r / \log 2$.

THEOREM 2. *Let the function (3) have a finite number of singularities in the interior of the unit circle. Then the function (4) is singular at every point of the imaginary axis.*

THEOREM 3. Let a be any real number, $\beta < a - \frac{1}{2}$, and $a_n = n^a + O(n^\beta)$. Then the function (4) is singular at every point of the imaginary axis.

The proof of the first two theorems is simple. In order to prove the third we have to consider the particular functions

$$(5) \quad f_{a,\lambda}(s) = \sum_{n=1}^{\infty} \binom{a}{n} \zeta^{(\lambda)}(ns) \quad (\lambda = 0, 1, 2, \dots),$$

where

$$(6) \quad \binom{a}{n} = \frac{(a+1)(a+2)\dots(a+n)}{n!} = \frac{\Gamma(a+n+1)}{n! \Gamma(a+1)}$$

and

$$(7) \quad \zeta^{(\lambda)}(s) = \sum_{m=2}^{\infty} (-\log m)^\lambda m^{-s},$$

so that $\zeta^{(0)}(s) = \zeta(s) - 1$ and $\zeta^{(\lambda)}(s)$ ($\lambda > 0$) is the λ -th derivative of $\zeta(s)$. For these functions we prove

THEOREM 4. Let k and l be integers, $k > 0$, $l \geq 1$, and let

$$(8) \quad |t_0| = \frac{2k\pi}{\log l}.$$

Then, if either

$$(a) \quad a > 0, \quad \lambda = 0$$

or

$$(b) \quad -1 < a \leq 0,$$

we have

$$(9) \quad \lim_{\sigma \rightarrow 0} \sigma^{a+1} |f_{a,\lambda}(\sigma + it_0)| > 0.$$

In case (b) it will even be found that this limit is infinity. (9) implies that $f_{a,\lambda}(s)$, as well as any function

$$(10) \quad f_1(s) = f_{a,\lambda}(s) + G(s),$$

where

$$(11) \quad G(\sigma + it_0) = o(\sigma^{-a-1}) \quad (\sigma \rightarrow 0),$$

is singular at the point it_0 . Hence, if $G(s)$ satisfies (11) for every $t_0 \neq 0$, the function (10) is singular at all the points of the imaginary axis, for the points it_0 that satisfy (8) for any k and l are dense everywhere on it. This will lead to the proof of Theorem 3.

The logarithm of the function $\prod_{m=2}^{\infty} [1/(1-m^{-s})]$, which has been considered by MacMahon and Oppenheim, satisfies the conditions of Theorem 3. Hence this function is also singular at all the points of the imaginary axis.

1. *Proof of the first two theorems.*

Proof of Theorem 1. By (3) and (4)

$$f(s) = g(2^{-s}) + \sum_{n=1}^{\infty} a_n [\zeta(ns) - 1 - 2^{-ns}].$$

Now, if $g(x)$ is singular at every point of the circle $|x| = r$, then $g(2^{-s})$ is singular at every point of the line $\sigma = -\log r / \log 2$.

On the other hand, by (2),

$$\sum_{n=1}^{\infty} a_n [\zeta(ns) - 1 - 2^{-ns}]$$

is obviously regular at least at every non-real point of that line.

Proof of Theorem 2. Let $re^{\phi i}$ be a singularity of $g(x)$ of smallest modulus, and let $r_1, r_2, \dots, r_{\mu}$ be all the moduli of singularities of $g(x)$ which lie between r and 1, so that

$$(12) \quad r < r_{\nu} < 1 \quad (\nu = 1, 2, \dots, \mu).$$

Then $g(m^{-s})$ is singular at all the points

$$(13) \quad s_{m,k} = \frac{-\log r - \phi i + 2k\pi i}{\log m} \quad (k = 0, \pm 1, \pm 2, \dots).$$

Now for every integer $q \geq 2$, by (3) and (4),

$$(14) \quad f(s) = \sum_{m=2}^q g(m^{-s}) + \sum_{n=1}^{\infty} a_n [\zeta(ns) - \sum_{m=1}^q m^{-ns}].$$

Hence a point $s_{q,k}$ which is neither the reciprocal of an integer nor a singularity of any of the functions $g(2^{-s}), g(3^{-s}), \dots, g[(q-1)^{-s}]$ is certainly a singularity of $f(s)$; for, by (2) and (13), the second sum in (14) is regular at such a point, and only the last term of the first sum is singular. Hence, if we put

$$(15) \quad v_{\nu} = \frac{\log r}{\log r_{\nu}} \quad (\nu = 1, 2, \dots, \mu),$$

$f(s)$ is singular at all the non-real points $s_{q,k}$ whose q is not a v_ν -th power of an integer; since, if $g(m^{-s})$ were singular at such a point, for any $m < q$, this would mean that $|m^{-s_{q,k}}| = m^{\log r / \log q} = r_\nu$, which implies $m^{v_\nu} = q$. Since, by (12) and (15),

$$v_\nu > 1 \quad (\nu = 1, 2, \dots, \mu),$$

there is an infinity of such integers q .

Hence, by (13), every point of the imaginary axis is a limiting point of singularities of $f(s)$.

2. Proof of Theorem 4.

2.1. *Elimination of the ζ -function from (5).* Since, by (6),

$$(16) \quad \sum_{n=1}^{\infty} \binom{\alpha}{n} x^n = (1-x)^{-\alpha-1} - 1 \quad (|x| < 1),$$

we have, by (5) and (7),

$$(17) \quad f_{a,\lambda}(s) = \sum_{n=1}^{\infty} \sum_{m=2}^{\infty} \binom{\alpha}{n} (-\log m)^\lambda (m^{-s})^n \\ = \sum_{m=2}^{\infty} (-\log m)^\lambda [(1-m^{-s})^{-\alpha-1} - 1] \quad (\sigma > 1).$$

Now let

$$(18) \quad H(x) = (\log x)^\lambda (1-x^{-s})^{-\alpha-1}, \quad h(x) = H(x) - (\log x)^\lambda,$$

and

$$(19) \quad F(s) = \int_{\frac{3}{2}}^{\infty} h(x) dx \quad (\sigma > 1).$$

Then, since

$$\int_{\frac{3}{2}}^{\infty} (\log x)^\lambda x^{-ns} dx = P(ns) \left(\frac{3}{2}\right)^{-ns} \quad (\sigma > 1, n \geq 1),$$

where

$$(20) \quad P(ns) = \sum_{\nu=0}^{\lambda} d_\nu (ns-1)^{-\nu-1},$$

$d_0, d_1, \dots, d_\lambda$ being constants, we have, by (16), (18), and (19),

$$(21) \quad F(s) = \sum_{n=1}^{\infty} \binom{\alpha}{n} P(ns) \left(\frac{3}{2}\right)^{-ns},$$

which shows that $F(s)$ is regular throughout the half-plane $\sigma > 0$, except at the points $1, \frac{1}{2}, \frac{1}{3}, \dots$

Putting

$$(22) \quad \psi(x) = x - [x] - \frac{1}{2},$$

we find by repeated partial integration

$$(23) \quad \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} H(x) dx = H(m) + \frac{1}{2} \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} H''(x) [\psi(x)]^2 dx \quad (m = 2, 3, \dots),$$

hence, by (18),

$$(24) \quad h(m) = \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} h(x) dx - \frac{1}{2} \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} H''(x) [\psi(x)]^2 dx \\ + \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} (\log x)^\lambda dx - (\log m)^\lambda,$$

and, since

$$\sum_{m=2}^{\infty} \left(\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} (\log x)^\lambda dx - (\log m)^\lambda \right)$$

converges, by (17), (18), and (19),

$$(25) \quad (-1)^\lambda f_{a,\lambda}(s) = \sum_{m=2}^{\infty} h(m) = F(s) - \frac{1}{2} \int_{\frac{1}{2}}^{\infty} H''(x) [\psi(x)]^2 dx + C_0 \quad (\sigma > 1),$$

where C_0 depends only on λ .

Now, if R is any positive number,

$$(26) \quad |H''(x)| < Cx^{-2}(\log x)^\lambda |1 - x^{-s}|^{-a-s} \quad (x > \frac{3}{2}, |s| < R, \sigma > 0),$$

where C depends only on a, λ , and R .

Hence

$$\int_{\frac{3}{2}}^{\infty} H''(x) [\psi(x)]^2 dx$$

converges uniformly in every closed bounded region of the half-plane $\sigma > 0$, and, by (25),

$$(27) \quad (-1)^\lambda f_{a,\lambda}(s) = C_0 + F(s) - \frac{1}{2} \int_{\frac{1}{2}}^{\infty} H''(x) [\psi(x)]^2 dx \quad (\sigma > 0).$$

2.2. *Decomposition of $f_{a,\lambda}(\sigma + it_0)$.* Let, henceforth (till the end of § 2),

$$(28) \quad a > -1, \quad s = \sigma + it_0, \quad t_0 \neq 0, \quad 0 < \sigma < \sigma'_0,$$

where $\sigma_0 = \sigma_0(t_0)$ is a sufficiently small number (so small that, for the

numbers μ, ν_1, ν_3 , defined later on, $\nu_3 + 1 < \nu_1 < \mu$); let

$$(29) \quad q = \exp \frac{2\pi}{|t_0|},$$

and let ν_1 be the integer for which

$$(30) \quad (\nu_1 - 1) q^{\nu_1 - 1} \sigma \leq 1 < \nu_1 q^{\nu_1} \sigma.$$

Let us call an integer m "neighboured" (to a power of q) if there is an integer ν such that

$$|m - q^\nu| < 1,$$

and let Σ'_m and Σ''_m denote sums in which m runs through the neighboured or the non-neighboured integers respectively. Then, if we put

$$(31) \quad \begin{cases} A = \sum'_{m=2}^{[q\nu_1]} H(m), & B = \sum'_{m=2}^{[q\nu_1]} \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} H(x) dx, \\ D = \sum''_{m=2}^{[q\nu_1]} \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} H''(x) [\psi(x)]^2 dx, & E = \int_{[q\nu_1] + \frac{1}{2}}^{\infty} H''(x) [\psi(x)]^2 dx, \end{cases}$$

we have, by (23) and (27),

$$(32) \quad (-1)^\lambda f_{a,\lambda}(s) = C_0 + F(s) + A - B - \frac{1}{2}D - \frac{1}{2}E.$$

In order to complete the proof, we have only to show that, if σ tends to 0,

$$(33) \quad F(s) - B - \frac{1}{2}D - \frac{1}{2}E = O(\nu_1^{\lambda-a-1} \sigma^{-a-1}),$$

and that if, moreover, t_0 satisfies (8),

$$(34) \quad \lim A \sigma^{a+1} > 0 \quad (a > \lambda),$$

$$(35) \quad \underline{\lim} |A| \sigma^{a+1} \nu_1^{a-\lambda} > 0 \quad (-1 < a \leq 0),$$

and $|A| \sigma \rightarrow \infty$ if $a = \lambda = 0$.

Theorem 4 follows immediately from the last four formulae.

2.3. Proof of (33).

2.31. The inequality for $F(s)$.

We have*, by (20),

$$|P(ns)| \leq \sum_{\nu=0}^{\lambda} |d_\nu| |nt_0|^{-\nu-1} < C_1 n^{-1},$$

* $C_1, C_2, \dots$ are used in this paragraph to denote (sufficiently large) positive constants which depend at most on a, λ , and t_0 .

and, by (6),

$$\frac{1}{n} \binom{a}{n} = \left(\frac{1}{a} + \frac{1}{n} \right) \binom{a-1}{n} \quad (a \neq 0), \quad \binom{0}{n} = 1, \quad \binom{a}{n} < C_2 n^a;$$

hence, by (21) and (16),

$$\begin{aligned} |F(s)| &< \left(\frac{1}{a} + 1 \right) C_1 \sum_{n=1}^{\infty} \binom{a-1}{n} \left(\frac{3}{2} \right)^{-n\sigma} \\ &= \left(\frac{1}{a} + 1 \right) C_1 \{ [1 - (\frac{3}{2})^{-\sigma}]^{-a} - 1 \} < C_3 \sigma^{-a} \quad (a > 0), \end{aligned}$$

$$|F(s)| < C_1 \sum_{n=1}^{\infty} n^{-1} \left(\frac{3}{2} \right)^{-n\sigma} = -C_1 \log [1 - (\frac{3}{2})^{-\sigma}] < C_3 \log \frac{1}{\sigma} \quad (a = 0),$$

$$|F(s)| < C_1 C_2 \sum_{n=1}^{\infty} n^{a-1} = C_3 \quad (a < 0).$$

Since, by (30), $v_1 = O[\log(1/\sigma)]$, these three inequalities imply that

$$(36) \quad F(s) = O(v_1^{\lambda-a-1} \sigma^{-a-1}).$$

2.32. *The inequality for E.* By (31), (22), and (26),

$$(37) \quad |E| < C_4 \int_{q^{v+\frac{1}{2}}}^{\infty} x^{-2} (\log x)^{\lambda} |1 - x^{-s}|^{-a-3} dx.$$

Now, if $\arg(x^{-s})$ is taken in the interval $(-\pi, \pi)$, by (28) and (29), we have, for every integer v

$$|\arg(x^{-s})| = |t_0 \log x - 2\pi v| = |t_0(\log x - v \log q)| \quad (q^{v-\frac{1}{2}} \leq x \leq q^{v+\frac{1}{2}}),$$

hence

$$(38) \quad |1 - x^{-s}| \geq \sin \left| \frac{1}{2} \arg x^{-s} \right| \geq C_5^{-1} |\log x - v \log q| \geq \frac{|x - q^v|}{C_5 q^{v+\frac{1}{2}}}$$

$$(q^{v-\frac{1}{2}} \leq x \leq q^{v+\frac{1}{2}});$$

and (if $v > 0$)

$$\begin{aligned} (39) \quad \int_{q^{v+\kappa}}^{q^{v+\frac{1}{2}}} x^{-2} (\log x)^{\lambda} |1 - x^{-s}|^{-a-3} dx &< C_6 q^{-2v} \nu^{\lambda} q^{v(a+3)} \int_{\kappa}^{\infty} y^{-a-3} dy \\ &= C_7 \nu^{\lambda} q^{v(a+1)} \kappa^{-a-2} \quad (0 < \kappa \leq q^{v+\frac{1}{2}} - q^v). \end{aligned}$$

On the other hand, if we put

$$(40) \quad \mu = \frac{q^{\frac{1}{2}} - 1}{\sigma},$$

we have

$$(41) \quad |1-x^{-s}| \geq 1-x^{-\sigma} > \begin{cases} C_8^{-1} \sigma \log x & (1 < x < q^{\mu+\frac{1}{2}}) \\ C_9^{-1} & (x > q^{\mu-\frac{1}{2}}). \end{cases}$$

Hence

$$(42) \quad \int_{q^{[\mu]+\frac{1}{2}}}^{\infty} x^{-2} (\log x)^{\lambda} |1-x^{-s}|^{-a-3} dx < C_9^{a+3} \int_1^{\infty} x^{-2} (\log x)^{\lambda} dx = C_{10}$$

and

$$(43) \quad \int_{q^{\nu}}^{q^{\nu+\kappa}} x^{-2} (\log x)^{\lambda} |1-x^{-s}|^{-a-3} dx < C_{11} \kappa q^{-2\nu} \nu^{\lambda} \sigma^{-a-3} \nu^{-a-3} \\ (\nu \leq \mu, \quad 0 < \kappa \leq q^{\nu+\frac{1}{2}} - q^{\nu}).$$

It is permissible to put $\kappa = \sigma \nu q^{\nu}$ if $\nu \leq \mu$, for then, by (40), $0 < \kappa \leq q^{\nu+\frac{1}{2}} - q^{\nu}$, and so we obtain from (39) and (43)

$$\int_{q^{\nu}}^{q^{\nu+\frac{1}{2}}} x^{-2} (\log x)^{\lambda} |1-x^{-s}|^{-a-3} dx < C_{12} q^{-\nu} \nu^{\lambda-a-2} \sigma^{-a-2} \quad (\nu \leq \mu).$$

Since we can treat the integral from $q^{\nu-\frac{1}{2}}$ to q^{ν} in the same way, we have

$$\int_{q^{\nu-\frac{1}{2}}}^{q^{[\nu]+\frac{1}{2}}} x^{-2} (\log x)^{\lambda} |1-x^{-s}|^{-a-3} dx < C_{13} \sigma^{-a-2} \sum_{\nu=\nu_1}^{\infty} \nu^{\lambda-a-2} q^{-\nu} \\ < C_{14} \sigma^{-a-2} \nu_1^{\lambda-a-2} q^{-\nu_1},$$

hence, by (37), (42), and (30),

$$(44) \quad |E| < C_{15} \nu_1^{\lambda-a-1} \sigma^{-a-1}.$$

2.33. The inequality for D .

By (31), (22), and the definition of Σ'' ,

$$(45) \quad |D| < \sum_{\nu=\nu_2}^{\nu_1} \int_{q^{\nu-1+\frac{1}{2}}}^{q^{\nu-\frac{1}{2}}} |H''(x)| dx,$$

where ν_2 is the smallest integer for which $q^{\nu_2} - q^{\nu_2-1} \geq 2$. By (26) and (38)

$$\int_{q^{\nu-1+\frac{1}{2}}}^{q^{\nu-\frac{1}{2}}} |H''(x)| dx < C_{16} q^{-2\nu} \nu^{\lambda} q^{\nu(a+3)} \int_{\frac{1}{2}}^{\infty} y^{-a-3} dy = C_{17} \nu^{\lambda} q^{\nu(a+1)},$$

$$\text{and similarly} \quad \int_{q^{\nu-\frac{1}{2}}}^{q^{\nu-\frac{1}{2}}} |H''(x)| dx < C_{17} \nu^{\lambda} q^{\nu(a+1)}.$$

Hence, by (45) and (30),

$$(46) \quad |D| < 2C_{17} \sum_{\nu=\nu_2}^{\nu_1} \nu^{\lambda} q^{\nu(a+1)} < C_{18} \nu_1^{\lambda} q^{\nu_1(a+1)} < C_{19} \nu_1^{\lambda-a-1} \sigma^{-a-1}.$$

2.34. *The inequality for B.* By (31) and the definition of Σ' ,

$$(47) \quad |B| < \int_{\frac{3}{2}}^{q^{\nu_3+\frac{1}{2}}} |H(x)| dx + \sum_{\nu=\nu_3+1}^{\nu_1} \int_{q^{\nu-\frac{3}{2}}}^{q^{\nu+\frac{3}{2}}} |H(x)| dx,$$

where ν_3 is the smallest integer for which $q^{\nu_3+\frac{1}{2}} - q^{\nu_3} \geq \frac{3}{2}$. Now by (18), (41), and (38),

$$\begin{aligned} \int_{q^{\nu-\frac{3}{2}}}^{q^{\nu+\frac{3}{2}}} |H(x)| dx &< \int_{q^{\nu-\kappa}}^{q^{\nu+\kappa}} (\log x)^\lambda (C_8^{-1} \sigma \log x)^{-\alpha-1} dx \\ &\quad + 2 \int_{q^{\nu+\kappa}}^{q^{\nu+\frac{3}{2}}} (\log x)^\lambda [C_5^{-1} q^{-\nu-\frac{1}{2}} (x-q^\nu)]^{-\alpha-1} dx \\ &< C_{20} \left(\kappa \nu^{\lambda-\alpha-1} \sigma^{-\alpha-1} + \nu^\lambda q^{\nu(\alpha+1)} \int_{\kappa}^{\frac{1}{2}} y^{-\alpha-1} dy \right) \quad (\nu_3 < \nu \leq \mu, \ 0 \leq \kappa \leq \tfrac{3}{2}). \end{aligned}$$

Putting here

$$\kappa = \begin{cases} \text{Min}(\frac{3}{2}, \nu q^\nu \sigma) & (\alpha \geq 0) \\ 0 & (\alpha < 0), \end{cases}$$

we obtain

$$(48) \quad \int_{q^{\nu-\frac{3}{2}}}^{q^{\nu+\frac{3}{2}}} |H(x)| dx < \begin{cases} C_{21} \nu^{\lambda-\alpha} q^\nu \sigma^{-\alpha} & (\alpha > 0) \\ C_{20} \nu^\lambda q^\nu \left(1 + \log \frac{3}{2\kappa}\right) < C_{21} \nu^\lambda q^\nu (1 + \nu_1 - \nu) & (\alpha = 0)^* \\ C_{21} \nu^\lambda q^{\nu(\alpha+1)} & (\alpha < 0) \end{cases} \quad (\nu_3 < \nu \leq \nu_1).$$

Similarly, since $q^{\nu+\frac{1}{2}} - q^\nu < \frac{3}{2}q$ for $\nu \leq \nu_3$,

$$(49) \quad \begin{aligned} \int_{q^{\nu-\frac{1}{2}}}^{q^{\nu+\frac{1}{2}}} |H(x)| dx &< C_{20} \left(\kappa \nu^{\lambda-\alpha-1} \sigma^{-\alpha-1} + \nu^\lambda q^{\nu(\alpha+1)} \int_{\kappa}^{\frac{3}{2}q} y^{-\alpha-1} dy \right) \\ &< \begin{cases} C_{22} \nu^{\lambda-\alpha} q^\nu \sigma^{-\alpha} & (\alpha > 0) \\ C_{22} \nu^\lambda q^\nu (1 + \nu_1 - \nu) & (\alpha = 0) \\ C_{22} \nu^\lambda q^{\nu(\alpha+1)} & (\alpha < 0) \end{cases} \quad (1 \leq \nu \leq \nu_3), \end{aligned}$$

and, if $\sqrt{q} > \frac{3}{2}$, by (18) and (38),

$$\int_{\frac{1}{2}}^{\sqrt{q}} |H(x)| dx < C_{23}.$$

* By (30) $\log 1/\kappa < \log(\nu_1 q^{\nu_1} \sigma) - \log(\nu q^\nu \sigma) = \log \nu_1 - \log \nu + (\nu_1 - \nu) \log q \leq (1 + \log q)(\nu_1 - \nu)$.

Hence, by (47) and (30),

$$(50) \quad |B| < \begin{cases} C_{24} \nu_1^{\lambda-a} q^{\nu_1} \sigma^{-a} & (a \geq 0) \\ C_{24} \nu_1^{\lambda} q^{\nu_1(a+1)} & (a < 0) \end{cases} < C_{25} \nu_1^{\lambda-a-1} \sigma^{-a-1}.$$

(33) follows from (36), (44), (46), and (50).

2.4. *Proof of (34).* By (31)

$$(51) \quad A\sigma^{a+1} = \sum_{m=2}^{[q^{\nu_1}]} \sigma^{a+1} H(m) = \sum_{\nu=1}^{\infty} b_{\nu},$$

where b_{ν} contains those terms of the preceding sum for which $q^{\nu-1} < m \leq q^{\nu}$, and $b_{\nu} = 0$ if there are no such terms (in particular for $\nu > \nu_1$). As there are at most two neighboured integers in the interval $(q^{\nu-1}, q^{\nu})$, we have, by (18) and (41),

$$|b_{\nu}| < 2C_8^{a+1}(\nu-1)^{\lambda-a-1}(\log q)^{\lambda-a-1} \quad (a > \lambda-1, \nu > 1).$$

Hence $\sum b_{\nu}$ converges uniformly in the case $a > \lambda$, and, by (51),

$$(52) \quad \lim_{\sigma \rightarrow 0} A\sigma^{a+1} = \sum_{\nu=1}^{\infty} \lim_{\sigma \rightarrow 0} b_{\nu} \quad (a > \lambda).$$

Now $\lim b_{\nu}$ is obviously either zero (if q^{ν} is not an integer) or $(\nu \log q)^{\lambda-a-1}$ (if q^{ν} is an integer), and the latter case actually occurs (for $\nu = k, 2k, \dots$) if t_0 satisfies (8). Hence (52) implies (34).

2.5. *Proof of (35).* In the case $-1 < a \leq 0$, the real part of $H(m)$ is, by (18), always positive, for $|\arg H(m)| < \frac{1}{2}\pi(a+1)$. Hence, if t_0 satisfies (8), by (31), since by (29) and (8) $l = q^k$,

$$(53) \quad \Re A > \sum_{n=1}^{[\nu_1/k]} H(l^n) = \sum_{n=1}^{[\nu_1/k]} (n \log l)^{\lambda} (1-l^{-n\sigma})^{-a-1} \\ > (\log l)^{\lambda-a-1} \sigma^{-a-1} \sum_{n=1}^{[\nu_1/k]} n^{\lambda-a-1}.$$

$$\text{Since} \quad \lim_{\nu_1 \rightarrow \infty} \nu_1^{a-\lambda} \sum_{n=1}^{[\nu_1/k]} n^{\lambda-a-1} > 0,$$

this proves (35) and completes the proof of Theorem 4.

3. Proof of Theorem 3.

LEMMA. *Let*

$$(54) \quad \gamma \geq -\frac{3}{2}, \quad \epsilon > 0, \quad s = \sigma + it_0, \quad t_0 \leq 0, \quad 0 < \sigma \leq \frac{1}{2}.$$

Then

$$(55) \quad \sum_{n=1}^{\infty} n^{\gamma} |\xi^{(\lambda)}(ns)| < C\sigma^{-\gamma-\frac{1}{2}-\epsilon},$$

where C depends only on λ , γ , ϵ , and t_0 .

Proof. It is known from the theory of the ξ -function that, with the above conditions*,

$$(56) \quad |\xi^{(\lambda)}(ns)| < C_1 n^{\frac{1}{2}+\epsilon}$$

and

$$(57) \quad |\xi^{(\lambda)}(ns)| < C_2 2^{-n\sigma} \quad [n > (2/\sigma)].$$

Hence

$$(58) \quad \sum_{n=1}^{[2/\sigma]} n^{\gamma} |\xi^{(\lambda)}(ns)| < C_1 \sum_{n=1}^{[2/\sigma]} n^{\gamma+\frac{1}{2}+\epsilon} < C_3 \sigma^{-\gamma-\frac{1}{2}-\epsilon}$$

and

$$(59) \quad \sum_{n=[2/\sigma]+1}^{\infty} n^{\gamma} |\xi^{(\lambda)}(ns)| < C_2 \sum_{n=1}^{\infty} n^{\gamma+\frac{1}{2}+\epsilon} 2^{-n\sigma} < C_2 2^{\sigma} \int_0^{\infty} x^{\gamma+\frac{1}{2}+\epsilon} 2^{-x\sigma} dx \\ = C_2 2^{\sigma} \sigma^{-\gamma-\frac{1}{2}-\epsilon} \int_0^{\infty} x^{\gamma+\frac{1}{2}+\epsilon} 2^{-x} dx < C_4 \sigma^{-\gamma-\frac{1}{2}-\epsilon}.$$

The lemma follows from (58) and (59).

We now return to the conditions of Theorem 3. Let

$$(60) \quad \lambda = \text{Max}(0, [-a]), \quad a' = a + \lambda, \quad a'_n = \frac{a_n n^{\lambda}}{\Gamma(a' + 1)},$$

and

$$(61) \quad f_1(s) = \sum_{n=1}^{\infty} a'_n \xi^{(\lambda)}(ns).$$

Then $\Gamma(a' + 1) f_1(s)$ is either the function $f(s)$ defined by (4) (if $\lambda = 0$), or its λ -th derivative (if $\lambda > 0$); it is therefore sufficient to show that $f_1(s)$ is singular at every point of the imaginary axis. As stated in the introduction, this will follow from the fact that the function

$$(62) \quad G(s) = f_1(s) - f_{a', \lambda}(s)$$

satisfies (11) with a' instead of a for every $t_0 \neq 0$, since, by (60), either $a' > 0$ and $\lambda = 0$, or $-1 < a' \leq 0$.

Since
$$a_n = n^a + O(n^{\beta})$$

and
$$\frac{\Gamma(a' + n + 1)}{n!} = n^{a'} + O(n^{a'-1}),$$

* $C_1, C_2, \dots$ are here sufficiently large numbers independent of n and σ .

we have, by (6) and (60),

$$(63) \quad a'_n = \binom{a'}{n} + O(n^\gamma),$$

where

$$(64) \quad \gamma = \text{Max}(-\frac{3}{2}, \beta + \lambda, a' - 1) < a' - \frac{1}{2}.$$

Putting $b_n = a'_n - \binom{a'}{n}$, we have, by (62), (61), and (5),

$$(65) \quad G(s) = \sum_{n=1}^{\infty} b_n \zeta^{(\lambda)}(ns),$$

and, by (63),

$$(66) \quad b_n = O(n^\gamma).$$

Using the lemma with $\epsilon = \frac{1}{2}(a' - \frac{1}{2} - \gamma)$, we obtain (11) with a' instead of a for every $t_0 \neq 0$ from (65), (66), and (64). This completes the proof.

THE SPECTRUM OF AN ARBITRARY FUNCTION

By NORBERT WIENER.

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Introduction.

1. The spectrum of a trigonometrical polynomial.
2. The spectrum of an arbitrary function of one variable.
3. The total spectral intensity.
4. The spectrum of an almost periodic function.
5. The Parseval theorem for almost periodic functions.
6. The Weierstrass theorem for almost periodic functions.

Introduction.

The present article is the outcome of a series of investigations concerning the generalizations of the Fourier integral which the author has published* in the course of the last few years. In these the leading motives have been to find a bond of union between Bohr's theory of almost periodic functions† and the Plancherel theory of the Fourier integral, and to include in this united theory an adequate mathematical account of such continuous spectra as that of white light.

A somewhat similar problem has been attacked by Hahn‡, who was, however, primarily interested in questions of ordinary convergence, while the present author concerns himself rather with theorems of the Hurwitz-Riesz-Fischer type. Our methods differ considerably.

The previous papers of the author suffered, on the one hand, from the presence of a number of superfluous conditions imposed on the functions considered, and, on the other, from a certain complication in the methods

* N. Wiener, "On the representation of functions by trigonometrical integrals", *Math. Zeitschrift*, 24 (1925), 575-617; "The harmonic analysis of irregular motion", *J. Math. and Phys.*, Massachusetts Institute of Technology, 1926.

† H. Bohr, "Zur Theorie der fastperiodischen Funktionen I, II, III", *Acta Math.*, 45-47 (1924-26).

‡ H. Hahn, "Über eine Verallgemeinerung des Fourierschen Integralformel", *Acta Math.*, 49 (1926), 301-353.

of proof. He therefore was glad to accept an invitation from Professors Hilbert and Courant in Göttingen to develop his theories in a course of lectures at the university. In the preparation of this course he received much valuable aid from Mr. K. Mahler. The necessary trip to Europe was made possible by the grant of a Guggenheim Fellowship. To the trustees of the Guggenheim Foundation and to his Göttingen colleagues he wishes to tender his most sincere thanks.

It was the intention of the author to develop the spectrum theory of this paper in book form. However, the methods have so simplified themselves in the course of his recent work that this procedure now seems to him inadvisable, if not impossible. Moreover, the theory is too young to be set down in more definitive form than that of a comprehensive article.

A recent article of Weyl* develops a new method of attacking the Bohr analogue of the Hurwitz-Parseval theorem for almost periodic functions. Like the present paper, and like an earlier investigation of Schurt†, Weyl's paper depends on the study of the function

$$g(t) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(x) \bar{f}(x-t) dx$$

derived from an almost periodic function $f(x)$. Weyl's method differs from that of this paper in that Weyl considers the trigonometrical components of an almost periodic function as arranged in the order of magnitude of their coefficients, while we choose the order of magnitude of their exponents. The Weyl theory is primarily based on the Schmidt-Hilbert theory of integral equations; the present theory has its roots in the Plancherel theory of the Fourier integral and the Hurwitz-Parseval-Riesz-Fischer theory of Fourier series.

1. The spectrum of a trigonometrical polynomial.

Let us consider the complex trigonometrical polynomial

$$f(t) = \sum_1^n A_k e^{i\Lambda_k t}$$

* H. Weyl, "Beweis des Fundamentalsatzes in der Theorie der fastperiodischen Funktionen" (aus einem schreiben an Herrn Harald Bohr), *Sitzungsber. der preussischen Akademie der Wiss. (Phys.-math. Klasse)*, 18, 19 (1926), 211-214. See also "Integralgleichungen und fastperiodische Funktionen", *Math. Annalen*, 97 (1926), 338-356.

† "Über die Entwicklung einer gegebenen Funktion nach den Eigenfunktionen eines positiv definiten Kerns", *Schwarz-Festschrift* (1914), 392-409 (404).

in the variable t . The individual term of $f(t)$ has a *frequency* characterized by Λ_k , an *amplitude* determined by $|A_k|$, and a *phase*, given by the argument of A_k . If we consider t as the time, and $f(t)$ as the displacement of a particle at time t , as in a sound vibration, the motion defined by $f(t)$ will have a certain spectrum. Now, in light spectra, as given by a prism or grating, and in the analogous sound spectra, amplitude and frequency play a definite rôle, but phase is neglected. A spectrum, indeed, is determined by a distribution of amplitude in frequency. It is thus a matter of some interest to find a mathematical expression depending on f , which shall determine the frequencies and amplitudes of the components of f , and shall be determined by them, without any reference to their phases.

The function

$$g(t) = \lim_{X \rightarrow \infty} \frac{1}{2X} \int_{-X}^X f(x) \bar{f}(x-t) \, dx$$

(where the bar denotes the conjugate) has the desired property. It is readily seen that

$$g(t) = \sum_1^n |A_k|^2 e^{i\Lambda_k t}.$$

The function $g(t)$ determines and is determined by the function

$$R(u) = \frac{1}{\pi} \int_{-\infty}^{\infty} g(t) \frac{\sin ut}{t} \, dt.$$

It is easily seen that, if $\Lambda_k \neq 0$,

$$\frac{1}{\pi} \int_{-\infty}^{\infty} e^{i\Lambda_k t} \frac{\sin ut}{t} \, dt = \begin{cases} -1 & (u < -|\Lambda_k|) \\ -\frac{1}{2} & (u = -|\Lambda_k|) \\ 0 & (-|\Lambda_k| < u < |\Lambda_k|) \\ \frac{1}{2} & (u = |\Lambda_k|) \\ 1 & (u > |\Lambda_k|) \end{cases}$$

and it is well known that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin ut}{t} \, dt = \operatorname{sgn} u.$$

Hence $R(u)$ is an increasing step-function which has a jump at $u = |\Lambda_k|$. If we group the Λ_k 's into pairs of corresponding negative and positive values, and if $\Lambda_m > 0$, $\Lambda_n = -\Lambda_m$, then the jump of R at Λ_m is $|A_m|^2 + |A_n|^2$. Here, of course, A_m or A_n may vanish. $R(0)$ is zero, and $R(+0)$ is $|A_0|^2$. R is moreover an odd function.

The function R gives us a direct picture of the spectrum of f . If $f(t)$ is taken as the velocity of a moving particle, $R(u_2) - R(u_1)$ may, in general, be considered to be proportional to the power of those frequencies in the vibration of the particle which are contained between u_2 and u_1 .

2. The spectrum of an arbitrary function of one variable.

In the definition of g and R , no use whatever is made of the harmonic resolution of f as a trigonometrical polynomial. It seems possible, therefore, to extend the definition of the intensity of the spectrum of f over the region (u_1, u_2) as $R(u_2) - R(u_1)$ to functions other than trigonometrical polynomials. Let us now investigate the conditions under which this can actually be done.

We shall make the assumption (A) that

$$g(t) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(x) \bar{f}(x-t) dx$$

exists for every t . By the Schwarz inequality,

$$\left| \int_{-T}^T f(x) \bar{f}(x-t) dx \right|^2 \leq \left[\int_{-T}^T |f(x)|^2 dx \int_{-T}^T |f(x-t)|^2 dx \right].$$

We have

$$\begin{aligned} \frac{1}{2T} \int_{-T}^T |f(x-t)|^2 dx &= \frac{1}{2T} \int_{-T-t}^{T-t} |f(x)|^2 dx \\ &= \frac{1}{2T} \int_{-T}^T |f(x)|^2 dx + \frac{1}{2T} \int_{-T-t}^{-T} |f(x)|^2 dx - \frac{1}{2T} \int_{T-t}^T |f(x)|^2 dx. \end{aligned}$$

Now

$$\begin{aligned} \frac{1}{2T} \int_{-T-t}^{-T} |f(x)|^2 dx &\leq \frac{1}{2T} \int_{-T-t}^{T+t} |f(x)|^2 dx - \frac{1}{2T} \int_{-T}^T |f(x)|^2 dx \\ &= \left(1 + \frac{t}{T}\right) \frac{1}{2(T+t)} \int_{-T-t}^{T+t} |f(x)|^2 dx - \frac{1}{2T} \int_{-T}^T |f(x)|^2 dx. \end{aligned}$$

Hence

$$\begin{aligned} &\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T-t}^{-T} |f(x)|^2 dx \\ &\leq \lim_{T \rightarrow \infty} \left(1 + \frac{t}{T}\right) \lim_{T \rightarrow \infty} \frac{1}{2(T+t)} \int_{-T-t}^{T+t} |f(x)|^2 dx - \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(x)|^2 dx = 0. \end{aligned}$$

Similarly,

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{T-t}^{T+t} |f(x)|^2 dx = 0.$$

Hence
$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(x-t)|^2 dx = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(x)|^2 dx.$$

From this we readily conclude that $|g(t)| \leq g(0)$. The function $g(t)$ is thus bounded. Indeed, we can go further, and assert that $g(t)$ is the limit of a uniformly bounded sequence of summable functions, and hence is summable over any finite range. From this it follows readily that $g(t)/t$ is summable over any finite range which does not include the origin and it is of summable absolute square from $-\infty$ to $-\epsilon$, and from ϵ to ∞ .

We now wish to introduce the chief theorems of the Plancherel theory of the Fourier transform*. We shall translate this into its complex form. If $f(x)$ is a complex function of the real variable x , summable over any finite range, and such that

$$\int_{-\infty}^{\infty} |f(x)|^2 dx$$

exists, then
$$F(u) = \text{l.i.m.}_{A \rightarrow \infty} \frac{1}{\sqrt{(2\pi)}} \int_{-A}^A f(x) e^{iux} dx$$

exists, and is summable over any finite range, and

$$\int_{-\infty}^{\infty} |F(u)|^2 du$$

exists and is equal to
$$\int_{-\infty}^{\infty} |f(x)|^2 dx.$$

Furthermore,
$$f(x) = \text{l.i.m.}_{A \rightarrow \infty} \frac{1}{\sqrt{(2\pi)}} \int_{-A}^A F(u) e^{-iux} du.$$

Here l.i.m. stands for "limit in mean" over $(-\infty, \infty)$.

Since $g(t)/t$ is summable over any finite range and of summable absolute square over $(-\infty, -\epsilon)$ and (ϵ, ∞) , there exists

$$\text{l.i.m.}_{N \rightarrow \infty} \frac{1}{\pi} \left[\int_{-N}^{-\epsilon} + \int_{\epsilon}^N \right] g(t) \frac{\sin ut}{t} dt$$

over the whole u -line. Thus there exists as a limit in the mean over any finite segment of the u -line

$$R(u) = \text{l.i.m.}_{N \rightarrow \infty} \frac{1}{\pi} \int_{-N}^N g(t) \frac{\sin ut}{t} dt.$$

* Hobson, *The theory of functions of a real variable*, 2nd edition, 2 (1926), 742-752.

We now wish to prove that $R(u)$ is a non-decreasing function. This is essential if we are to consider $R(u_2) - R(u_1)$ as the total spectral power of f in the region (u_1, u_2) . To prove this, we introduce the auxiliary functions

$$f_P(t) = f(t) \quad [|t| \leq P]; \quad f_P(t) = 0 \quad [|t| > P];$$

$$g_P(t) = \frac{1}{2P} \int_{-\infty}^{\infty} f_P(x) \bar{f}_P(x-t) dx;$$

$$R_P(u) = \frac{1}{\pi} \int_{-\infty}^{\infty} g_P(t) \frac{\sin ut}{t} dt.$$

The integrals here are infinite integrals only in appearance; it is easily seen that we are justified in writing

$$\begin{aligned} \frac{dR_P(u)}{du} &= \frac{1}{\pi} \int_{-\infty}^{\infty} g_P(t) \cos ut dt \\ &= \frac{1}{2\pi P} \int_{-\infty}^{\infty} \cos ut dt \int_{-\infty}^{\infty} f_P(x) \bar{f}_P(x-t) dx \\ &= \frac{1}{2\pi P} \int_{-\infty}^{\infty} f_P(x) dx \int_{-\infty}^{\infty} \bar{f}_P(x-t) \cos ut dt \\ &= \frac{1}{2\pi P} \int_{-\infty}^{\infty} f_P(x) dx \int_{-\infty}^{\infty} \bar{f}_P(\xi) \cos u(x-\xi) d\xi \\ &= \frac{1}{2\pi P} \left\{ \left| \int_{-\infty}^{\infty} f_P(x) \cos ux dx \right|^2 + \left| \int_{-\infty}^{\infty} f_P(x) \sin ux dx \right|^2 \right\} \\ &\geq 0. \end{aligned}$$

Hence $R_P(u)$ is a monotone non-decreasing function of u for each P .

The function $g_P(t)$ converges boundedly to $g(t)$ as $P \rightarrow \infty$. Hence over the intervals $(-\infty, -\epsilon)$ and (ϵ, ∞) , $g_P(t)/t$ converges in the mean to $g(t)/t$. It follows from Plancherel's theory of the Fourier transform that

$$\text{l.i.m.}_{P \rightarrow \infty} \frac{1}{\pi} \left[\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right] g_P(t) \frac{\sin ut}{t} dt = \text{l.i.m.}_{N \rightarrow \infty} \frac{1}{\pi} \left[\int_{-N}^{-\epsilon} + \int_{\epsilon}^N \right] g(t) \frac{\sin ut}{t} dt.$$

From this we readily conclude that over any finite range

$$R(u) = \text{l.i.m.}_{P \rightarrow \infty} R_P(u).$$

We see at once, since every R_P is non-decreasing, and since therefore $\frac{1}{2\eta} \int_{u-\eta}^{u+\eta} R_P(\lambda) d\lambda$ is monotone, that its limit, $\frac{1}{2\eta} \int_{u-\eta}^{u+\eta} R(\lambda) d\lambda$ is likewise

monotone. By a well known theorem, however*, it is possible to choose $R(u)$

$$\lim_{\eta \rightarrow 0} \frac{1}{2\eta} \int_{u-\eta}^{u+\eta} R(\lambda) d\lambda = R(u)$$

almost everywhere. If we define $R(u)$ as this limit wherever this limit exists, and choose the values of $R(u)$ suitably at all other points, it is easy to see that we can so determine $R(u)$ that it shall be monotone non-decreasing. It will always be possible to put $R(0) = 0$.

Let us choose $R(u)$ in this manner. Let us put

$$\rho(u) = R(u) - \frac{1}{\pi} \int_{-M}^M g(t) \frac{\sin ut}{t} dt.$$

The function $\rho(u)$ is of limited total variation over every finite interval, and by a familiar theorem†, its Fourier integral converges everywhere to $\frac{1}{2}\{\rho(u+0) + \rho(u-0)\}$. It is easy to conclude from these properties that, as an ordinary infinite integral,

$$\frac{R(u+0) + R(u-0)}{2} = \frac{1}{\pi} \int_{-\infty}^{\infty} g(t) \frac{\sin ut}{t} dt.$$

Inasmuch as $R(u)$ is of limited total variation, it can have at most a denumerable set of discontinuities, all of the first kind. This is of fundamental importance when we came to the study of almost periodic functions.

3. The total spectral intensity.

It is manifest that $\lim_{u \rightarrow \infty} R(u)$, or, as we shall write it, $R(\infty)$, if it exists, is a measure of the total spectral intensity of f . It is therefore a matter of some interest to determine, first, whether $R(\infty)$ is finite, and, secondly, what its relation with $f(t)$ may be.

We have, as an appeal to the Plancherel theory will show,

$$\begin{aligned} \frac{1}{S} \int_0^S R(u) du &= \frac{1}{\pi S} \int_{-\infty}^{\infty} \frac{g(t)}{t} dt \int_0^S \sin ut du \\ &= \frac{1}{\pi S} \int_{-\infty}^{\infty} g(t) \frac{1 - \cos St}{t^2} dt. \end{aligned}$$

* Hobson, 1, 541.

† Hobson, 2, 741.

Since R is monotone increasing, $R(\infty)$ will exist when, and only when,

$$\lim_{S \rightarrow \infty} \frac{1}{S} \int_0^S R(u) du$$

exists, and will assume the same value. Manifestly, however,

$$\begin{aligned} \frac{1}{S} \int_0^S R(u) du &\leq \max |g| \frac{1}{\pi S} \int_{-\infty}^{\infty} \frac{1 - \cos St}{t^2} dt \\ &= g(0). \end{aligned}$$

Thus $R(\infty)$ always exists. If $g(t)$ is continuous at $t = 0$, an argument of the Fejér type will show that

$$\begin{aligned} R(\infty) &= \lim_{S \rightarrow \infty} \frac{1}{S} \int_0^S R(u) du = g(0) \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(t)|^2 dt. \end{aligned}$$

We thus see that, under this assumption, the mean square of the modulus of f gives the total spectral intensity.

We shall call the assumption that $g(t)$ is continuous at $t = 0$ assumption (B). As an example of a function for which it is not fulfilled, we may put $f(t) = \sin t^2$. Here

$$g(t) = 0 \quad [t \neq 0];$$

$$g(0) = \frac{1}{2}.$$

We see that R vanishes identically, and $R(\infty) \neq g(0)$.

4. The spectrum of an almost periodic function.

In this and the following two paragraphs we shall confine our attention to the almost periodic functions of Bohr. A continuous function $f(t)$ is said to have a *translation number* τ pertaining to the quantity ϵ if, for every value of t ,

$$|f(t+\tau) - f(t)| \leq \epsilon.$$

The function $f(t)$ is said to be *almost periodic* if, for every positive ϵ , there exists a positive number l_ϵ , such that every interval $(a, a+l_\epsilon)$ contains at least one translation number τ pertaining to ϵ . The following

facts are capable of an *elementary* proof: the sum or product of a finite number of almost periodic functions is almost periodic, and so is any constant multiple of an almost periodic function; the limit of a uniformly convergent series of almost periodic functions is almost periodic; almost periodic functions are uniformly continuous. If $f(t)$ is almost periodic,

$$M\{f\} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_x^{x+T} f(t) dt$$

exists as a uniform limit in z . The sum of a uniformly convergent series of sines and cosines of multiples of t is almost periodic. If $f(t)$ is almost periodic, $g(t) = M_x\{f(x)\bar{f}(x-t)\}$ is also almost periodic. If $f(t)$ is almost periodic and is greater than or equal to zero, and $M\{f\} = 0$, $f(t)$ is identically zero. We now wish to derive from these facts a proof of Bohr's analogue to Parseval's theorem.

Let $f(t)$ be an almost periodic function; then our conditions (A) and (B) are fulfilled, as we may easily show. Let us form $g(t)$ and $R(u)$ as before. Let $R(+0) = a_0$, and let the discontinuities of R for positive arguments, at most a denumerable set, be at $u = \lambda_1, \lambda_2, \dots$. Let

$$a_n = R(\lambda_n + 0) - R(\lambda_n - 0).$$

All the coefficients a_n are positive, and

$$\sum_0^\infty a_k \leq R(\infty) = M\{|f|^2\}.$$

Let us form the function

$$\gamma(t) = \Re g(t) - \sum_0^\infty a_k \cos \lambda_k t,$$

where $\Re$ denotes the real part of the function. We have, as a simple computation will show,

$$\frac{2}{\pi} \int_0^\infty \gamma(t) \frac{\sin ut}{t} dt = R(u) - R_1(u),$$

where the odd function $R_1(u)$ consists of the sum of all the jumps of R with abscissae less than u , together with half the jump (if any) with abscissa u . That is

$$R_2(u) = \frac{2}{\pi} \int_0^\infty \gamma(t) \frac{\sin ut}{t} dt$$

is a continuous function of limited total variation, say V .

Consider
$$\frac{1}{2\epsilon} \int_{-\infty}^{\infty} |\Phi(u+\epsilon) - \Phi(u-\epsilon)| du.$$

This may be written

$$\begin{aligned} & \frac{1}{2\epsilon} \sum_{-\infty}^{\infty} \int_{2k\epsilon}^{2(k+1)\epsilon} |\Phi(u+\epsilon) - \Phi(u-\epsilon)| du \\ &= \frac{1}{2\epsilon} \int_0^{2\epsilon} \sum_{-\infty}^{\infty} |\Phi\{u+(2k+1)\epsilon\} - \Phi\{u+(2k-1)\epsilon\}| du \\ &\leq \max_u \sum_{-\infty}^{\infty} |\Phi\{u+(2k+1)\epsilon\} - \Phi\{u+(2k-1)\epsilon\}|, \end{aligned}$$

and this does not exceed the total variation of Φ over $(-\infty, \infty)$.

Similarly,
$$\frac{1}{2\epsilon} \int_A^{\infty} |\Phi(u+\epsilon) - \Phi(u-\epsilon)| du$$

does not exceed the total variation of Φ over $(A-\epsilon, \infty)$, and

$$\frac{1}{2\epsilon} \int_{-\infty}^{-A} |\Phi(u+\epsilon) - \Phi(u-\epsilon)| du$$

does not exceed the total variation of Φ over $(-\infty, -A+\epsilon)$.

We have

$$\begin{aligned} & \frac{1}{2\epsilon} \int_{-\infty}^{\infty} [R_2(u+\epsilon) - R_2(u-\epsilon)]^2 du \\ &\leq \text{Max}_{A>u>-A} |R_2(u+\epsilon) - R_2(u-\epsilon)| \frac{1}{2\epsilon} \int_{-\infty}^{\infty} |R_2(u+\epsilon) - R_2(u-\epsilon)| du \\ &\quad + \text{Max} |R_2(u_1) - R_2(u_2)| \frac{1}{2\epsilon} \left[\int_A^{\infty} + \int_{-\infty}^{-A} \right] |R_2(u+\epsilon) - R_2(u-\epsilon)| du. \end{aligned}$$

The second term contains as a factor a quantity not exceeding the total variation of R_2 over $(A-\epsilon, \infty)$ and $(-\infty, \epsilon-A)$, taken together, and can be made arbitrarily small for all values of ϵ less than some fixed quantity by taking A large enough. When this is done, the first term, which is less than or equal to

$$V. \max_{A>u>-A} |R_2(u+\epsilon) - R_2(u-\epsilon)|,$$

can be made arbitrarily small by taking ϵ small enough. It follows,

from the Fourier transform theory, that

$$\begin{aligned} 0 &= \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_{-\infty}^{\infty} [R_2(u+\epsilon) - R_2(u-\epsilon)]^2 du \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_{-\infty}^{\infty} \left[\frac{4}{\pi} \int_0^{\infty} \gamma(t) \frac{\sin \epsilon t}{t} \cos ut dt \right]^2 du \\ &= \lim_{\epsilon \rightarrow 0} \frac{4}{\pi \epsilon} \int_{-\infty}^{\infty} [\gamma(t)]^2 \frac{\sin^2 \epsilon t}{t} dt. \end{aligned}$$

Now, $\frac{\sin^2 \epsilon t}{\epsilon^2 t^2} > \frac{4}{\pi^2}$ for $|\epsilon t| \leq \frac{\pi}{2}$. Hence

$$0 = \lim_{\epsilon \rightarrow 0} \frac{16\epsilon}{\pi^3} \int_{-(\pi/2\epsilon)}^{\pi/2\epsilon} [\gamma(t)]^2 dt.$$

It follows that

$$M\{|\gamma(t)|^2\} = 0.$$

However, $\Re g(t)$ and $\sum_0^{\infty} a_k \cos \lambda_k t$ are both almost periodic, and hence the same is true of $\gamma(t)$. It follows that $\gamma(t)$ is identically zero, that

$$\Re g(t) = \sum_0^{\infty} a_k \cos \lambda_k t,$$

and that

$$R(u) = R_1(u).$$

Since $R_1(u)$ increases solely by jumps, we have

$$M\{|f(t)|^2\} = R(\infty) = R_1(\infty) = \sum_0^{\infty} a_k.$$

In physical language, the entire power of the spectrum of an almost periodic function is contained in a denumerable set of lines.

5. The Parseval theorem for almost periodic functions.

We now wish to seek for a means of expressing the various coefficients a_k directly in terms of f . We have

$$\Re g(t) = \sum_0^{\infty} a_k \cos \lambda_k t,$$

where this series converges uniformly and absolutely. Accordingly,

$$M\{\Re g(t) \cos \lambda_j t\} = \frac{1}{2} a_j \quad [j \neq 0];$$

$$M\{\Re g(t)\} = a_0.$$

However, since it follows at once from the definition of $g(t)$ that $g(-t) = \bar{g}(t)$,

$$M\{g(t) \cos \lambda_j t\} = \frac{1}{2}a_j.$$

That is, because of the uniform existence of means of almost periodic functions,

$$\begin{aligned} a_j &= 2 \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \cos \lambda_j t dt \lim_{U \rightarrow \infty} \frac{1}{2U} \int_{-U}^U f(x) \bar{f}(x-t) dx \\ &= 2 \lim_{U \rightarrow \infty} \frac{1}{2U} \int_{-U}^U f(x) dx \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \bar{f}(x-t) \cos \lambda_j t dt \\ &= 2 \lim_{U \rightarrow \infty} \frac{1}{2U} \int_{-U}^U f(x) dx \lim_{V \rightarrow \infty} \frac{1}{2V} \int_{-V}^V \bar{f}(y) \cos \lambda_j (x-y) dy \\ &= 2 \lim_{U \rightarrow \infty} \frac{1}{2U} \int_{-U}^U f(x) \cos \lambda_j x dx \lim_{V \rightarrow \infty} \frac{1}{2V} \int_{-V}^V \bar{f}(y) \cos \lambda_j y dy \\ &\quad + 2 \lim_{U \rightarrow \infty} \frac{1}{2U} \int_{-U}^U f(x) \sin \lambda_j x dx \lim_{V \rightarrow \infty} \frac{1}{2V} \int_{-V}^V \bar{f}(y) \sin \lambda_j y dy \\ &= 2 |M\{f(x) \cos \lambda_j x\}|^2 + 2 |M\{f(x) \sin \lambda_j x\}|^2 \\ &= \frac{1}{2} |M\{f(x)(e^{i\lambda_j x} + e^{-i\lambda_j x})\}|^2 + \frac{1}{2} |M\{f(x)(e^{i\lambda_j x} - e^{-i\lambda_j x})\}|^2 \\ &= |M\{f(x) e^{i\lambda_j x}\}|^2 + |M\{f(x) e^{-i\lambda_j x}\}|^2 \quad [j \neq 0]. \end{aligned}$$

Similarly,

$$a_0 = |M\{f(x)\}|^2.$$

Hence

$$M\{|f(t)|^2\} = \sum_0^\infty a_k = |M\{f(x)\}|^2 + \sum_1^\infty |M\{f(x) e^{i\lambda_j x}\}|^2 + |M\{f(x) e^{-i\lambda_j x}\}|^2.$$

This is a precise analogue to the Hurwitz-Parseval theorem for periodic functions, and is the well known fundamental theorem of Bohr.

6. The Weierstrass theorem for almost periodic functions*.

Let $f(z)$ be almost periodic. Consider

$$\phi(x) = \max_t |f(x+t) - f(t)|.$$

* The method of the present section has much resemblance to that used by Weyl in the paper already cited. The function $\phi(x)$ is S. Bochner's "Verschiebungsfunktion", [S. Bochner, "Beiträge zur Theorie der fastperiodischen Funktionen", *Math. Annalen*, 96 (1926), 119-147].

We have

$$\begin{aligned} |\phi(x+\tau)-\phi(x)| &\leq \max_t |f(x+t+\tau)-f(t+\tau)| - |f(x+t)-f(t)| \\ &\leq \max_t |f(x+t+\tau)-f(x+t)| + \max_t |f(t+\tau)-f(t)| \\ &\leq 2 \max_t |f(t+\tau)-f(t)|. \end{aligned}$$

Hence any displacement number for f pertaining to ϵ is a displacement number for ϕ pertaining to 2ϵ , and $\phi(x)$ is almost periodic.

It is easy to show that any polynomial in an almost periodic function is almost periodic, and hence by Weierstrass's theorem and the fact that the uniform limit of an almost periodic function is almost periodic, that any continuous function of an almost periodic function is almost periodic. Let $H(u)$ be a continuous non-negative function of u which is 1 when $0 \leq u \leq \frac{1}{2}\epsilon$, between 1 and 0 when $\frac{1}{2}\epsilon \leq u \leq \epsilon$, and 0 when $u \geq \epsilon$. Let

$$\psi_\epsilon(x) = \frac{H[\phi(x)]}{MH[\phi(x)]}.$$

Since $H[\phi(x)]$ is somewhere positive, and it is everywhere non-negative and almost periodic, it is easy to show that $MH[\phi(x)]$ cannot vanish. The function $\psi_\epsilon(x)$ is almost periodic.

Let
$$f_\epsilon(x) = M[f(t)\psi_\epsilon(x-t)].$$

The existence and almost periodic character of $f_\epsilon(x)$ are proved without difficulty. The definition of ψ_ϵ ensures that

$$|f(x)-f(t)| \leq \epsilon$$

if $\psi_\epsilon(x-t) \neq 0$. Hence, since $f_\epsilon(x)$ is a mean of these values of $f(t)$,

$$\max_x |f(x)-f_\epsilon(x)| \leq \epsilon.$$

Similarly, if
$$f^{(\epsilon)}(x) = M[f_\epsilon(t)\psi_\epsilon(x-t)],$$

$f^{(\epsilon)}$ exists and is almost periodic, and

$$\max_x |f^{(\epsilon)}(x)-f_\epsilon(x)| \leq \epsilon.$$

Hence
$$\max_x |f^{(\epsilon)}(x)-f(x)| \leq 2\epsilon.$$

We have
$$f^{(\epsilon)}(x) = M_t \left[\psi_\epsilon(x-t) M_\tau [f(\tau) \psi_\epsilon(t-\tau)] \right].$$

Bearing in mind the uniformity properties of almost periodic functions, we have

$$f^{(\epsilon)}(x) = M \left[f(\tau) M [\psi_{\epsilon}(x-t) \psi_{\epsilon}(t-\tau)] \right].$$

However, by § 4,

$$M [\psi_{\epsilon}(x-t) \psi_{\epsilon}(t-\tau)] = \sum_0^{\infty} a_k \cos \lambda_k(x-\tau),$$

where all the coefficients a_k are positive, and $\sum_0^{\infty} a_k$ converges. Hence

$$f^{(\epsilon)}(x) = M \left[f(\tau) \sum_0^{\infty} a_k \cos \lambda_k(x-\tau) \right].$$

Since
$$\left| \frac{1}{2T} \int_{-T}^T f(\tau) \cos \lambda_k(x-\tau) d\tau \right| \leq \max |f(\tau)|,$$

it follows that we can invert the order of M and Σ , and that

$$f^{(\epsilon)}(x) = \sum_0^{\infty} a_k \left\{ \cos \lambda_k x M \{ f(\tau) \cos \lambda_k \tau \} + \sin \lambda_k x M \{ f(\tau) \sin \lambda_k \tau \} \right\}.$$

Inasmuch as
$$\left| M \left\{ f(\tau) \frac{\sin \lambda_k \tau}{\cos \lambda_k \tau} \right\} \right| \leq |f(\tau)|,$$

$f^{(\epsilon)}(x)$ is the sum of an absolutely convergent series of trigonometric terms. That is to say, we can choose N so large that

$$\max_x \left| f^{(\epsilon)}(x) - \sum_1^N a_k \left\{ \cos \lambda_k x M \{ f(\tau) \cos \lambda_k \tau \} + \sin \lambda_k x M \{ f(\tau) \sin \lambda_k \tau \} \right\} \right| \leq \epsilon,$$

and hence that

$$\max_x \left| f(x) - \sum_1^N a_k \left\{ \cos \lambda_k x M \{ f(\tau) \cos \lambda_k \tau \} + \sin \lambda_k x M \{ f(\tau) \sin \lambda_k \tau \} \right\} \right| \leq 3\epsilon.$$

In other words, we have proved Bohr's approximation theorem, to the effect that it is possible to approximate uniformly to any desired degree of accuracy to an almost periodic function by means of trigonometrical polynomials.

DIVERGENT CONTINUED FRACTIONS AND POLAR SINGULARITIES

By R. WILSON.

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PART II. *Boundary pole multiple.*

1. *Introduction.* In a previous paper[†], I have discussed the behaviour of a class of continued fractions at points of divergence, and also at points on the boundary of the region of representation. The continued fractions are those of the Padé table, and, when representing a function meromorphic in a region containing the origin, the continued fraction of the ν -th row converges at all points within the circle passing through the ν -th pole, except at enclosed poles[‡].

In the earlier paper only the case of a *simple* boundary pole was considered, and the nature of the divergence was determined at the boundary pole and also at the enclosed poles, simple, multiple, or with index-points of equal moduli.

In the present paper the discussion is extended to the case of a multiple boundary pole.

[†] R. Wilson, "Divergent continued fractions and polar singularities", *Proc. London Math. Soc.* (2), 26 (1927), 159-168.

[‡] R. Montessus de Ballore, "Sur les fractions continues algébriques", *Rend. di Palermo*, 19 (1905), 1-73.

The previous paper thus forms a necessary introduction to the methods and notation of the present paper, and references to it will bear an asterisk.

2. *Multiple poles.* A pole of order m will replace m simple poles in the earlier discussion, and it is thus necessary to develop the mechanism so as to be able to deal with multiple poles.

This is accomplished in stages.

First, only poles of the special form $B_s/(1-\beta_s z)^m$ are considered, and the function $f(z)$ of 2*, (3) is generalized to include poles of this type, as follows :

$$f(z) \equiv \sum_s B_s / (1 - \beta_s z)^m + \phi(z), \quad (1)$$

where $\phi(z)$ is holomorphic within a circle of radius $1/\rho$, with

$$\rho < |\beta| < \dots < |\beta_{r+1}| < |\beta_r| < \dots < |\beta_2| < |\beta_1|,$$

and the number of poles, counted according to their multiplicity, is $\nu+1$ (see § 8).

Since $f(z) \equiv c_0 + c_1 z + c_2 z^2 + \dots + c_n z^n + \dots$ [as in 2*, (1)]

we have

$$c_n = \sum_s^{-m} c_n B_s \beta_s^r + M_n \rho^{n+1},$$

where

$$^{-m}c_n = m(m+1) \dots (m+n-1)/n!$$

and $|M_n| < M$, a finite constant.

The earlier method is then generalized; $P_{n,\nu}(z)$ and $Q_{n,\nu}(z)$ are evaluated (for n large) and $\lim_{n \rightarrow \infty} P_{n,\nu}(z)/Q_{n,\nu}(z)$ is discussed at

(a) enclosed poles;

(b) the boundary pole;

(c) at other points on the bounding circle;

for the case in which the enclosed and boundary poles are of the above type of multiple pole. Enclosed simple poles occur in particular when $m = 1$.

* The circle of radius $1/\rho$ contains the first $\nu+1$ poles of $f(z)$, but excludes all other singularities of the function.

Finally, it is shown that the results are unaffected by the substitution of

$$\frac{B_{s,1}}{(1-\beta_s z)} + \frac{B_{s,2}}{(1-\beta_s z)^2} + \dots + \frac{B_{s,m}}{(1-\beta_s z)^m} \quad \text{for} \quad \frac{B_{s,m}}{(1-\beta_s z)^m}$$

in the case of every pole.

In the discussion which follows

$$\frac{B_{s,1}}{(1-\beta_s z)}, \quad \frac{B_{s,2}}{(1-\beta_s z)^2}, \quad \dots, \quad \frac{B_{s,m}}{(1-\beta_s z)^m}$$

will be termed elements of the multiple pole, and, for convenience, B_s will be written for $B_{s,m}$, when possible, as in (1).

3. *The effect on $Q_{n,\nu}(z)$.* The contributions in the coefficients due to the element $B_s/(1-\beta_s z)^m$ will occupy m columns in each of the two determinants Δ_{1Q} , Δ_{2Q} , whose product determines $Q_{n,\nu}(z)$.

These will be written

$$\left. \begin{array}{cccccc} 0 & \dots & 0 & & -^m c_{n-\nu} B_s & \dots & -^1 c_{n-\nu} B_s \\ -^1 c_0 \beta_s^{n-\nu+1} & \dots & -^1 c_{m-1} \beta_s^{n-\nu+1} & & -^m c_{n-\nu+1} B_s \beta_s & \dots & -^1 c_{n-\nu+1} B_s \beta_s \\ -^2 c_0 \beta_s^{n-\nu+2} & \dots & -^2 c_{m-1} \beta_s^{n-\nu+2} & & -^m c_{n-\nu+2} B_s \beta_s^2 & \dots & -^1 c_{n-\nu+2} B_s \beta_s^2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -^\nu c_0 \beta_s^n & \dots & -^\nu c_{m-1} \beta_s^n & & -^m c_n B_s \beta_s^n & \dots & -^1 c_n B_s \beta_s^n \end{array} \right\}, \quad (2)$$

The following results are useful in evaluating the corresponding terms in the product of the two determinants.

$$\left. \begin{array}{l} -^m c_{n+1} = -^m c_n + -^{m+1} c_n + \dots + -^1 c_n, \\ -^m c_{n+1} = -^{m+1} c_{n+1} + -^{m+1} c_n + \dots + -^{m+1} c_1 + -^{m+1} c_0. \end{array} \right\}$$

From these follow successively

$$\left. \begin{array}{l} -^1 c_0 \cdot -^m c_{n-\nu} + \dots + -^1 c_{m-1} \cdot -^1 c_{n-\nu} = -^m c_{n-\nu+1}, \\ -^2 c_0 \cdot -^m c_{n-\nu} + \dots + -^2 c_{m-1} \cdot -^1 c_{n-\nu} = -^m c_{n-\nu+2}, \\ -^3 c_0 \cdot -^m c_{n-\nu} + \dots + -^3 c_{m-1} \cdot -^1 c_{n-\nu} = -^m c_{n-\nu+3}, \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \end{array} \right\}, \quad (3)$$

Using (3), it is clear that in the product of Δ_{1Q} and Δ_{2Q} the contributions from (2) produce terms

$$-^m c_{n-\nu+1} B_s \beta_s^{n-\nu+1} \text{ in } c_{n-\nu+1}, \quad \dots, \quad -^m c_{n-\nu+t} B_s \beta_s^{n-\nu+t} \text{ in } c_{n-\nu+t}, \quad \dots$$

2 K 2

Hence, with a multiple pole of order p at $1/\beta_r$ and a boundary pole of order k at $1/\beta$,

$$Q_{n,\nu}(1/\beta_r) = \frac{1}{B_r \beta_r^\nu} \Delta_{1Q} \times \Delta_{2Q} [1 + K_n(\rho/\beta)^n] \quad (\text{see } 3^*)$$

$$= \frac{1}{B_r \beta_r^\nu} \begin{vmatrix} \dots & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \dots & -^1c_0 \beta_s^{n-\nu+1} & \dots & -^1c_{m-1} \beta_s^{n-\nu+1} & \dots & -^1c_0 \beta_r^{n-\nu+1} & \dots & \\ \dots & -^2c_0 \beta_s^{n-\nu+2} & \dots & -^2c_{m-1} \beta_s^{n-\nu+2} & \dots & -^2c_0 \beta_r^{n-\nu+2} & \dots & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ \dots & -^\nu c_0 \beta_s^n & \dots & -^\nu c_{m-1} \beta_s^n & \dots & -^\nu c_0 \beta_r^n & \dots & \end{vmatrix} \\ \begin{vmatrix} \dots & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \dots & -^1c_{\mu-1} \beta_r^{n-\nu+1} & \dots & -^1c_0 \beta_r^{n-\nu+1} & \dots & -^1c_{k-1} \beta_r^{n-\nu+1} & \dots & \\ \dots & -^2c_{\mu-1} \beta_r^{n-\nu+2} & \dots & -^2c_0 \beta_r^{n-\nu+2} & \dots & -^2c_{k-1} \beta_r^{n-\nu+2} & \dots & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ \dots & -^\nu c_{\mu-1} \beta_r^n & \dots & -^\nu c_0 \beta_r^n & \dots & -^\nu c_{k-1} \beta_r^n & \dots & \end{vmatrix} \\ \times \begin{vmatrix} \dots & -^m c_{n-\nu} B_s & \dots & -^1 c_{n-\nu} B_s & \dots & -^p c_{n-\nu} B_r & \dots & \\ \dots & -^m c_{n-\nu+1} B_s \beta_s & \dots & -^1 c_{n-\nu+1} B_s \beta_s & \dots & -^p c_{n-\nu+1} B_r \beta_r & \dots & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ \dots & -^m c_n B_s \beta_s^\nu & \dots & -^1 c_n B_s \beta_s^\nu & \dots & -^p c_n B_r \beta_r^\nu & \dots & \\ \dots & -^1 c_{n-\nu} B_r & \dots & -^k c_{n-\nu} B & \dots & -^1 c_{n-\nu} B & \dots & \\ \dots & -^1 c_{n-\nu+1} B_r \beta_r & \dots & -^k c_{n-\nu+1} B \beta & \dots & -^1 c_{n-\nu+1} B \beta & \dots & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ \dots & -^1 c_n B_r \beta_r^\nu & \dots & -^k c_n B \beta^\nu & \dots & -^1 c_n B \beta^\nu & \dots & \end{vmatrix} \\ \times [1 + K_n(\rho/\beta)^n], \quad (4)$$

where $|K_n| < K$, a finite constant.

These non-vanishing determinants will be evaluated later.

4. *The effect on $P_{n,\nu}(z)$.* The minors of the top row terms

$$\psi_{n-t}(1/\beta_r)/\beta_r^t \quad (t = 0, 1, \dots, \nu)$$

of $P_{n,\nu}(1/\beta_r)$ are the same as those of $Q_{n,\nu}(1/\beta_r)$.

Take the contributions (common multiplier omitted)

$$^{-m}c_{n-\nu+1}(\beta_s/\beta_r)^{n-\nu+1}/\beta_r^\nu, \dots, \quad ^{-m}c_{n+1}(\beta_s/\beta_r)^{n+1}$$

in

$$\psi_{n-t}(1/\beta_r)/\beta_r^\nu, \dots, \psi_n(1/\beta_r).$$

This set gives to $P_{n,\nu}(1/\beta_r)$

$$\Delta'_{1P} \times \Delta_{2P} [1 + K_n(\rho/\beta)^n] / \beta_r^\nu,$$

where $\Delta_{2P} = \Delta_{2Q}$, $|K_n| < K$, a finite constant, and Δ'_{1P} is Δ_{1Q} modified as follows.

The zeros in (4) of the first row of Δ_{1Q} above the columns (2) are replaced by $^{-1}c_0\beta_s^{n-\nu+1}$, ..., $^{-1}c_{m-1}\beta_s^{n-\nu+1}$, giving the columns

$$\begin{array}{ccccccc} ^{-1}c_0\beta_s^{n-\nu+1} & \dots & ^{-1}c_{m-1}\beta_s^{n-\nu+1} & & & & \\ ^{-1}c_0\beta_s^{n-\nu+1} & \dots & ^{-1}c_{m-1}\beta_s^{n-\nu+1} & & & & \\ ^{-2}c_0\beta_s^{n-\nu+2} & \dots & ^{-2}c_{m-1}\beta_s^{n-\nu+2} & & & & \\ \dots & \dots & \dots & \dots & \dots & & \\ ^{-\nu}c_0\beta_s^n & \dots & ^{-\nu}c_{m-1}\beta_s^n & & & & \end{array}$$

It is easy to show that $\lim_{n \rightarrow \infty} \Delta'_{1P}/\Delta_{1Q}$ exists and is finite, and so, for the finite number of contributions due to enclosed poles other than that at $1/\beta_r$, $\lim_{n \rightarrow \infty} (\Sigma \Delta'_{1P})/\Delta_{1Q}$ is finite. Hence the contributions to $P_{n,\nu}(1/\beta_r)$ due to enclosed poles other than that at $1/\beta_r$ are of the same order as $Q_{n,\nu}(1/\beta_r)$.

It will now be shown that the contribution to $P_{n,\nu}(1/\beta_r)$ due to the pole at $1/\beta_r$ is of a higher order than $Q_{n,\nu}(1/\beta_r)$.

Consequently, all contributions from internal poles (other than that at $1/\beta_r$), whether simple or multiple, are ignorable, as in 4 (6)*, 6* and 7*, and this irrespective of the multiplicity of the boundary pole.

5. *Evaluation of $P_{n,\nu}(1/\beta_r)$.* The contribution in $\psi_{n-t}(1/\beta_r)/\beta_r^\nu$ due to the pole of order p at $1/\beta_r$ is $^{-p-1}c_{n-t}B_r/\beta_r^\nu$.

Hence

$$P_{n,\nu}(1/\beta_r) = \Delta_{1P} \{ \Delta_{2P} \times [1 + G_n(\rho/\beta)^n] + H_n \Delta_{2Q} \},$$

where $|G_n| < G$, $|H_n| < H$, finite constants.

$$\Delta_{1P} \times \Delta_{2P} = \frac{1}{|\beta_r|} \begin{vmatrix} \dots 0 & \dots 0 & \dots 0 \\ \dots -^1c_0\beta_s^{n-\nu+1} & \dots -^1c_{m-1}\beta_s^{n-\nu+1} & \dots -^1c_0\beta_r^{n-\nu+1} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots -^\nu c_0\beta_s^n & \dots -^\nu c_{m-1}\beta_s^n & \dots -^\nu c_0\beta_r^n \\ \dots 0 & \dots 0 & \dots 0 & 1 \\ \dots -^1c_{p-1}\beta_r^{n-\nu+1} & \dots -^1c_0\beta_r^{n-\nu+1} & \dots -^1c_{k-1}\beta_r^{n-\nu+1} & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots -^\nu c_{p-1}\beta_r^n & \dots -^\nu c_0\beta_r^n & \dots -^\nu c_{k-1}\beta_r^n & 0 \end{vmatrix} \\ \times \begin{vmatrix} \dots -^m c_{n-\nu} B_s & \dots -^1 c_{n-\nu} B_s & \dots -^p c_{n-\nu} B_r \\ \dots -^m c_{n-\nu+1} B_s \beta_s & \dots -^1 c_{n-\nu+1} B_s \beta_s & \dots -^p c_{n-\nu+1} B_r \beta_r \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots -^m c_n B_s \beta_s^\nu & \dots -^1 c_n B_s \beta_s^\nu & \dots -^p c_n B_r \beta_r^\nu \\ \dots -^1 c_{n-\nu} B_r & \dots -^k c_{n-\nu} B & \dots -^1 c_{n-\nu} B & -^{p-1} c_{n-\nu} B_r \\ \dots -^1 c_{n-\nu+1} B_r \beta_r & \dots -^k c_{n-\nu+1} B \beta & \dots -^1 c_{n-\nu+1} B \beta & -^{p-1} c_{n-\nu+1} B_r \beta_r \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots -^1 c_n B_r \beta_r^\nu & \dots -^k c_n B \beta^\nu & \dots -^1 c_n B \beta^\nu & -^{p-1} c_n B_r \beta_r^\nu \end{vmatrix} \tag{6}$$

where Δ_{1P} , Δ_{2Q} are matrices of order $\nu+1$ in row and $\nu+2$ in column.
The product can thus be expanded as the sum of $\nu+2$ products of corresponding determinants of order $\nu+1$.
For the purpose of determining the values of these separate bi-products, two lemmas will be quoted.

LEMMA 1. *The value of the determinant*

$$\begin{vmatrix} -^1c_0 & \dots & -^1c_{m_1-1} & \dots & -^1c_0 & \dots & -^1c_{m_t-1} \\ -^2c_0\beta_1 & \dots & -^2c_{m_1-1}\beta_1 & \dots & -^1c_0\beta_t & \dots & -^2c_{m_t-1}\beta_t \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -^\nu c_0\beta_1^{\nu-1} & \dots & -^\nu c_{m_1-1}\beta_1^{\nu-1} & \dots & -^\nu c_0\beta_t^{\nu-1} & \dots & -^\nu c_{m_t-1}\beta_t^{\nu-1} \end{vmatrix}$$

in which any m may be unity, is $\Pi(\beta_r - \beta_s)^{m_r m_s} \Pi \beta_r^{m_r (m_r - 1)/2}$

LEMMA 2. *The determinant Δ formed from a succession of sets of columns such as*

$$\begin{array}{cccccc} -^a c_{n-\nu}, & -^{a+1} c_{n-\nu}, & \dots, & -^2 c_{n-\nu}, & -^1 c_{n-\nu}, \\ -^a c_{n-\nu+1} \alpha, & -^{a+1} c_{n-\nu+1} \alpha, & \dots, & -^2 c_{n-\nu+1} \alpha, & -^1 c_{n-\nu+1} \alpha, \\ -^a c_{n-\nu+2} \alpha^2, & -^{a+1} c_{n-\nu+2} \alpha^2, & \dots, & -^2 c_{n-\nu+2} \alpha^2, & -^1 c_{n-\nu+2} \alpha^2, \\ \dots & \dots & \dots & \dots & \dots \\ -^a c_n \alpha^\nu, & -^{a+1} c_n \alpha^\nu, & \dots, & -^2 c_n \alpha^\nu, & -^1 c_n \alpha^\nu, \end{array}$$

typified by the number pairs $a, \alpha; b, \beta; c, \gamma; \dots x, \xi; \dots y, \eta; \dots k, \kappa$ (in which any x may be unity), together with the set l, λ ;

$$\begin{array}{cccccc} -^{l-1} c_{n-\nu}, & \dots, & -^{r-1} c_{n-\nu}, & -^{r+1} c_{n-\nu}, & \dots, & -^1 c_{n-\nu}, \\ -^{l-1} c_{n-\nu+1} \lambda, & \dots, & -^{r-1} c_{n-\nu+1} \lambda, & -^{r+1} c_{n-\nu+1} \lambda, & \dots, & -^1 c_{n-\nu+1} \lambda, \\ -^{l-1} c_{n-\nu+2} \lambda^2, & \dots, & -^{r-1} c_{n-\nu+2} \lambda^2, & -^{r+1} c_{n-\nu+2} \lambda^2, & \dots, & -^1 c_{n-\nu+2} \lambda^2, \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -^{l-1} c_n \lambda^\nu, & \dots, & -^{r-1} c_n \lambda^\nu, & -^{r+1} c_n \lambda^\nu, & \dots, & -^1 c_n \lambda^\nu \end{array}$$

satisfies the relation

$$\Delta = \frac{n^{l-r+1}}{(l-r+1)!} \Pi(\eta-\xi)^{yz} \Pi \xi^{x(x-1)/2} \left[1 + O\left(\frac{1}{n}\right) \right].$$

Only products in which a β column is omitted from Δ_{1P} will be considered. They are of a higher order in this factor than the others, and from (4) and (6) it is clear that the ratio of any such factor to Δ_{1Q} is of the order $(\beta_r/\beta)^n$.

It is now possible to compare the second factors by means of Lemma 2. From (6), we see that there are k of these (corresponding each to the omission of one β column in the first factor), and that their orders vary through integral powers of n .

The survivor is clearly that in which the column

$$-^1 c_{n-\nu} B, \dots, -^1 c_n B \beta^\nu$$

is omitted. This determines the surviving biproduct term $\bar{\Delta}_{1P} \times \bar{\Delta}_{2P}$.

Comparing $\bar{\Delta}_{1P}$ with Δ_{1Q} , it follows from Lemma 1 that

$$\frac{\bar{\Delta}_{1P}}{\Delta_{1Q}} = \left(\frac{\beta_r}{\beta} \right)^{n-\nu+1} \frac{\beta_r^{p-1}}{\beta^{k-1}} \frac{\prod' (\beta_r - \beta_s)^n}{\prod' (\beta - \beta_s)^n} (\beta_r - \beta)^{k-p},$$

where terms in $(\beta_r \sim \beta)$ are omitted in Π' .

Comparing $\bar{\Delta}_{2P}$ with Δ_{2Q} , it follows from Lemma 2 that

$$\frac{\bar{\Delta}_{2P}}{\Delta_{2Q}} = \frac{n^{k-1}}{(k-1)!} \frac{B_r}{B} \frac{\beta_r^n}{\beta_{k-1}} \frac{\prod_s'(\beta_r-\beta_s)^m}{\prod_s'(\beta-\beta_s)^m} (\beta_r-\beta)^{k-p-1}.$$

Hence, from (4) and (6),

$$\frac{P_{n,\nu}(1/\beta_r)}{Q_{n,\nu}(1/\beta_r)} = \frac{n^{k-1}}{(k-1)!} \frac{B_r^2}{B} \frac{\beta_r^{2p-1}}{\beta^{2k-2}} \frac{\prod_s'(\beta_r-\beta_s)^{2m}}{\prod_s'(\beta-\beta_s)^{2m}} \frac{[1+O(1/n)]}{(\beta_r-\beta)^{2p-2k+1}},$$

and therefore, writing $R_{n,\nu}(z)$ for $P_{n,\nu}(z)/Q_{n,\nu}(z)$,

$$\lim_{n \rightarrow \infty} \frac{R_{n,\nu}(1/\beta_r)}{n^{k-1}(\beta_r/\beta)^n} = \frac{B_r^2}{B(k-1)!} \frac{\beta_r^{2p-\nu}}{\beta^{2k-\nu-1}} \frac{\prod_s'(\beta_r-\beta_s)^{2m}}{\prod_s'(\beta-\beta_s)^{2m}} (\beta_r-\beta)^{2k-2p-1}. \quad (7)$$

It will be seen that results (8)* and (10)* of the first paper follow at once as special cases of this.

6. *Value at the boundary pole.* The right-hand column of Δ_{2P} in (6) now becomes $^{-k-1}c_{n-\nu}B, \dots, ^{-k-1}c_nB\beta^\nu$, and the surviving biproduct is again that for which the column $^{-1}c_{n-\nu}B, \dots, ^{-1}c_nB\beta^\nu$ is omitted.

In this case, however, $\bar{\Delta}_{1P}$ and $\bar{\Delta}_{2P}$ have values different from those found in Section 5.

Thus
$$\frac{\bar{\Delta}_{1P}}{\Delta_{1Q}} = B$$

and
$$\frac{\bar{\Delta}_{2P}}{\Delta_{2Q}} = \left(\frac{n^k}{k!}\right)\left[1+O\left(\frac{1}{n}\right)\right].$$

Hence, from (4) and (6),

$$\frac{P_{n,\nu}(1/\beta)}{Q_{n,\nu}(1/\beta)} = \left(\frac{Bn^k}{k!}\right)\left[1+O\left(\frac{1}{n}\right)\right],$$

and therefore
$$\lim_{n \rightarrow \infty} \frac{R_{n,\nu}(1/\beta)}{n^k} = \frac{B}{k!}. \quad (8)$$

The result of Section 8* of the first paper is a special case of this.

7. *Passage from the element to the multiple pole.* The substitution in (1) of

$$\frac{B_{s,1}}{(1-\beta_s z)} + \frac{B_{s,2}}{(1-\beta_s z)^2} + \dots + \frac{B_{s,m}^1}{(1-\beta_s z)^m} \quad \text{for} \quad \frac{B_{s,m}}{(1-\beta_s z)^m}$$

necessitates a modification in the terms of Δ_{2P}, Δ_{2Q} .

The columns [in (4) and (6)],

$$\begin{array}{ccccccc} -^m c_{n-\nu} B_{s,m} & & \dots & & -^1 c_{n-\nu} B_{s,m}, \\ -^m c_{n-\nu+1} B_{s,m} \beta_s & & \dots & & -^1 c_{n-\nu+1} B_{s,m} \beta_s, \\ \dots & & \dots & & \dots & & \dots \\ \dots & & \dots & & \dots & & \dots \\ -^m c_n B_{s,m} \beta_s^\nu & & \dots & & -^1 c_n B_{s,m} \beta_s^\nu \end{array}$$

become

$$\begin{array}{ccccccc} \sum_{t=1}^m -^t c_{n-\nu} B_{s,t}, & \sum_{t=1}^{m-1} -^t c_{n-\nu} B_{s,t+1}, & \dots, & \sum_{t=1}^2 -^t c_{n-\nu} B_{s,t+m-2}, & -^1 c_{n-\nu} B_{s,m} \\ \sum_{t=1}^m -^t c_{n-\nu+1} B_{s,t} \beta_s, & \sum_{t=1}^{m-1} -^t c_{n-\nu+1} B_{s,t+1} \beta_s, & & & \\ & \dots, & \sum_{t=1}^2 -^t c_{n-\nu+1} B_{s,t+m-2} \beta_s, & -^1 c_{n-\nu+1} B_{s,m} \beta_s \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \sum_{t=1}^m -^t c_n B_{s,t} \beta_s^\nu, & \sum_{t=1}^{m-1} -^t c_n B_{s,t+1} \beta_s^\nu, & \dots, & \sum_{t=1}^2 -^t c_n B_{s,t+m-2} \beta_s^\nu, & -^1 c_n B_{s,m} \beta_s^\nu. \end{array}$$

Successive subtractions from the right-hand side at once reduce this to the previous set. The operation applies equally to determinants and matrices, and hence the proofs given cover the cases of enclosed poles, the pole at which evaluation is taken, and the boundary pole, not only for *elements*, but for *complete multiple poles*.

The results (7) and (8), proved for elements, thus hold for multiple poles of general form.

This extension will be used in all subsequent work, and the results obtained will hold for functions with multiple poles of general form.

8. *The case of a stationary boundary.* In the discussion of a multiple boundary pole, a case arises which has no counterpart in the discussion for a simple boundary pole. For, if the boundary pole is of order k , the bounding circle, determined as in Section 2, is also the boundary for the continued fractions of the $k-1$ rows antecedent, so that the bounding circle remains stationary for k rows of the Padé table.

A generalization of the preceding work is necessary to deal with this case. The discussion is similar to that of Sections 3-6, but the values of Δ_{1P} , Δ_{1Q} , Δ_{2P} , Δ_{2Q} for $P_{n,\nu-\tau}$, $Q_{n,\nu-\tau}$ ($\tau < k$) are now $\nu+1-\tau$ rowed matrices. Their expressions are similar in form to those given in (4) and (6). In the evaluation determinants similar to those of Lemma 2, but more complicated, are used. The results are given here, but the values

of the non-vanishing constants are not stated, owing to their cumbersome nature.

The method gives rise to the following results, corresponding to (7) and (8) respectively :

$$\lim_{n \rightarrow \infty} \frac{R_{n,\nu-\tau}(1/\beta_\tau)}{n^{k-2\tau-1}(\beta_\tau/\beta)^n} = \text{constant}, \tag{9}$$

$$\lim_{n \rightarrow \infty} \frac{R_{n,\nu-\tau}(1/\beta)}{n^k} = \text{constant}. \tag{10}$$

These results hold for $\tau = 0, 1, 2, \dots, k-1$.

9. *Value on the bounding circle.* It is required to evaluate $P_{n,\nu}(z)/Q_{n,\nu}(z)$ where $|z\beta|=1$ and $z \neq 1/\beta$.

Methods similar to those of Sections 3-5 give

$$\begin{aligned} Q_{n,\nu}(z) = & \begin{vmatrix} \dots & 0 & \dots & 0 & \dots & 0 & \dots & 0 & 1 \\ \dots & -^1c_0\beta_s^{n-\nu+1} & \dots & -^1c_{m-1}\beta_s^{n-\nu+1} & \dots & -^1c_0\beta^{n-\nu+1} & \dots & -^1c_{k-1}\beta^{n-\nu+1} & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & -^\nu c_0\beta_s^n & \dots & -^\nu c_{m-1}\beta_s^n & \dots & -^\nu c_0\beta^n & \dots & -^\nu c_{k-1}\beta^n & 0 \end{vmatrix} \\ \times & \begin{vmatrix} \dots & -^m c_{n-\nu} B_s & \dots & -^1 c_{n-\nu} B_s & \dots & -^k c_{n-\nu} B & \dots & -^1 c_{n-\nu} B & z^\nu \\ \dots & -^m c_{n-\nu+1} B_s \beta_s & \dots & -^1 c_{n-\nu+1} B_s \beta_s & \dots & -^k c_{n-\nu+1} B \beta & \dots & -^1 c_{n-\nu+1} B \beta & z^{\nu-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & -^m c_n B_s \beta_s^\nu & \dots & -^1 c_n B_s \beta_s^\nu & \dots & -^k c_n B \beta^\nu & \dots & -^1 c_n B \beta^\nu & 1 \end{vmatrix} \\ & \times [1 + K_n(\rho/\beta)^n], \end{aligned}$$

where $|K_n| < K$, a constant.

This is a product of two $\nu+2$ columned matrices, and is evaluated as in the cases of Δ_{1P} , Δ_{2P} of Section 5.

The contributions to the top row terms of $P_{n,\nu}(z)$,

$$z^\nu \psi_{n-\nu}(z), \quad z^{\nu-1} \psi_{n-\nu+1}(z), \quad \dots, \quad \psi_n(z),$$

arise as in Section 4.

They are of three types :

(a) The set of terms such as

$$z^\nu \cdot -^m c_{n-\nu+1}(\beta_s z)^{n-\nu+1}, \quad \dots, \quad -^m c_{n+1}(\beta_s z)^{n+1},$$

in which each set is multiplied by the corresponding finite common factor.

(b) The leading terms

$$\frac{z^\nu \cdot B_s}{(1 - \beta_s z)^m}, \quad \frac{z^{\nu-1} \cdot B_s}{(1 - \beta_s z)^m}, \quad \dots, \quad \frac{B_s}{(1 - \beta_s z)^n}.$$

All the above sets are drawn from (5) and are summed over the enclosed poles and the boundary pole *in all their elements*.

(c) The contributions due to the remainder function $\phi(z)$ [see (1)].

Evaluating a typical term of (a), we have

$$P'_{n,\nu}(z) = \frac{z^{n+1}}{(1 - \beta_s z)^m} \times \left[\begin{array}{cccc} \dots & {}^{-1}c_0 \beta_s^{n-\nu+1} & \dots & {}^{-1}c_{m-1} \beta_s^{n-\nu+1} & \dots & 0 \\ \dots & {}^{-1}c_0 \beta_s^{n-\nu+1} & \dots & {}^{-1}c_{m-1} \beta_s^{n-\nu+1} & \dots & {}^{-1}c_0 \beta_r^{n-\nu+1} \\ \dots & {}^{-2}c_0 \beta_s^{n-\nu+2} & \dots & {}^{-2}c_{m-1} \beta_s^{n-\nu+2} & \dots & {}^{-2}c_0 \beta_r^{n-\nu+2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & {}^{-\nu}c_0 \beta_s^n & \dots & {}^{-\nu}c_{m-1} \beta_s^n & \dots & {}^{-\nu}c_0 \beta_r^n \\ \dots & 0 & \dots & 0 & \dots & 0 \\ \dots & {}^{-1}c_{p-1} \beta_r^{n-\nu+1} & \dots & {}^{-1}c_0 \beta_s^{n-\nu+1} & \dots & {}^{-1}c_{k-1} \beta_s^{n-\nu+1} \\ \dots & {}^{-2}c_{p-1} \beta_r^{n-\nu+2} & \dots & {}^{-2}c_0 \beta_s^{n-\nu+2} & \dots & {}^{-2}c_{k-1} \beta_s^{n-\nu+2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & {}^{-\nu}c_{p-1} \beta_r^n & \dots & {}^{-\nu}c_0 \beta_s^n & \dots & {}^{-\nu}c_{k-1} \beta_s^n \end{array} \right] \times \left[\begin{array}{cccc} \dots & {}^{-m}c_{n-\nu} B_s & \dots & {}^{-1}c_{n-\nu} B_s & \dots & {}^{-p}c_{n-\nu} B_r \\ \dots & {}^{-m}c_{n-\nu+1} B_s \beta_s & \dots & {}^{-1}c_{n-\nu+1} B_s \beta_s & \dots & {}^{-p}c_{n-\nu+1} B_r \beta_r \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & {}^{-m}c_n B_s \beta_s^\nu & \dots & {}^{-1}c_n B_s \beta_s^\nu & \dots & {}^{-p}c_n B_r \beta_r^\nu \\ \dots & {}^{-1}c_{n-\nu} B_r & \dots & {}^{-k}c_{n-\nu} B & \dots & {}^{-1}c_{n-\nu} B \\ \dots & {}^{-1}c_{n-\nu+1} B_r \beta_r & \dots & {}^{-k}c_{n-\nu+1} B \beta & \dots & {}^{-1}c_{n-\nu+1} B \beta \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & {}^{-1}c_n B_r \beta_r^\nu & \dots & {}^{-k}c_n B \beta^\nu & \dots & {}^{-1}c_n B \beta^\nu \end{array} \right] \times [1 + G_n(\rho/\beta)^n],$$

where $|G_n| < G$, a finite constant.

The contributions are summed over the range in (5) (excepting the leading term), and over the enclosed poles and the boundary pole—a finite summation.

Then, as in (7), but with Q, P' playing the parts of P, Q ,

$$\lim_{n \rightarrow \infty} \frac{P'_{n,\nu}(z)}{Q_{n,\nu}(z)} \bigg/ \frac{(z\beta)^{n+1}}{n^{k-1}} = C'(z),$$

where $C'(z)$ is a function of z .

Similarly, the contributions arising from (b), summed over the enclosed and boundary poles in all their elements, give

$$\lim_{n \rightarrow \infty} \frac{P'_{n,\nu}(z)}{Q_{n,\nu}(z)} = \sum_s \left\{ \sum_{t=1}^m \frac{B_{s,t}}{(1-\beta_s z)^t} \right\}.$$

Finally, the contributions from (c) can be expressed as

$$\begin{aligned} P''_{n,\nu}(z) = & \left| \begin{array}{cccccc} \dots & 0 & \dots & \dots & 0 & \dots & 0 & \dots & 1 \\ \dots & -^1c_0\beta_s^{n-\nu+1} & \dots & -^1c_{m-1}\beta_s^{n-\nu+1} & \dots & -^1c_0\beta^{n-\nu+1} & \dots & -^1c_{k-1}\beta^{n-\nu+1} & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & -^\nu c_0\beta_s^n & \dots & -^\nu c_{m-1}\beta_s^n & \dots & -^\nu c_0\beta^n & \dots & -^\nu c_{k-1}\beta^n & 0 \end{array} \right| \\ & \times \left| \begin{array}{cccccc} \dots & -^m c_{n-\nu} B_s & \dots & -^1 c_{n-\nu} B_s & \dots & \dots & \dots & \dots & \dots \\ \dots & -^m c_{n-\nu+1} B_s \beta_s & \dots & -^1 c_{n-\nu+1} B_s \beta_s & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & -^m c_n B_s \beta_s^\nu & \dots & -^1 c_n B_s \beta_s^\nu & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & -^k c_{n-\nu} B & \dots & -^1 c_{n-\nu} B, & \dots & z^\nu \phi_{n-\nu}(z) & \dots & \dots & \dots \\ \dots & -^k c_{n-\nu+1} B \beta & \dots & -^1 c_{n-\nu+1} B \beta, & \dots & z^{\nu-1} \phi_{n-\nu+1}(z) & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & -^k c_n B \beta^\nu & \dots & -^1 c_n B \beta^\nu, & \dots & \phi_n(z) & \dots & \dots & \dots \end{array} \right| \\ & \times [1 + K_n(\rho/\beta)^n], \end{aligned}$$

where $\phi_{n-\nu}(z)$ represents the first $n-\nu+1$ terms in the expansion of $\phi(z)$, and $|K_n| < K$, a finite constant.

Since
$$z^{\nu-t} \phi_{n-\nu+t}(z) = z^{\nu-t} \phi_{n-\nu}(z) + H_n(z\rho)^n,$$

where $|H_n| < H$, a finite constant,

$$P''_{n,\nu}(z) = Q_{n,\nu}(z) [\phi_{n-\nu}(z) + L_n(z\rho)^n + N_n(\rho/\beta)^n],$$

where $|L_n| < L$, $|H_n| < H$, two finite constants, and, therefore,

$$\lim_{n \rightarrow \infty} \frac{P'''_{n,v}(z)}{Q_{n,v}(z)} = \lim_{n \rightarrow \infty} \phi_{n-v}(z) = \phi(z).$$

Hence
$$\lim_{n \rightarrow \infty} R_{n,v}(z) = \sum_s \left\{ \sum_{t=1}^m \frac{B_{s,t}}{(1-\beta_s z)^t} \right\} + \phi(z) = f(z) \quad (k > 1).$$

For a simple boundary pole $k=1$, and the result ceases to hold, for the contribution from (a) is not ignorable in this case (see 9*), since $|z\beta|=1$.

The corresponding results for the continued fractions of the $k-1$ rows antecedent are stated below, the above case occurring, in particular, when $\tau=0$.

In the general case it is found that

$$\frac{P_{n,v-\tau}(z)}{Q_{n,v-\tau}(z)} = \left[\frac{(z\beta)^{n+\nu}}{n^{k-2\tau+1}} C(z) + f(z) \right] \left[1 + O\left(\frac{1}{n}\right) \right],$$

where $C(z)$ is a function of z .

Hence, for any given point on the circle $|z\beta|=1$, except $z=1/\beta$,

$$\lim_{n \rightarrow \infty} R_{n,v-\tau}(z) = f(z) \quad \text{for } \tau < \frac{1}{2}(k-1),$$

is indeterminate but finite for $\tau = \frac{1}{2}(k-1)$, and

$$\lim_{n \rightarrow \infty} \frac{R_{n,v-\tau}(z)}{n^{2\tau-k+1}(z\beta)^{n+1}} = C(z) \quad \text{for } \tau > \frac{1}{2}(k-1).$$

These results hold for $\tau=0, 1, 2, \dots, k-1$.

The case $\tau = \frac{1}{2}(k-1)$ corresponds to that of a simple boundary pole for which $k=1$ and $\tau=0$.

10. *Summary of results.* The summary forms an extension of that given in the earlier paper, and some of the results are generalizations of those there stated for a simple boundary pole.

For purposes of comparison, we first give certain known results on the Taylor expansion of a function whose circle of convergence is defined by a single multiple pole. These results have an added interest in the fact that the first row of the Padé table consists of the successive partial sums of the Taylor series.

Suppose the pole at $1/\beta$ to be nearest the origin and of order k . If

$$S_n(z) \equiv c_0 + c_1 z + c_2 z^2 + \dots + c_n z^n,$$

then†

$$\lim_{n \rightarrow \infty} \frac{S_n(1/\beta)}{n^k} = \frac{B}{k!},$$

where B is the coefficient of $1/(1-\beta z)^k$ in the principal part of the Laurent expansion of $f(z)$ in the neighbourhood of $1/\beta$.

Further, if z be any other point on the circle of convergence $|z\beta|=1$, it is readily seen that

$$\lim_{n \rightarrow \infty} \frac{S_n(z)}{n^{k-1}(z\beta)^{n+1}} = C(z),$$

where $C(z)$ is a function of z . For $k=1$, $S_n(z)$ remains finite, but does not tend to a limit at any point on the circle of convergence except at the point $1/\beta$ discussed above.

We now consider the case of the Padé continued fraction for the $v+1$ -th row, defined within the circle $|z\beta|=1$ by $R_{n,v}(z)$, where the boundary pole at $1/\beta$ is of order k .

This case differs from the preceding in the essential particular that the pole at $1/\beta$ is *not necessarily* nearest to the origin; in fact, the bounding circle contains poles the sum of whose multiplicities (including the boundary pole) is equal to $v+1$.

The order of divergence at the boundary pole is given by

$$\lim_{n \rightarrow \infty} \frac{R_{n,v}(1/\beta)}{n^k} = \frac{B}{k!},$$

as in (8).

The continued fractions of the preceding $k-1$ rows have the same bounding circle and boundary pole. The above result is therefore generalized in the following:

$$\lim_{n \rightarrow \infty} \frac{R_{n,v-\tau}(1/\beta)}{n^k} = \text{constant} \quad (\tau = 0, 1, 2, \dots, k-1),$$

as in (10).

The value at any other point on the bounding circle $|z\beta|=1$ is, in the first case,

$$\lim_{n \rightarrow \infty} R_{n,v}(z) = f(z);$$

and, in the more general case,

$$\lim_{n \rightarrow \infty} R_{n,v-\tau}(z) = f(z) \quad \text{for } \tau < \frac{1}{2}(k-1);$$

it is indeterminate but finite for $\tau = \frac{1}{2}(k-1)$;

and

$$\lim_{n \rightarrow \infty} \frac{R_{n,v-\tau}(z)}{n^{2\tau-k-1}(z\beta)^{n+1}} = C(z) \quad \text{for } \tau > \frac{1}{2}(k-1).$$

† P. Dienes, *Leçons sur les singularités des fonctions analytiques*, 48.

One case remains to be discussed which does not arise in the Taylor expansion of a function, namely the value at an enclosed pole. Taking the pole to be at $1/\beta_r$, the result is as follows: The order of divergence at an enclosed pole is independent of the multiplicity or of the equality in moduli of the index-points of the enclosed poles, and is given by

$$\lim_{n \rightarrow \infty} \frac{R_{n,\nu}(1/\beta_r)}{n^{k-1}(\beta_{r'}\beta)^n} = \text{constant}.$$

In (9) is given the corresponding result for $R_{n,\nu-\tau}(1/\beta_r)$, $\tau = 0, 1, 2, \dots, k-1$, and in the first case the exact value of the constant is given by (7).

In comparison with the results obtained in the first paper [see (10)*], the increased effect of the dominance of the boundary pole, due to its multiplicity, is everywhere displayed, and, in the details of evaluation, the index-points $1/\beta_r$, $1/\beta$, and the coefficients B_r , B play parts similar to those arising from a simple boundary pole.

ON THE MULTIPLE TANGENTS AND MULTISECANTS OF
SCROLLS IN HIGHER SPACE

By C. G. F. JAMES*.

[Read 10 March, 1927.]

THE formulae of the present paper were obtained simultaneously with those of a recent paper† and by the same method. They were not, however, included owing to the fact that reference was not available at the time to the necessary formulae for the intersection of line systems in higher space. These formulae have now been obtained‡, and are applied in the present note. The problems discussed refer to the numerical properties of scrolls in higher space, more especially those of its systems of lines which have a given number of contacts and intersections with the scroll. A scroll in S_8 has, in general, no trisecants or higher secants, and a scroll in S_7 has no tangents which meet it again, so that the discussion is confined to spaces of dimension five to seven, and for the most part only the results are stated. These are obtained from the functional method of Severi, as explained fully in the first paper cited, the theoretical basis being similarly constructed, and we shall employ the notation explain therein without further reference. So, also, for the theorems of line geometry employed, and the general definitions of the characteristics of such systems, the reader is referred to the second of my two papers.

A scroll of order n and rank (of a general section) N is denoted as in I by (n, N) , and for each formula the result must be known (as a rule)

* Mr. James died 10 May, 1926.

† "Some formulae for scrolls in ordinary space", *Trans. Camb. Phil. Soc.*, 23, No. 8 (1926), 201. This paper is referred to as I.

‡ "Formulae for the intersection of line systems in higher space", *Proc. London Math. Soc.* (2), 24 (1925), 359. This paper is referred to as II.

for two special cases. These are obtained from :

(i) Results in a paper* by Severi for the scrolls (4, 6), (5, 8) and (5, 10) of S_4 .

(ii) The fact that a normal rational scroll has no multiseccants other than chords, since neither has a rational normal curve.

(iii) The principle that a scroll of order n cannot be met by a line in n points, nor can a non-rational scroll be met by a line in as many as $n-1$ points.

The special cases used, as well as any giving verifications, are quoted after each result, save when these are already known.

1. *Space of four dimensions.* Notice first of all that in this space the scroll (n, N) or S has $\frac{1}{2}(n^2+n-3N)$ improper double points, through which pass two generators†. In the remaining spaces the scroll has, in general, no double points. The chord system forms a hypercongruence (∞^4) , (II, page 375) whose order, namely the order of the cone of its lines through a point, equals $\frac{1}{2}(n^2-n-N)$; and whose class, namely the number of its lines in a plane, equals $\frac{1}{2}(n^2-n)$. From these we build up the formulae which follow.

The order of the cone of trisecants through a point of S is equal to

$$\frac{1}{2}(n^2-n-N)-(n-2) \quad (1)$$

[(2, 2), (3, 4), (4, 6)]. The order of the cone of lines through a double point and bisecant elsewhere equals

$$\frac{1}{2}(n^2-n-N)-(2n-5) \quad (2)$$

[(4, 6)]. *There is a complex (∞^3) of trisecants, of which the order and class (namely respectively the number through a point, and the order of the scroll of lines in a space S_3) are equal to*

$$m_3 \equiv \frac{1}{6}n(n-1)(n-2) - \frac{1}{2}nN - n + 2N^\dagger, \quad (3)$$

$$n_3 \equiv \frac{1}{3}n(n-1)(n-2) - \frac{1}{2}nN + N^\S, \quad (4)$$

* "Intorno ai punti doppi . . .", *Rend. di Palermo*, 25 (1901), 33. This paper we refer to as S.

† Severi, *loc. cit.*, or by the functional method as $\int nn'$.

‡ See Severi, *loc. cit.* Verifications: (2, 2), (3, 4), (4, 6). A well known result for the trisecants of a space curve.

§ Salmon-Rogers, *Geometry of three dimensions*, 2 (London, 1915), 5th ed., 93.

respectively. The number of lines through an ordinary point and a double point of S and elsewhere trisecant are respectively

$$m_3 = \left\{ \frac{1}{2}n(n-1) - \frac{1}{2}N - 2n + 5 \right\}, \quad (5)$$

$$m_3 = \{n(n-1) - N - 5n + 14\}, \quad (6)$$

the determining special cases being respectively (3, 4) [Verification (4, 6)] and (4, 6). There is a congruence (∞^2) of quadrisecants describing a form of order m_4 (by definition the order of the congruence), where

$$m_4 = 3 \binom{n}{4} - 2 \binom{n}{2} + 3Nn - N \binom{n}{2} + \frac{1}{8}N^2 + 3n - \frac{29}{4}N \quad (7)$$

[(4, 6), (5, 8), (5, 10); also (2, 2), (3, 4)]. The class or the number of quadrisecants in a space S_3 is known*, and need not be recorded here.

There is a scroll of quintisecants of order

$$m_5 = 5 \binom{n}{5} - 3 \binom{n}{3} + 8 \binom{n}{2} - 15n - 3 \binom{N}{2} + \frac{1}{4}nN^2 \\ - N \left\{ \frac{3}{2} \binom{n}{3} - 4 \binom{n}{2} + \frac{19}{2}n - \frac{51}{2} \right\} \quad (8)$$

[(3, 4), (4, 6), (5, 8), (5, 10), (6, 12)]. It also vanishes for the rational sextic scroll, though this does not fall into one of the cases mentioned above. To determine the order of the curve of intersection with S the following lemmas are required:—

I. *The order of the curve of intersection of the scroll S with quadrisecants which also belong to a hypercomplex (∞^5) of degree k (namely the order of the cone of rays through a point is equal to k) is*

$$k \{4n_4 + nf_5\}, \quad (9)$$

where n_4 is the class of the system of quadrisecants and f_5 is the number in formula† (5).

For since the number sought is equal to the number of lines common to the hypercomplex and a certain scroll, it contains k as a factor, and the other factor involves the characteristics of S only. (See II.) Choose, then, the hypercomplex as the system of lines meeting one or other of k planes, and consider one of the planes ϖ . Taking the intersections of the curve of intersection of quadrisecants meeting ϖ with a space through ϖ , we obtain the second factor in (9).

* Salmon-Rogers, *op. cit.*, 2, 96.

† And generally in future f_x denotes the number in equation (x).

II. *The order of the curve of intersection of S with trisecants in a hypercongruence* (μ, ν) is $3n_3\mu + nk_3\nu$, where k_3 is the multiplicity of a curve of order n and rank N in S_3 on its trisecant scroll, namely*

$$\frac{1}{2}n(n-1) - \frac{1}{2}N - n + 2,$$

or f_1 .

For, as in Lemma I (proof), the number sought is a linear function of μ and ν , and the hypercongruence can therefore be represented by the system of lines meeting one or other of μ lines, or lying in one or other of ν spaces.

III. *The order of the curve of section of S and chords belonging to a complex (μ, ν) is $\mu\{n(n-1) - N\} + \nu n(n-1)$.*

Here the system can be represented by that of all lines either passing through one of μ points, or belonging to one or other of ν linear complexes in as many spaces S_3 .

IV. *The order of the curve of intersection with S of lines meeting it and belonging to a congruence (μ, ν) is $\mu\nu$, namely the curve is the intersection of S with the form generated by the congruence.*

Using these results the curve of intersection is found to be of order

$$\begin{aligned} 15 \binom{n}{5} - 3 \binom{n}{3} + 6 \binom{n}{2} - 15n + \frac{5}{8}nN^2 - 7 \binom{N}{2} \\ - N \left\{ 4 \binom{n}{3} - 9 \binom{n}{2} + \frac{69}{4}n - \frac{79}{2} \right\} \end{aligned} \quad (10)$$

$[(4, 6), (5, 8), (5, 10), (6, 10), (6, 12)]$.

Finally we must find the number a_5 of intersections of the curve with a fixed generator. This number is of importance, since from it and the corresponding number for a second curve can be read off the number of intersections of the two curves, the genus of either, and their classes of immersion in S (namely the orders of the forms of tangent planes to S in points of the respective curves. See I, II, and S.). Note, first, that the lines meeting a generator g and S twice outside g form a congruence of order $f_1 + \frac{1}{2}(n-2)(n-3)$, since their locus is cut by a plane through g in g counted f_1 times, and in $\frac{1}{2}(n-2)(n-3)$ further lines joining the $n-2$ residual intersections with S in pairs. The class is f_1 . Similarly the complex of chords, one intersection of which lies on g , is of order

* Namely here as elsewhere of order μ and class ν .

$n-2$ and class $n-1$. Using these and proceeding as for corresponding problems in S_3 (Paper I) we obtain

$$a_5 = 3 \binom{n}{4} - 4 \binom{n}{3} + 5 \binom{n}{2} - 10n - N \left\{ \binom{n}{2} - \frac{9}{2}n + \frac{59}{4} \right\} + \frac{1}{8}N^2 + 35 \quad (11)$$

$[(4, 6), (5, 8), (5, 10), (6, 10), (6, 12), \text{ and without precision } (3, 4)]$.

Finally the number of sextisecants of S is equal to

$$m_6 = 5 \binom{n}{6} - 4 \binom{n}{4} + 12 \binom{n}{3} - 29 \binom{n}{2} + 50n - \frac{1}{8} \left(\frac{N}{3} \right) + \frac{97}{8} \left(\frac{N}{2} \right) \\ - \frac{1105}{16}N + \frac{1}{4}N^2 \binom{n}{2} - \frac{3}{2}N^2n + N \left\{ 4 \binom{n}{3} - 9 \binom{n}{2} - \frac{3}{2} \binom{n}{4} + \frac{47}{2}n \right\} \quad (12)$$

$[(4, 6), (5, 8), (5, 10), (6, 10), (6, 12), (7, 12), (7, 14), (7, 16)]$.

2. We pass on to problems concerned with lines which are tangent to S . In the first place the tangents form a complex $\{2(N-n), N\}$ (see S_1). Note generally before passing that in the higher spaces this number gives the order of the locus of the tangent lines. We have immediately:

The number of bitangents of S is equal to*

$$m_{2-2} = 2(N-n)^2 + \frac{1}{2}N^2 + 34n - 27N. \quad (13)$$

It vanishes for the scrolls $(4, 6)$ and $(5, 10)$, and a verification is given by the fact that all quadrisecants of $(5, 8)$ lie in a plane cutting S in a trinodal quartic curve, so that m_{2-2} equals four.

There is a scroll of stationary tangents of order $An + BN$, which must equal zero for $(3, 4)$, and, a second case being not available, the result is kept indeterminate and used to find the number of such tangents elsewhere meeting the scroll, a number which is zero for the scrolls $(4, 6)$ and $(5, 10)$ and six for $(5, 8)$, namely the number of inflexions of a rational quartic curve. (Cf. *supra*.) Using these it is found that the order of the scroll of stationary tangents, say m_3 , is

$$3(3N - 4n), \quad (14)$$

and the number of stationary tangents elsewhere secant is

$$m_{3:1} = 3n(3N - 4n) + 66n - 48N. \quad (15)$$

* In this paper the index $\alpha-\beta:\alpha$ refers to lines having an α -point, a β -point, and α simple intersections, and $\alpha:\alpha$, $\alpha-\beta:$, $\alpha:$ explain themselves.

The order of the curve of contact of these stationary tangents is of the form $An + BN$. Now (Severi, Paper S.) a rational quartic scroll has its trisecants in the planes of one system of a quadric cone, projecting it from its unique node A , such planes cutting S in a cubic curve with a node at A . Each such plane contains three generators of S_3 , and taking an arbitrary section we have on a quadric curves of type* $(3, 3)$ and $(3, 1)$ sections of S_3 and S respectively. Two such curves would, in general, cut in twelve points, but since they have stationary contact at each distinct common point, the order of the curve sought is four. So also the trisecants of the scroll $(5, 10)$ lie in the planes of a quintic scroll (*i.e.* of planes), and each such plane cuts S in a cubic curve†, and S_3 in nine lines. As before, this gives 15 as the order of the curve of contact of the two scrolls. Hence, on substitution, *the order of the curve of contact of the stationary tangents is found to be*

$$4N - 5n. \quad (16)$$

It meets each generator in a single point.

We pass to the consideration of the ordinary tangents which meet S again in one or more points. *There is first a congruence of ordinary tangents elsewhere secant of order and class*

$$m_{2:1} = 3Nn - 2n^2 + 10n - 12N, \quad (17)$$

$$n_{2:1} = Nn + 4n - 6N, \quad (18)$$

respectively, the latter being the number of tangents of a space curve elsewhere secant. $m_{2:1}$ vanishes for $(3, 4)$, while for $(4, 6)$ the locus of the lines of the congruence is the above quadric cone counted six times, and for $(5, 10)$ the quintic plane-scroll counted six times. At any point of the original scroll there are $n - 3$ sheets tangent to each other and $2(N - n) - 2$ distinct sheets; namely, the former number is the number of tangents at a point elsewhere secant, and the latter the number of lines through a point of S elsewhere tangent to S . *The scroll of tangents to S which are elsewhere bisecant to it is of order*

$$m_{2:2} = 2Nn(n - 1) - 17Nn - 3\binom{N}{2} + \frac{135}{2}N - 6\binom{n}{3} + 24\binom{n}{2} - 54n \quad (19)$$

[vanishing for $(3, 4)$, $(4, 6)$, $(5, 10)$, and equal to six for $(5, 8)$]. To determine the order of the curve of intersection we require Lemmas III

* A curve of type (a, b) on a quadric is one which cuts the generators of the respective systems in a, b points.

† Notice that the scroll is double on the form of planes.

and IV and also

LEMMA V. *The curve of intersection of lines meeting S , elsewhere tangent, and belonging to an arbitrary hypercomplex of degree k , is of order*

$$k[n_{2:1} + n(2N - 2n - 2)]. \quad (20)$$

The proof is as for Lemma I. Using these results *the order of the curve of intersection of these 2 : 2-tangents is determined as*

$$16 \binom{n}{2} - 12 \binom{n}{3} - 4 \binom{N}{2} - 16nN + 3Nn(n-1) - 24n + 46N \quad (21)$$

[vanishing for (4, 6) and (5, 10) and for the scroll (5, 8), consisting of the quartic curve in the singular plane counted four times].

Consider now for a moment the scroll of lines meeting a generator g of S and elsewhere tangent to it. A space through g cuts S in g and in a residual curve which meets a plane through g in $n-2$ points, and which therefore meets g in a single point. It is therefore rational and of rank $N-2$. $N-4$ of its tangents therefore meet g outside their common point. The scroll which we are now considering has g of multiplicity $2(N-n)-2$, and is met by an arbitrary space through g in $N-4$ lines. It is therefore of order $3N-2n-6$. From this result and the functional method it is found that *the curve of intersection of 2 : 2-tangents cuts each generator in $a_{2:2}$ points, where*

$$a_{2:2} = 3Nn - 2n^2 + 8n - 17N + 30 \quad (22)$$

[(4, 6), (5, 10), (5, 8)]. We pass on to consider the order of the curve of contact of the 2 : 2-tangents. For this purpose two further lemmas are required.

LEMMA VI. *The order of the curve of contact of tangents to S which belong to a hypercongruence (μ, ν) is $\mu N + \nu n$.*

The proof is as for Lemma II.

LEMMA VII. *The order of the curve of contact of 2 : 1-tangents which belong to a k -tic hypercomplex is $k(nN + n^2 + n - 6N)$.*

The proof is as for Lemmas I and V. Using these *the order of the curve of contact of the 2 : 2-tangents is equal to*

$$N \binom{n}{2} - 6Nn - \binom{N}{2} + 3 \binom{n}{3} + 4 \binom{n}{2} - 18n + \frac{5}{2}3N \quad (23)$$

[(4, 6), (5, 10), (5, 8)]. To determine the number of points in which it cuts each generator we need only note that the order of the scroll of tangents in points of a generator and elsewhere secant is $2(n-3)$, for

there are $n-3$ sheets tangent along the generator, while a space through the generator cuts it in $n-3$ lines, and thence, from the functional method, *the curve cuts each generator in $a'_{2:2}$ points, where*

$$a'_{2:2} = n(n-1) - \frac{1}{2}N - 6n + 15 \quad (24)$$

$[(4, 6), (5, 10), (5, 8)]$.

Lastly the number of tangents to S which meet it thrice elsewhere is

$$m_{2:3} = N \left\{ 4 \binom{n}{3} - 16 \binom{n}{2} + \frac{11}{2}n - 227 \right\} - \binom{N}{2} \{ 3n - 22 \} \\ - 8 \binom{n}{4} + 36 \binom{n}{3} - 104 \binom{n}{2} + 200n \quad (25)$$

$[(4, 6), (5, 8), (5, 10), (6, 10), (6, 12)]$.

3. There is another class of problem which may be considered in connexion with scrolls by means of the functional method, namely the problem of determining the nature of the system of spaces (of various dimensions) containing three or more generators. In dual form this is a problem connected with a series of ∞^1 spaces of dimension $r-2$ in S_r . In fact, any numbers connected with a series of spaces of any dimension can, as we have already pointed out, be discussed by the functional method and, in particular, those connected with its multisection spaces of various dimensions. This generalized subject has yet to be fully discussed, and has not been undertaken by the author, the more especially as formulae are not yet available for the intersection of two systems of spaces of arbitrary dimension in a higher space*. To complete, however, the case of scrolls in S_4 , the principal subject of this note, the following formulae were calculated.

(i) *The class of the developable of ∞^1 spaces S_3 , each containing three generators of a scroll (n, N) in S_4 , equals†*

$$\binom{n}{3} - \frac{1}{2}Nn - n + 2N \quad (26)$$

$[3, 4), (4, 6), (5, 10)]$.

* The general method of obtaining such formulae is as for line systems in my paper II, and an account will probably be published at a future date.

† This number is equal to m_3 , as indeed it must be, for the present number is equal to the number of sets of three coplanar generators of the scroll which results on projecting S from any point, while m_3 is the number of its triple points, and these are equal in virtue of the self-duality of scrolls in ordinary space. It is also equal to the number of trisecants of a curve of order n and rank N in S_4 , though I am unable to say whether this has any geometrical significance.

(ii) *The number of spaces containing four generators of the same scroll equals*

$$\binom{n}{4} - 2\binom{n}{2} - N\left\{\frac{1}{2}\binom{n}{2} - 2n + \frac{29}{4}\right\} + \frac{1}{8}N^2 + 5n. \quad (27)$$

The special cases are given by considering the residual section by such a space, so that the latter number must vanish for (3, 4), (4, 6), (5, 8), (5, 10), (6, 12), (7, 16), noting that the residual section must be a directrix curve of the scroll, and therefore of its genus. It also vanishes for (6, 10), (7, 12), and (8, 18), for which this test only holds if we exclude that the residual section be plane, but if this were the case any space through the plane would have the property in question, and the scroll would not be of general type.

4. *Space of five dimensions.* In this, as in the remaining spaces, we denote by $\{x\}$ a line system of dimension $\{x\}$. Starting with lines which cut the scroll S in distinct points in general there is a $\{4\}$ of chords, whose characteristics are three in number, namely m , the order, or the number of lines through a point, and equal to the number of double points on the projection into S_4 , namely $\frac{1}{2}(n^2 + n - 3N)$; n_1 , the first class, or the order of the scroll of its lines in a S_4 , namely $\frac{1}{2}n(n-1) - \frac{1}{2}N$; and n_2 , the second class, or the number of its lines in a S_3 , namely $\frac{1}{2}n(n-1)$. The $\{6\}$ of lines meeting a second scroll (n' , N') has characteristics (n' , 0), and its lines meeting a plane form a $\{4\}$ of characteristics (n' , n' , n'). Hence by the generalized Halphen theorem (James, Paper II) *the ∞^2 trisecants generate a construct of dimension three and order*

$$m'_3 = 3\binom{n+1}{3} - \binom{n}{2} - 2Nn - 3n + 6N \quad (28)$$

$[(2, 2), (3, 4), (4, 6)]$. The number of its lines in a S_4 is equal to*

$$n'_3 = \binom{n}{3} - \frac{1}{2}Nn - n + 2N. \quad (29)$$

The multiplicity of S on the form, namely the number of lines through one of its points elsewhere bisecant, is equal to the number of double points of a scroll ($n-1$, $N-2$) in S_4 , or

$$\frac{1}{2}(n^2 - n - 3N + 6). \quad (30)$$

* Severi, "Sopra alcune singolarità delle curve di un iperspazio", *Mem. Torino* (2), 51 (1902), 81.

The number of quadrisecants is

$$m'_4 = 3 \binom{n+1}{4} - 5 \binom{n}{2} - \frac{1}{2}N \left\{ 4 \binom{n}{2} - 9n \right\} - \frac{5}{2} \binom{N}{2} - 39n + \frac{1}{4}5N \quad (31)$$

[(4, 6), (5, 10), and without precision (3, 4)].

5. We pass on to consider the system of tangents. They form a construct V_4 of order $2(N-n)$, which as a system of lines is of order $2(N-n)$ (by definition) and class N (this being the order of the scroll in a space). The first problem is to find the order of the scroll of tangents elsewhere secant, say $m'_{2,1}$. Noting that the secants of a scroll (n', N') meeting a S_3 form a $\{5\}$ of characteristics (n', n') , and using formulae in my Paper II and the functional method, we find that

$$m'_{2,1} = 3Nn - 2n^2 + 26n - 24N \quad (32)$$

[(3, 4), (4, 6)]. Now the order of the V_3 of tangents in points of a curve section of S is the class of immersion of the section in S , a number conserved on projection, and hence equal* to $\mu + a(N-n)$ or N , μ being the order of the curve on projection into S_3 or n , and a the number of points in which each generator of the projected scroll meets the curve, namely unity. The order of the curve of contact is thence of the form $nN + An + BN$, subject to $2A + 3B + 12 = 0$, from the case (4, 6); though, no other special case being available†, the result is completed by a method considered below. Since the tangents along a generator form a special bilinear congruence of ordinary space the curve meets each generator in $n-4$ points. The order of the curve of intersection is $2nN - 2n^2 + An + BN$, subject to $2A + 3B + 8 = 0$, being undetermined for the same reason as before. It cuts each generator in $2N - 2n - 4$ points.

The final determination of these orders is carried out as follows. Let c denote an arbitrary section of S , A any point on it. Through A pass a certain number of trisecants: let the points where any one cuts S again be B, B' , and let the generators through B, B' meet c in C, C' respectively. Consider the correspondence so set up between A and all these points C, C' . The first index is

$$x = 2 \cdot \frac{1}{2}(n^2 - n - 3N + 6), \quad (i)$$

while the second is equal to the order of the curve of residual section of trisecants, one of whose intersections is on a fixed generator g . This

* See Paper I.

† The case (3, 4), though suitable for the order of the scroll, cannot give a special case for the order of the curve of contact on account of the existence of the directrix.

curve meets g in $n-4$ points, and is cut by a S_4 through g in

$$n(n-1)-N-(4n-10),$$

being twice the number of apparent double points of a curve of order $n-2$ and of the same genus as c . The second index is therefore

$$y = n^2 - 4n - N + 6, \quad (\text{ii})$$

The united points are the points on c at which a tangent meets the curve again, whose number is therefore $x+y+2kp$, where k is the valency of the correspondence and p the genus of c , namely of S . We must therefore have

$$2n^2 - 5n - 4N + 12 + k(N - 2n + 2) = nN + An + BN,$$

from which it follows that $k = n-6$, and the order of the curve of contact of tangents elsewhere secant equals

$$nN + 9n - 10N. \quad (33)$$

Consider, secondly, the correspondence between the various points C, C' of indices each equal to y , and valency K (say). The united points give the points of intersection of the same tangents, and hence

$$2n^2 - 8n - 2N + 12 + K(N - 2n + 2) = 2nN - 2n^2 + A'n + B'N,$$

whence $K = 2n-6$, and the order of the curve of intersection of tangents elsewhere secant equals

$$2nN - 2n^2 + 8n - 8N. \quad (34)$$

These curves respectively meet a given generator in $n-4, 2(N-n-2)$ points*.

There are a finite number of stationary tangents in number $A(4N-6n)$, where the determination of A depends on the knowledge of a second special case, or we can find it as follows. Take any section of S , and let us associate thereon points P, Q , when through them pass generators g, h such that a tangent to S in some point of g meets h , and let the valency be γ . The united points are the points of intersection of stationary tangents and are in number

$$(n-4) + 2(N-n-2) + 2\gamma p,$$

or

$$2N - n - 8 + \gamma(N - 2n + 2).$$

* By the usual functional calculations.

Comparing with the previous form it is seen that γ must equal four, and hence *the number of stationary tangents equals* $3(2N-3n)$.

6. *Space of six dimensions.* The chords of S form a system whose characteristics are the same as those recorded in S_5 , while the lines meeting a scroll S' and a space S_4 form a system $\{6\}$ of characteristics $(n', n', 0)$. *The trisecants therefore form a scroll of order*

$$m_3'' = 2 \binom{n}{3} + 2 \binom{n}{2} - 2Nn - 8n + 10N \quad (35)$$

$[(3, 4), (4, 6), (5, 8)]$ *The order of the curve of intersection is*

$$3 \binom{n}{3} + 2 \binom{n}{2} - \frac{5}{2}Nn - 6n + 10N, \quad (36)$$

and it meets each generator in a_3'' *points, where*

$$a_3'' = \frac{1}{2}(n^2 - 3n - 3N) + 7 \quad (37)$$

$[(4, 6), (5, 8)]$. The only tangent problem is to determine *the number of tangents which meet the scroll again*, which is

$$m_{2:1}'' = 2Nn - 2n^2 + 26n - 20N, \quad (38)$$

the determining special cases being again $(4, 6)$, $(5, 8)$.

7. *In space of seven dimensions the number of trisecants of a scroll S is*

$$m_3''' = \binom{n}{3} + n^2 - \frac{3}{2}Nn - 11n + 10N, \quad (39)$$

the special cases being $(4, 6)$, $(5, 8)$, $(6, 10)$.

8. We now return to the lower spaces and consider *the enumerative properties of the various systems of lines under investigation in connexion with a given curve on the scroll S* . (Cf. I, § 16, *et seq.*) The curve C_μ considered throughout is taken to be of order μ , and to meet an arbitrary generator of S in a points, and therefore given in general by these numbers. It is of rank*

$$R_\mu = 2\mu a - na^2 + a(N - n), \quad (40)$$

* This follows from the analogous result proved in I, for the rank of a curve on a scroll of ordinary space. It may also be obtained directly by the functional method to be explained. The value of h_μ may also be so obtained independently. We do not require the modification for proper multiple points.

and the number of its apparent double points, $\frac{1}{2} \{ \mu(\mu-1) - R_\mu \}$, is denoted by h_μ . It is also supposed not to pass through any improper double points of S . With minor exceptions only the spaces S_4 and S_5 need be considered. The suffix $\alpha : \beta$ refers as before to lines having an α -point contact and β intersections with S other than in points of C_μ , and a bracketed upper index (i) denotes that they meet C_μ and have at the intersection i intersections with S . The several such indices refer to distinct intersections with the curve. The spaces S_4, S_5 are distinguished as before by the addition of an accent in the latter case. The problems fall into two classes. Those of the first deal with lines meeting C_μ once only, the point of intersection in question absorbing one intersection with S only, and when the solutions are finite in number the result can be written down from the number of intersections of C_μ and a suitable curve of the set already considered, noting that curves (μ, a) and (μ', a') on S cut in

$$\mu a' + \mu' a - n a a'$$

points. In S_4 the cases are :

$m_{4,1}^{(1)}$, the number of lines meeting C_μ elsewhere quadrisecant [from the curve of intersections of quintisecants, § 1, equations (10), (11)].

$m_2^{(2)}$, the number of lines tangent to S in points of C_μ and meeting S twice elsewhere [from the curve of contact of the 2:2-tangents, § 2, equations (21), (22)].

$m_{2,1}^{(1)}$, the number of lines meeting C_μ elsewhere tangent and unisecant to S (from the curve of intersection of the last-named tangents).

$m_1^{(3)}$, the number of stationary tangents whose points of contact lie on C_μ [from the curve of contact of stationary tangents, § 2, equation (16)].

The analogous results in S_5 are :

$m_1^{(2)}$, the number of tangents to S in points of C_μ elsewhere secant to S (from the curve of contact of 2:1-tangents); and

$m_{2,1}^{(1)}$, the number of lines meeting C_μ elsewhere tangent to S (from the curve of intersection of the same lines).

In S_6 we have the unique result :

$m_2^{(1)}$, the number of lines meeting C_μ and elsewhere chord to S (from the curve of § 6).

We shall not write down any of these results explicitly. The remaining results of this class can be deduced from the work of the previous

sections without use of degeneration methods, and these we shall consider. They can all be obtained by the degeneration method, and in some cases more rapidly. One or two cases will be briefly discussed by each method.

The formulae of the second class refer to lines which are at least chords of C_μ , and for these at present the degeneration process is essential. In the first place the scroll S is replaced by two scrolls as before, and the curve (μ, a) by curves (μ, a) , (μ', a) , one on each of the partial scrolls. μ , n , and N being additive characteristics, a functional relation of the form

$$f(n+n', N+N', \mu+\mu', a)$$

$$= f(n, N, \mu, a) + f(n', N', \mu', a) + G(n, N, \mu; n', N'; \mu', a)$$

is thereby obtained, where $f(n, N, \mu, a)$ denotes the expression sought. The solution of this equation is recorded in I (with references). It contains an indeterminate part $A_a\mu + B_an + C_aN$, where A_a , B_a , and C_a are undetermined functions of a . We consider secondly the case in which C_μ is replaced by curves (μ, a) and (μ', a') on S (which remains unaltered). This gives a functional equation of the usual type

$$f(n, N, \mu+\mu', a+a')$$

$$= f(n, N, \mu, a) + f(n, N, \mu', a') + g(\mu, a; \mu', a'; n, N),$$

having the usual solution in terms of μ and a , with undetermined members $A'\mu + B'a$, where A' and B' are now undetermined functions of (n, N) . Comparing the two solutions the form of $f(n, N, \mu, a)$ is obtained with an undetermined part $A\mu + Ban + CaN$, the new A , B , C being mere numbers, determined finally by suitable special cases. The solution in this form is denoted by

$$\iint G(n, N, \mu; n', N', \mu'; a) :: g(\mu, a; \mu', a'; n, N).$$

If the lines in the discussion meet C_μ once only the expression sought must clearly be of the form $\mu f_1(n, N) + a f_2(n, N)$, the coefficients of μ and a being functions of n and N only. We can thus pass from the result of the first calculation to the final form by noting that A_a must be a constant and B_a , C_a must be multiples of a^* .

* In particular, when the lines meet S once only and that in a point of C_μ , the result must be of the form $\mu(An + BN) + a(A'n + B'N)$. Using special cases this gives the value $\mu + a(4N - 6n)$ for the number of lines having stationary contact with S in points of C_μ , agreeing with the result obtained as stated above.

The necessary special cases are given by comparison with the results for space sections and in some cases for generators, and, of course, from the considerations already used. Further, we may consider curves of the cubic scroll, which meet the directrix in (say κ) fewer points than does a general line of those under consideration. (The former intersections are $\mu - 2a$ in number.) The result must in such cases vanish identically when μ has any value less than $2a + \kappa$.

9. *Space of four dimensions. Preliminary results.* There is a complex of lines meeting C_μ and elsewhere secant to S , whose order (or the number of its lines through a fixed external point) is equal to

$$\mu(n-2) - a(N-n). \quad (41)$$

Similarly through an arbitrary point of S pass

$$\mu(n-3) - a(N-n-1) \quad (41 a)$$

lines meeting C_μ and S elsewhere, and there are

$$\mu(n-4) - a(N-n-1) \quad (41 b)$$

such lines passing through a double point. These results are quickly proved by the functional method, or from the intersections* of C_μ with the curves of intersection of the cones of lines through such points elsewhere chord to S . The class of the complex is $\mu(n-1)$.

In all higher spaces there is a construct of such lines whose order is given by (41).

There is, secondly, in any space a congruence (∞^2) of lines tangent to S in points of C_μ . Its order equals the class of immersion of C_μ in S , namely $\mu + a(N-n)$ (I, § 4, Lemma III).

10. Before dealing with problems in which C_μ occurs we must consider some more preliminary results, restricted now to space S_4 .

I. *On the scroll of lines tangent to S , meeting it elsewhere, and meeting a given plane ω .* It is of order $m_{2:1} + n_{2:1}$ [Equations (17), (18)], namely

$$4Nn - 2n^2 + 14n - 18n, \quad (42)$$

* Thus through an external point passes such a cone of order h , namely $\frac{1}{2}n(n-1) - \frac{1}{2}N$, cutting S in a curve of order $2h$, which meets each generator in $n-2$ points. In the other cases the curve passes through the vertex of the cone. The theorems can also be found by projection, using for example the Lemma I of § 4, in I.

and touches S in a curve of order

$$nN + 4n - 6N + n(n-3), \quad (43)$$

for a space through the plane cuts the curve in $nN + 4n - 6N$ points* outside ϖ , and in n points in ϖ each $(n-3)$ -fold. For a similar reason the order of the curve of intersection with S is $nN + 4n - 6N + n\{2(N-n)-2\}$, namely

$$3nN - 2n^2 + 2n - 6N. \quad (44)$$

The "a" (say A_1) of this latter curve is equal to the order of the scroll of lines meeting a generator g and elsewhere tangent to S , namely (see below, § 2, Lemma V)

$$A_1 = 3N - 2n - 6. \quad (45)$$

Since there are $n-3$ tangents at a point of S elsewhere secant, the scroll of such tangents in points of a generator g is a scroll of S_3 of order $2(n-3)$. The nature of the line g itself as a multiple directrix line for the scroll is that which occurs on a degenerate scroll formed by $n-3$ reguli with g as common directrix. Hence, by an analogous argument, the "a" (say A_2) for the curve of contact [Equation (43)] equals $2(n-3)$.

II. *On the scroll of tangents in points of a space section elsewhere secant to S .* From the sections by the space itself its order is equal to $n(n-3) + nN + 4n - 6N$, namely

$$n^2 + nN + n - 6N, \quad (46)$$

and the order of the curve of section is $n_3 + nN + 4n - 6N$, namely, by equation (16),

$$nN - n - 2N. \quad (47)$$

It meets each generator in $a_3 + N - 4$ points, namely $N - 3$ points.

III. *On the scroll of lines meeting a space section and elsewhere tangent to S .* Since the curve of section is $(2N - 2n - 2)$ -fold† on the scroll, it is seen, as in II, that its order is $n(2N - 2n - 2) + nN + 4n - 6N$, namely

$$3nN - 2n^2 + 2n - 6N, \quad (48)$$

and the order of the curve of contact is given by (47). The curve meets each generator in $a_3 + n - 3$, or $n - 2$ points.

* This expression gives the number of tangents of a curve of order n and rank N in S_3 which meet it again.

† For the number of lines through a point of S elsewhere tangent is $2N - 2n - 2$.

IV. *On the scroll of lines tangent in points of a generator g and elsewhere cutting S .* We have already noted that this scroll (which lies on an ordinary space defined by g and its successive generator) is of order $2(n-3)$. The order of the curve of intersection is equal to the number of intersections of a generator with the curve of contact in III, namely $n-2$ (or directly). The present curve meets each other generator once (in the point where the latter line meets the space of g and its succeeding generator).

V. Similarly *the scroll of order $3N-2n-6$ of lines meeting a generator g and elsewhere tangent to S touches S along a curve of order $N-3$, meeting each other generator once.*

11. *On the scroll of lines meeting C_μ and elsewhere tangent.* The properties of this scroll are at once read off from the intersections of C_μ and curves in I, II, and IV of the last section giving :

The order of the scroll in question is

$$m_2^{(1)} = \mu(3N-2n-6) + a(8n-6N); \quad (49)$$

the curve of contact is of order

$$n_2^{(1)} = \mu(N-3) + a(2n-2N), \quad (49 a)$$

and it meets each generator in $a_2^{(1)}$, or $\mu-2a$ points. The multiplicity of C_μ is $2(N-n-1)$.

So also *the scroll of lines tangent to S in points of C_μ and elsewhere unisecant to S is of order*

$$m_1^{(2)} = 2\mu(n-3) + a(nN-n^2+7n-6N), \quad (50)$$

$$\text{while} \quad n_1^{(2)} = \mu(n-2) + a(nN-n^2-2N+n), \quad (50 a)$$

$$a_1^{(2)} = \mu + a(N-n-3). \quad (50 b)$$

These results are verified by their vanishing for curves on the cubic scroll when $\mu = 2a$. They can also be obtained by the functional method; for example, $m_2^{(1)}$ and $m_1^{(2)}$ are respectively

$$\iint \{2\mu + a(N-n)\} n' + \{2\mu' + a(N'-n')\} n :: 0,$$

$$\iint \mu' \{2(N-n) + N\} + \mu \{2(N'-n') + N'\} :: 0.$$

12. *On the congruence of lines meeting C_μ and S twice elsewhere.* We start with the scroll of order m_3+n_3 [§ 1, equations (3), (4)] of trisecants of S meeting a fixed line l . The curve of intersection is of order $3n_3$. To find the number of points in which it cuts a generator g we must consider the section of S to be the space (lg) , which cuts it, in fact, in a curve of order $n-1$ and rank $N-2$ meeting g once. The number sought is the order of the scroll of chords of this curve meeting l , and this is easily found to be

$$n^2-4n+5-\frac{1}{2}N. \quad (51)$$

This curve therefore meets C_μ in

$$n_2^{(1)} = \mu(n^2-4n+5-\frac{1}{2}N) + a(n^2-3n+3N-nN), \quad (52)$$

which is the order of the congruence. Its class (or the number of its lines in a space) is $\mu(h-n+2)$. The multiplicity of C_μ is $h-n+2$, and that of an arbitrary point on S $\mu(n-3)-a(N-n-1)$, by equation (41a). We have finally to consider *the order of the scroll of lines meeting C_μ and elsewhere trisecant to S* . The following preliminary results must first be considered.

I. *On the scroll of lines meeting a plane and quadrisecant to S .* It is of order m_4+n_4 , where n_4 denotes the number of quadrisecants of a space curve of order n and rank N , namely

$$n_4 = \frac{1}{24}(-n^4+18n^3-71n^2+78n-48nh+132h+12h^2).$$

By considering the section of the figure by a space through the plane the order of the curve of intersection is seen to equal nf_5+4n_5 , where f_5 denotes the number of lines through a point of S elsewhere trisecant. The curve meets each generator in points in number equal to the order of the scroll of lines meeting a generator g and elsewhere trisecant, which by the usual argument equals f_5+d , where d denotes (temporarily) the difference between the order of the trisecant scroll of the curve of order $n-1$ and rank $N-2$, section of S and a space through g , and the multiplicity of this curve thereon. These three numbers are therefore

(i) *Order of the scroll.*

$$\frac{1}{4}N^2-\frac{1}{4}N(3n^3-19n+40)+\frac{1}{24}(5n^4-30n^3+31n^2+90n).$$

(ii) *Order of the curve of intersection.*

$$\frac{1}{2}N^2-\frac{1}{2}N(3n^2-15n+22)+\frac{1}{2}(n^4-6n^3+11n^2-6n).$$

(iii) "a" of the same curve.

$$\frac{1}{2}(n^3 - 7n^2 + 18n) - \frac{1}{2}N(2n - 9) - 15. \quad \text{Formulae* (53)}$$

II. On the scroll of lines meeting a space section of S and elsewhere meeting S thrice. The order of this scroll is evidently given by [53(ii)] (as may be seen independently). It follows from the usual argument that it cuts S in a curve of order $f_{23} + 12n_5$, or†

$$N^2 - \frac{1}{2}N(5n^2 - 17n + 12) + \frac{1}{2}(2n^4 - 11n^3 + 23n^2 - 26n). \quad (54)$$

The curve meets each generator in points whose number is the order of the curve of residual intersection of the scroll of lines meeting it and S thrice elsewhere, namely $a'_{2:2} + d$, or†

$$\frac{1}{2}(2n^3 - 13n^2 + 29n - 30) - \frac{1}{2}N(3n - 11). \quad (55)$$

III. On the scroll of lines meeting a generator and elsewhere trisecant. This scroll has been considered several times already, its order being given by [53(iii)], while it cuts S again in a curve of order f_{55} . Finally this curve cuts another generator in points whose number A is sought. In the space of the two generators we have a residual curve of intersection of order $n - 2$ cutting each of them once. A equals the number of isolated chords of the curve meeting each generator. The curve being of class $N - 4$, this is

$$A = n^2 - 7n + 15 - \frac{1}{2}N, \quad (56)$$

Finally, considering the intersection of C_μ with the curves of intersection in I, II, and III we obtain respectively :

(i) The order of the scroll of lines meeting C_μ and elsewhere trisecant to S is

$$m_3^{(1)} = \mu \left\{ \frac{1}{2}(n^3 - 7n^2 + 18n) - \frac{1}{2}N(2n - 9) - 15 \right\} \\ + \alpha \left\{ \frac{1}{2}N^2 - \frac{1}{2}N(n^2 - 6n + 22) + \frac{1}{2}(n^3 - 7n^2 + 24n) \right\}. \quad (57 a)$$

The curve of residual intersection is of order

$$n_3^{(1)} = \mu \left\{ \frac{1}{2}(2n^3 - 13n^2 + 29n) - \frac{1}{2}N(3n - 11) - 15 \right\} \\ + \alpha \left\{ N^2 - \frac{1}{2}N(2n^2 - 6n + 12) + n^3 - 3n^2 + 2n \right\}, \quad (57 b)$$

* The calculations are verified by the fact that these three expressions vanish for the scrolls (3, 4), (4, 6), (5, 10); and are equal to 1, 4, and 1 for the scroll (5, 8), which agrees with the fact that the scroll in question reduces to a pencil in the singular plane containing a quartic curve (Severi, *loc. cit.*).

† These expressions vanish for (4, 6) and (5, 10), and a further check is given by the fact that for the scroll (5, 8) the present scroll consists of n pencils in the singular quartic plane, the numbers being therefore respectively 16 and 4. The same checks are used for the scroll next to be considered.

and it meets each generator in $a_3^{(1)}$ points, where

$$a_3^{(1)} = \mu(n^2 - 7n + 15 - \frac{1}{2}N) + \frac{1}{2}a(n^2 - n - 30 + 11N - 2Nn). \quad (57 c)^*$$

13. We pass on now to consider problems relating to chords, etc., of C_μ for which the degeneration processes are necessary. The simplest is to find the number of tangents in points of C_μ meeting it again (the only case involving contacts with S). Since the form of tangents in points of the curve is of order $\mu + a(N - n)$, the function G of § 8 is

$$\{\mu + a(N - n)\} \mu' + \mu \{\mu' + a(N' - n')\}.$$

In the second degenerate figure (§ 8) the function g equals the number of points in which $C_{\mu'}$ cuts the curve of intersection of the scroll $S_1^{(2)}$ for the curve C_μ , together with the analogous number given by interchanging the curves. Hence, by (50a) and (50b),

$$g(\mu, a; \mu', a'; n, N) = 2\mu\mu' + (\mu a' + \mu' a)(N - n - 5) + 2aa'(4n - 2N).$$

Integrating and comparing the results, we find that the number sought $m^{(2)(1)}$ is of the form

$$2 \binom{\mu}{2} + \mu a(N - n - 5) + \binom{a}{2} (8n - 4N) + A\mu + Ban + CaN.$$

For a space section ($\mu = n$, $a = 1$) the result must be equal to $nN + 4n - 6N$, and the result must vanish identically for a cubic scroll when $\mu = 2a$. These conditions are consistent, and the final result is

$$m^{(1)(2)} = 2 \binom{\mu}{2} + \mu a(N - n - 5) + 2 \binom{a}{2} (4n - 2N) + 10an - 6aN. \quad (58)$$

14. On the scroll of chords of C_μ meeting S elsewhere. The calculation proceeds according to the general theory, and the following are the contributions to the function G in the first degeneration.

(i) The order of the scroll of chords of C_μ on S meeting S' , namely $h_\mu + \frac{1}{2}\mu(\mu - 1)$, where [Cf. equation (40)],

$$h_\mu = \frac{1}{2}\mu(\mu - 1) - \mu a + \frac{1}{2}a^2n - \frac{1}{2}a(N - n). \quad (59)$$

* In addition to the verifications of the last note it is to be noted that these numbers vanish for curves on (3, 4), for which $\mu = 2a$; and for (5, 8) we have an independent verification from the fact that the scroll dealt with is formed by $\mu - a$ pencils in the singular plane, $\mu - a_1$ being the number of intersections of C_μ with the quartic of S in this plane.

† In the first place a multiple of this number, but the calculations show that the lines in question count once only.

(ii) The order of the scroll of lines meeting both curves and S outside C_μ . This is written down as the order of the scroll of lines of the complex in § 9 [Equation (41) et seq.] meeting C_μ on S' , namely (Paper II)

$$\mu' \{ \mu(2n-3) - a(N-n) \}.$$

(iii) The order of the scrolls which analogously arise when in (i) and (ii) the scrolls S and S' are interchanged.

Turning to the second figure, in which the scroll S remains unaltered, we require the order of the scroll of lines meeting C_μ , $C_{\mu'}$ and S elsewhere. To do this we consider the scroll of lines meeting a plane ϖ , C_μ , and S twice outside the curve. Its order, which is not required, can be written down by § 12. Further we have

LEMMA I. *The order of the curve of intersection is*

$$\mu \{ (n-1)(n-2) - N + 2 \},$$

since it does not cut ϖ and the lines which lie in a space through ϖ are the lines issuing from the μ intersections of the space and C_μ and elsewhere chord to the section of S .

LEMMA II. *The curve meets each generator in $\mu(2n-6) - a(N-4)$ points, for this number is the order of the scroll of lines meeting a fixed generator, C_μ , and S elsewhere. (Consider, as usual, a space passing through the generator in question.)*

Using these lemmas, we find that the curve C_μ cuts the curve dealt with therein in

$$\mu\mu'(2n-6) - (a\mu' + a'\mu)(N-4) + aa'n(n-3)$$

points, which is the value of $g(\mu, a; \mu', a'; n, N)$. The symmetry between C_μ and $C_{\mu'}$ verifies the calculations. Integrating, comparing, and using the same special cases as in the preceding section*, the order sought is found to be

$$m_1^{(1)(1)} = \binom{\mu}{2} (2n-6) + \frac{1}{2} a^2 n(n-3) - \mu a(N-4) - \frac{1}{2} a(nN - n^2 + 7n - 6N) \quad (60)$$

We pass on to find the order of the curve of residual intersection. The contributions to G arising from the scrolls (i) and (ii) above are from

(i) the intersection of S' and the form of order h_μ of chords of C_μ , a curve of order $h_\mu n'$; and

* The value for a space section is three times the order of the trisecant scroll of a space curve of order n and rank N . $m_1^{(1)(1)}$ vanishes for every curve on (3, 4).

(ii) the residual intersection with S of the form of common secants of C_μ and $C_{\mu'}$, a curve of order $\mu\mu'n - \mu\mu' - ann'$.

To find the value of $g(\mu, a; \mu', a'; n, N)$ for the new figure, we consider *the scroll of lines meeting a section of S , C_μ , and S again*. By the usual method this scroll is seen to be of order

$$n\{\mu(n-3) - a(N-n-1)\} + \mu \cdot (h-n+2).$$

It cuts S again in a scroll of order $b+2\mu \cdot (h-n+2)$, where b is the number of tangents in points of a section meeting C_μ outside the section. In fact, b equals the number of intersections of C_μ and the curve of § 10, *Lemma II*, so that finally, on reduction, the curve at present in question is found to be of order

$$\mu(n^2 - 3n + 1) + a(2n - 2N).$$

It cuts each generator in

$$\mu(n-2) - a(n-1) \quad (61)$$

points*, since such (by the usual argument) is the order of the curve of residual intersection of the scroll of lines meeting a generator, C_μ , and S elsewhere. The intersections of $C_{\mu'}$ and the curve are therefore in number

$$\mu\mu'(n-2) - (a\mu' + \mu a')(n-1) + aa'(n^2 - 2N + n),$$

giving $g(\mu, a; \mu', a'; n, N)$ for the present problem.

The value for a section is $n(h-n+2)$, the curve consisting of the section itself counting according as the number of lines through one of its points elsewhere chord. It vanishes for curves on (3, 4) for which $\mu = 2a$ or $2a+1$. Finally *the order of the curve of residual intersection* of chords of C_μ meeting S again is*

$$\binom{\mu}{2}(n-2) + \frac{1}{2}a^2(n^2 + n - 2N) - a\mu(n-1) + \frac{1}{2}(n^2 - nN - n + 2N)a. \quad (62)$$

There remains only to determine the number of points in which it cuts each generator. For this purpose the same degenerate figure is taken as in § 8, and selecting a generator of the first scroll S a functional equation

$$f(n+n', N+N', \mu+\mu', a) = f(n, N, \mu, a) + H(n, N, \mu; n', N', \mu'; a)$$

is read off, $f(n, N, \mu, a)$ being the number sought. Writing $n = N = \mu = 0$, and then changing accented for unaccented symbols, the solution is

$$f(n, N, \mu, a) = H(0, 0, 0; n, N, \mu; a) + A_a,$$

* In the first place $\mu - 2a + (\mu - a)(n - 3)$.

(A_a being a function of a only), the solution being assumed *a priori* to exist. The treatment of the second degenerate figure is unaltered. Comparing the two solutions, an expression for the number sought is obtained with an undetermined part $A+Ba$, A and B being pure numbers. In the present case considering lines meeting a generator on S , C_μ , and $C_{\mu'}$, together with lines meeting g and chord to $C_{\mu'}$, the function H is seen to be $(\mu-a)\mu'+h'_\mu$, giving the solution $h_\mu-a\mu+A_a$. For the second degeneration we consider the scroll of lines meeting a generator g , C_μ , and S again, whose order is given by Lemma II of this section. It meets S again in a curve of order f_{61} , and there remains to find the number of points in which this curve cuts a second generator g' . Taking the section by the space of these two lines, we see that this number is $\mu-2a$. The intersection of $C_{\mu'}$ and this curve gives

$$g(\mu, a; \mu', a'; n, N) = \mu\mu' - 2(a\mu' + a'a) + aa'(n+1).$$

Solving and using the same cases* as for the order of the curve in (62), we find that *the curve of residual intersection of chords of C_μ meeting S again meets each generator of S in*

$$\binom{\mu}{2} - 2a\mu + \frac{1}{2}n(a^2+a) - \frac{1}{2}aN + \frac{1}{2}a^2 + \frac{3}{2}a \quad (63)$$

points.

Similarly from the first figure the multiplicity of the curve C_μ is seen to be a solution of the modified functional equation, where

$$H \equiv \{\mu'(\mu'-2) - a(N'-n')\} + n'(\mu-1) + \mu'(n-1),$$

by choosing a point Q on C_μ , and reading off the number of lines meeting C_μ and S' , $C_{\mu'}$ and S' , or $C_{\mu'}$ and S (respectively) and passing through Q . In the second figure a point on C_μ is selected, and the number sought is seen to be a solution of

$$f(n, N, \mu+\mu', a+a') = f(n, N, \mu, a) + \mu'(n-2) - a'(N-n),$$

namely, it is of the form

$$\mu(n-2) - a(N-n) + A,$$

where A is a function of n and N only.

Since the result must vanish identically on a cubic scroll we have finally :

The multiplicity of C_μ on the scroll under discussion is

$$(\mu-1)(n-3) - a(N-n-1). \quad (64)$$

* The value for a space section is $h-n+2$.

Another verification for the set of four formulae for this scroll is given for convenience in the next section.

15. There are two further cases to be dealt with, in each of which the solutions are in finite number. The first is to *determine the number of trisecants of C_μ which are not generators of S* . The number sought is seen to be equal to

$$\iint h_{\mu\mu'} + h_{\mu'\mu} : g(\mu, a; \mu', a'; n, N),$$

where $g(\mu, a; \dots)$ is the sum of the number of intersections of $C_{\mu'}$, with the curve of intersection of the scroll of chords of C_μ meeting S again, together with the analogous number resulting from the interchange of C_μ and $C_{\mu'}$. This number can be written down immediately from the previous section, and, after solving each equation and comparing, the expression sought is found to be of the form

$$\begin{aligned} & \binom{\mu}{3} - 2a \binom{\mu}{2} + \frac{1}{2}\mu(a^2 + 3a) + \frac{1}{2}n\mu(a^2 + a) - \frac{1}{2}aN\mu \\ & - 2N \binom{a+1}{3} + a^2(N - 2n) + A\mu + Ban + CaN \quad (65). \end{aligned}$$

Since the trisecants of a space curve are infinite in number, no use can be made of a space section as a special case, and alternatives must therefore be sought. As already explained, the result must vanish for curves on a cubic scroll for which $\mu = 2a$, $2a+1$, or $2a+2$. Useful verifications of the higher terms are thereby supplied, and we derive, in addition,

$$A = 0, \quad 3B + 4C = -2.$$

For scrolls other than the cubic scroll we must always have $\mu > 2a$, for otherwise the residual section by the space of two generators, which is variable except in the case named, would not be cut by the curve. This is absurd, since these sections describe the whole of S . Consider, then, the elliptic quintic scroll (5, 10). Its trisecants are known to lie in planes cutting S in cubic curves*. These cubic curves cut C_μ in $\mu - 2a$ points, and evidently C_μ can have no trisecants if μ equal $2a+1$ or $2a+2$. Before using these as special cases, the existence of suitable curves must be verified. Taking first the case $\mu = 2a+1$, we

* Severi, *loc. cit.* The same is true for a quartic scroll, though the cubic curves are not cut by C_μ in $\mu - a$ points. Since $\mu > 2a$, the only case arising in which C_μ cuts such curves in less than three points is $\mu = 3$, $a = 1$, namely, the cubic curves themselves.

notice that the residual section by the space of two generators is an elliptic curve met once by C_μ , which is therefore elliptic. Since the points of C_μ and the lines of the scroll are in $(a, 1)$ correspondence, it follows from the well known theorem of Zeuthen* that the series of points of C_μ on the generators of S has no double points, namely $2\mu(a-1)-5a(a-1)=0$, whence $a=2$ and $\mu=5$. Such a curve is the section of S by a cubic form, passing through a space section and five generators, and the dimensions involved show that such curves effectively exist. Taking, secondly, $\mu=2a+2$, for which there are a large number of cases, the expression (65) must vanish identically. This verifies again the higher terms, and the coefficient of a gives $5B+10C=0$. Thus, finally, $B=-2$, $C=1$, the values being verified by the curve† $\mu=5$, $a=2$. Finally notice that when $a=1$ or 2 for a curve on any scroll, no generators of the scroll can be trisecants of the curve, and the expression (65) must reduce to the expression for the number of trisecants of an arbitrary curve of its order and rank even when the curve is specialized by lying on S . If R denotes the rank of the curve, then by using Zeuthen's theorem for the correspondence between the curve and the scroll (as an entity ∞^1), it is easily found in the respective cases that‡ $N-2n=R-2\mu$, and $N-3n=\frac{1}{2}R-2\mu$. If we use these, (65) with $B=-2$, $C=1$ reduces in each case to

$$\binom{\mu}{3}-\mu+2R-\frac{1}{2}R\mu,$$

the result in question§. The number of trisecants of C_μ other than those which lie on S is given by (65), with $A=0$, $B=-2$, $C=1$.

16. The last case to be considered is on the number of chords of C_μ which meet S twice elsewhere. The contributions of various types to the function G of the general functional method are as follows:

(i) The number of lines chord to C_μ and meeting S' twice. This is written down from the main formula of II, used to give the number of lines common to the congruence of chords of C_μ and the hypercomplex of chords of S , namely $h\mu \cdot h'+\frac{1}{2}\mu(\mu-1)\frac{1}{2}n(n-1)$.

* *Math. Annalen*, 3 (1870), 150-156.

† Similarly this curve can have no chords meeting S again, thereby giving a verification of the equations (60), (62)-(64).

‡ Or from equation 40. Inversely Zeuthen's Theorem gives a proof of this result in the general case.

§ Severi, "Sopra alcune singolarita della curve ...", *Mem. Torino* (2), 51 (1902), §8.

(ii) The number of lines meeting C_μ , $C_{\mu'}$, and chord to S , which obviously is $\mu'f_{51}$.

(iii) The number of lines meeting both curves and both scrolls. This is derived as the number of lines common to the complex of lines meeting C_μ and S and the similar complex for $C_{\mu'}$ and S' (§ 9), namely

$$\mu\mu'(n-1)(n'-1) + \{\mu(n-2) - a(N-n)\} \{\mu'(n'-2) - a(N'-n')\}.$$

(iv) The number of chords of C_μ meeting both S and S' , namely $n'.m_1^{(1)} [\text{Equation (60)}]$.

(v) The numbers analogous to those in (i), (ii), and (iv) in which S and S' and simultaneously C_μ and $C_{\mu'}$ are interchanged.

The second function $g(\mu, a; \mu', a'; n, N)$ is equal to the number of intersections of $C_{\mu'}$ with the curve of the preceding section [Equation (62)], which, on calculation, is found to equal*

$$\begin{aligned} \mu\mu'(n^2 - 7n + 15 - \tfrac{1}{2}N) + \tfrac{1}{2}(\mu a' + \mu' a)(n^2 - n - 2Nn - 30 + 11N) \\ + \tfrac{1}{2}aa' \{ 2N^2 - N(5n + 12) + n^3 - 5n^2 + 34n \}. \end{aligned}$$

The final answer must vanish identically for all curves on the quartic and elliptic quintic scrolls, and finally

The number of chords of C_μ meeting S twice outside this curve equals

$$\begin{aligned} \binom{\mu}{2} (n^2 - 7n + 15 - \tfrac{1}{2}N) + \tfrac{1}{2}\mu a (n^2 - n - 30 - 2Nn + 11N) \\ + \tfrac{1}{4}a^2 (n^3 - 5n^2 + 34n - 12N - 5Nn + 2N^2) \\ + \tfrac{1}{2}a \left\{ 3 \binom{n}{3} - 12 \binom{n}{2} + 27n - \tfrac{1}{2}N(n^2 - 12n + 54) + \tfrac{1}{2}N^2 \right\}. \quad (66) \end{aligned}$$

It vanishes identically for those curves on the cubic scroll for which $\mu = 2a$ or $2a+1$, and equals $\tfrac{1}{2}(\mu-a)(\mu-a-1)$ for curves on the rational quintic scroll (since C_μ cuts the singular quartic in $\mu-a$ points, and the line joining two such points counts as a chord of C_μ meeting S twice again). Finally, for a space section it reduces to $6n_4$. All these cases supply verifications of the formula.

17. *Formulae for space of five dimensions in connexion with an arbitrary curve C_μ on the scroll.* Apart from those in § 8 there are but two cases to consider, of which the first deals with *the scroll of lines*

* Verifications of the results of the previous section and the calculation are supplied by the symmetry of this expression with respect to the two curves entering.

meeting C_μ and S twice elsewhere. Although direct methods are applicable, the results are most rapidly obtained by the functional method (bearing in mind that the solutions involve μ and a linearly). The order of the scroll is equal to

$$\iint n' \{ \mu(n-1) + \mu(n-2) - a(N-n) \} \\ + n \{ \mu'(n'-1) + \mu'(n'-2) - a(N'-n') \} + (h\mu' + h'\mu) :: 0.$$

For a section by a space of four dimensions (hereafter termed a space for brevity) the result must be equal to $na+3\beta$, where β denotes the number of its lines in a S_4 through the section, and a denotes the multiplicity of the section. The former is the number of trisecants of a curve of order n and rank N in S_4 , namely f_{29} . The latter is the number of lines passing through a point of S and meeting it twice again, and is therefore equal to the number of improper nodes of a scroll $(n-1, N-2)$ in S_4 , or $\frac{1}{2}(n^2-n-3N+6)$. Hence $an+3\beta = n^3-2n^2-3Nn+n+6N$. The order which we are seeking must vanish identically for curves on a quartic scroll, and is finally found to be

$$m_2^{1(1)} = \mu(n^2-3n-2N+8) + a(n^2-nN-7n+6N). \quad (67)$$

The coefficient of μ is the order of the scroll A (say) of lines meeting a generator g , and elsewhere bisecant to S . It is given directly as $a+(h-2n+5)$, the second term being the number of lines meeting g and a residual section by a space through g in two points.

The order of the curve of intersection is similarly

$$\iint [2h\mu' + 2h'\mu + n' \{ \mu(n-2) - a(N-n) \} + n \{ \mu'(n'-2) - a(N'-n') \} \\ + (\mu nn' - \mu n') + (\mu' nn' - \mu' n')] :: 0,$$

the various components representing

(i) the curve of intersection of S' with lines meeting C_μ and S' twice, etc.;

(ii) the curve of intersection with S' of the form of lines meeting C_μ and S again, etc.; and

(iii) the curve of intersection of S' with the form of lines of order meeting C_μ and S' (residual to C_μ itself), etc.

When applied to a space section the result must reduce to $f_{32}+6\beta$, namely $n^3-3n^2-2nN+5n+2N$, while it must vanish identically for the scroll (4, 6); whence finally the order of the curve of intersection of lines

meeting C_μ and elsewhere chord to S is

$$\mu(n^2 - 4n - N + 6) + a(n^2 - Nn - n + 2N). \quad (68)$$

The coefficient of μ is the order of the curve of intersection of the scroll A (above), and can be alternatively obtained as $n - 4 + 2(h - 2n + 5)$. The curve named meets a second generator in $\int^* n = n - 4$ points, and the number of points in which it cuts C_μ can be read off. This is, in fact, the number of points in which the curve of Equation (68) cuts a given generator, so that *the curve of intersection of lines meeting C and elsewhere chord to S meets each generator in*

$$\mu(n - 4) - a(N - 6) \quad (69)$$

points.

18. The concluding problem is to find the number of lines meeting C_μ twice and S elsewhere. The function G of the general integration is immediately written down, namely

$$(h_\mu n' + h_\mu n) + \mu' \{ \mu(n - 2) - a(N - n) \} + \{ \mu'(n' - 2) - a(N' - n') \} \mu.$$

So also the function $g(\mu, a; \mu', a'; n, N)$ equals the number of intersections of C_μ with the curve of intersection of the lines meeting C_μ and elsewhere chord to S , and from the formulae of § 17 is written down as

$$\mu\mu'(n - 4) - (a\mu' + a'\mu)(N - 6) + aa'(n^2 - 7n + 2N),$$

being verified by its symmetry with respect to the two curves entering. For $\mu = n$ and $a = 1$ the result must reduce to 3β (§ 17), and it must vanish identically for all curves on the quartic scroll (and for those on the cubic scroll of S_4 for which $\mu = 2a$ or $\mu = 2a + 1$). The result on calculation is :

The number of chords of C_μ meeting S again is

$$\begin{aligned} \binom{\mu}{2} (n - 4) - a\mu(N - 6) + \frac{1}{2}a^2(n^2 - 7n + 2N) \\ + \frac{1}{2}a(n^2 - Nn - 13n + 10N). \end{aligned} \quad (70)$$

NOTE ON UNIFORM SUMMABILITY

By W. L. FERRAR.

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Introduction.

The theory of Cesàro summability of series $\Sigma a_n f_n$ has been very fully and methodically treated, notably in a thesis “Studier over Cesàro’s Summabilitetsmetode”, by A. F. Andersen*. On the other hand, *uniform* summability of such a series has never been systematically considered, and the only published result of a general character is that given by Bohr† in 1909.

Some recent work on the Cardinal Function of Interpolation‡ has shown that the uniformity theory has interesting applications to this function. Accordingly I propose to set out, as shortly as possible, those results which can be grouped around what Andersen calls the “Bohr-Hardy’ske Sætning”.

Although it is practically certain that the theorems could be extended to non-integral values, it seems doubtful whether the latter have, in the present connexion, sufficient separate importance to warrant a repetition and elaboration of the heavy algebra of Andersen’s methods.

1. *Preliminary lemmas.*

Throughout this paper k, λ denote positive integers or zero. The variable x has a closed domain D ; a_n does not, unless the contrary is expressly stated, depend on x ; f_n or u_n denote functions of x .

* *Jul. Gjellerups Forlag* (Copenhagen, 1921).

† *Göttinger Nachrichten (Math. Phys. Klasse)*, 1909, 247–262.

‡ J. M. Whittaker, *Proc. Edinburgh Math. Soc.* (2), 1 (1927), 42. The reference there given is not adequate. Theorem II of the present paper is necessary.

LEMMA A*. For a given x , $|f_n| < K$, $\sum n^k |\Delta^{k+1} f_n| < K$ imply

$$n^\lambda \Delta^\lambda f_n \rightarrow 0 \quad (\lambda = 1, 2, \dots, k).$$

LEMMA B. $\sum n^k |\Delta u_n|$ uniformly convergent, $u_n \rightarrow 0$ (not necessarily uniformly), imply $\sum n^{k-1} |u_n|$ uniformly convergent, when k is any positive integer.

This is a slight modification of a lemma given by Bromwich†.

Since $k \geq 1$, $\sum |\Delta u_n|$ converges. Write

$$U_n = |\Delta u_n| + |\Delta u_{n+1}| + \dots$$

Then, for $p > n$,

$$\begin{aligned} U_n - U_p &\geq |\Delta u_n + \Delta u_{n+1} + \dots + \Delta u_{p-1}| \\ &= |u_n - u_p|. \end{aligned}$$

But, since both U_p and $u_p \rightarrow 0$ as $p \rightarrow \infty$, the above gives

$$U_n \geq |u_n|. \quad (1)$$

Again, for $\nu > \mu$,

$$\begin{aligned} \sum_{\mu}^{\nu} n^k |\Delta u_n| &= \sum_{\mu}^{\nu} n^k (U_n - U_{n+1}) \\ &= \sum_{\mu}^{\nu} \{n^k - (n-1)^k\} U_n + (\mu-1)^k U_{\mu} - \nu^k U_{\nu+1}. \end{aligned} \quad (2)$$

But

$$\begin{aligned} \nu^k U_{\nu+1} &= \nu^k \{|\Delta u_{\nu+1}| + |\Delta u_{\nu+2}| + \dots\} \\ &< \sum_{\nu+1}^{\infty} n^k |\Delta u_n|. \end{aligned}$$

Let μ_0 be such that, for every x , $\nu \geq \mu_0$ implies

$$\sum_{\nu+1}^{\infty} n^k |\Delta u_n| < \epsilon.$$

Then, from (2), $\mu \geq \mu_0$,

$$\sum_{\mu}^{\nu} \{n^k - (n-1)^k\} U_n < 2\epsilon. \quad (3)$$

Further, since $\lim \{n^k - (n-1)^k\} / kn^{k-1} = 1$,

we have, for $n \geq n_0$, $\frac{1}{2}kn^{k-1} < n^k - (n-1)^k$. (4)

* Andersen, Thesis, 34, where the theorem includes non-integral values of k .

† *Math. Annalen*, 65 (1908), 350-369 (361).

Accordingly, if μ_1 be the greater of μ_0 and n_0 , and $\mu \geq \mu_1$,

$$\begin{aligned} \sum_{\mu} n^{k-1} |u_n| &\leq \sum_{\mu} n^{k-1} U_n \\ &< (2/k) \sum_{\mu} \{n^k - (n-1)^k\} U_n \\ &< 4\epsilon/k. \end{aligned}$$

Moreover, μ_1 is independent of x .

LEMMA C. $|f_n| < K$, $\sum n^k |\Delta^{k+1} f_n|$ uniformly convergent, imply $\sum n^\lambda |\Delta^{\lambda+1} f_n|$ uniformly convergent, $\lambda = 0, 1, \dots, k$.

This follows at once by using Lemmas A and B.

LEMMA D. The conditions of Lemma C imply $f_n \rightarrow f$ uniformly.

For $\sum |\Delta f_n|$ uniformly convergent implies $\sum \Delta f_n$ uniformly convergent, i.e. $f_0 - f_n$ tends uniformly to a limit.

2. The Bohr-Hardy theorem.

THEOREM I*. $\sum a_n$ bounded (C, k) , $|f_n|$ uniformly bounded and tending to zero, $\sum n^k |\Delta^{k+1} f_n|$ uniformly convergent, imply $\sum a_n f_n$ uniformly summable (C, k) . Its sum is that of the uniformly convergent series

$$\sum S_n^k(a) \Delta^{k+1} f_n.$$

COROLLARY. If a involve x , the only added condition is

$$|S_n^k(a)| < Kn^k,$$

where the same K is valid for every x .

THEOREM II. $\sum a_n$ summable (C, k) to sum S , $|f_n|$ uniformly bounded, $\sum n^k |\Delta^{k+1} f_n|$ uniformly convergent, imply $\sum a_n f_n$ uniformly summable (C, k) . Its sum is that of either of the uniformly convergent series

$$Sf + \sum S_n^k(a) \Delta^{k+1} f_n,$$

$$Sf_0 + \sum s_n^k(a) \Delta^{k+1} f_n,$$

where $f_n \rightarrow f$ (cf. Lemma D) and s is derived from the series

$$(a_0 - S) + a_1 + a_2 + \dots$$

* Apart from the implications of Lemmas C and D, this is Bohr's theorem of 1909.

As we see from (H 2), page 280, we require, in order that ${}_0T_n^k/A_n^k$ may tend to its limit uniformly:

(i)* $\sum s_j^k \Delta^{k+1} f_j$ uniformly convergent;

(ii) $\sum j^k |\Delta^{k+1} f_j| < K$, the same K for all x ;

while, in order that ${}_i T_n^k/A_n^k$ may tend uniformly to zero ($i \neq 0$), we need

(iii) $\sum j^{k-i} |\Delta^{k+1-i} f_j|$ uniformly convergent, $i = 1, 2, \dots, k$.

Under our hypotheses, $|S_n^k| < Kn^k$, and thus (i) and (ii) are satisfied, while (iii) is satisfied in virtue of Lemma C.

2.3. *Proof of Theorem II.* By Lemma D, our hypotheses imply $f_n \rightarrow f$ uniformly. Write $f_n = f + g_n$. The uniform summability of $\sum a_n f_n$ to the first form of the sum in Theorem II follows immediately. The second form of the sum is obtained thus:

Let s_n^k, t_n^k be derived respectively from the series

$$(a_0 - S) + a_1 + a_2 + \dots, \quad (a_0 - S)f_0 + a_1 f_1 + \dots$$

Then

$$T_n^k = a_n f_n + (k+1, k) a_{n-1} f_{n-1} + \dots + (k+n, k) a_0 f_0,$$

$$t_n^k = a_n f_n + (k+1, k) a_{n-1} f_{n-1} + \dots + (k+n, k) (a_0 f_0 - S f_0).$$

Therefore

$$T_n^k = t_n^k + (k+n, k) S f_0.$$

Working in § 2.2 with s_n^k, t_n^k and using (β) instead of (α) to deal with the error terms,

$$t_n^k/A_n^k \rightarrow \sum s_n^k \Delta^{k+1} f_n$$

uniformly, and the second form of the sum in Theorem II follows.

2.4. *An application * of Theorem II.* The necessary and sufficient condition that the series

$$\sum_{n=0}^{\infty} a_n \frac{\sin \pi(x-n)}{\pi(x-n)}$$

should be uniformly summable (C, k) in any finite region of the x plane (points $x = n$ being excluded) is that the series

$$\sum (-)^n a_n/n$$

should be summable (C, k) .

* J. M. Whittaker, *loc. cit.*

Write

$$u_n(x) = \frac{n \sin \pi x}{x - n}.$$

Then, whether $f_n = u_n(x)$ or $1/u_n(x)$,

$$n^k |\Delta^{k+1} f_n| = O(1/n^2),$$

and the conditions of Theorem II are satisfied in so far as f_n is concerned; the result stated is an immediate consequence.

Thus we can construct an integral function, represented by the sum (C, k) of a series, with given values a_n at the points $x = n$, provided that $\Sigma(-)^n a_n/n$ is summable (C, k) .

3. Extension of Lemma A to uniform convergence.

Two fundamental results in the ordinary summability theory are Lemmas A and C, the latter without "uniformly" of course. Lemma A also extends to uniform convergence. We have, in fact,

LEMMA A₁. $|f_n| < K$, $\Sigma n^k |\Delta^{k+1} f_n|$ uniformly convergent, imply $n^\lambda \Delta^\lambda f_n \rightarrow 0$ uniformly, $\lambda = 1, 2, \dots, k$.

Let Σa_n be any series summable (C, k) to sum S . Then, putting $f_0 = f_1 = \dots = f_n$ in § 2.3,

$$S_n^k = s_n^k + (k+n, k) S.$$

Hence, from the second form of the sum in Theorem II, uniform convergence to the limit being understood throughout,

$$\lim T_n^k / A_n^k = \lim \sum_{j=0}^n S_j^k \Delta^{k+1} f_j + S \lim \left[f_0 - \sum_{j=0}^n (k+j, k) \Delta^{k+1} f_j \right].$$

Comparing this with the first form of the sum in Theorem II, we get

$$f = f_0 - \lim \sum_{j=0}^n (k+j, k) \Delta^{k+1} f_j. \quad (7)$$

Now

$$(i) \quad \sum_{j=0}^n (j+\lambda, \lambda) \Delta^{\lambda+1} f_j = \sum_{j=0}^n (\lambda+j-1, \lambda-1) \Delta^\lambda f_j - (n+\lambda, \lambda) \Delta^\lambda f_{n+1};$$

(ii) by repeated applications of (i);

$$\sum_{j=0}^n (k+j, k) \Delta^{k+1} f_j = f_0 - \{1 + (n+1, 1) \Delta + \dots + (n+k, k) \Delta^k\} f_{n+1}.$$

Accordingly, since $f_n \rightarrow f$, (7) gives

$$\lim \{ (n+1, 1) \Delta + (n+2, 2) \Delta^2 + \dots + (n+k, k) \Delta^k \} f_{n+1} = 0. \quad (8)$$

We have thus shown that the hypotheses of Lemma A_1 imply (8). But if we combine our result with that of Lemma C, we get successively $\lambda = 1, 2, \dots, k$,

$$\lim (n+1, 1) \Delta f_{n+1} = 0, \quad \lim \{(n+1, 1) \Delta + (n+2, 2) \Delta^2\} f_{n+1} = 0, \quad \dots$$

Remembering that uniform convergence has been implied throughout this section, we see that the lemma is now established.

4. Series of decreasing degree of indetermination.

Consider the series

$$\Sigma S_n^{k-\lambda} \Delta^{k+1-\lambda} f_n = \Sigma b_n \phi_n \quad (\text{say}).$$

The sum $(C, \lambda-1)$ of this series is given by

$$\lim (\tau_n^{\lambda-1} / A_n^{\lambda-1})$$

where τ is formed from $\Sigma b_n \phi_n$ in the way T was formed from $\Sigma a_n f_n$. By using Vandermonde's theorem for the formal algebra $[S_j^{\lambda-1}(b) \equiv S_j^k(a)]$, and Lemmas A_1 and C for remainder terms and results concerning convergence, it is quite simple to prove

THEOREM III. *With the hypotheses of either Theorem I or of Theorem II, the series*

$$\Sigma a_n f_n, \quad \Sigma S_n^0 \Delta f_n, \quad \Sigma S_n^1 \Delta^2 f_n, \quad \dots, \quad \Sigma S_n^{k-1} \Delta^k f_n, \quad \Sigma S_n^k \Delta^{k+1} f_n$$

are respectively uniformly summable (C, k) , $(C, k-1)$, $(C, k-2)$, ... convergent, absolutely convergent, and, with the exception of $\Sigma a_n f_n$ in the case $f_n \rightarrow f \neq 0$, have the same sum.

A theorem equivalent to the above, for ordinary summability and for k non-integral, appears in this volume of the *Proceedings*, page 57, in a paper by A. F. Andersen.

5. Convergence factors.

It will be seen that Theorems I and II are related in the same way as are the Abel and Dirichlet tests for convergence. Some writers, while enunciating a theorem corresponding to the one, give no indication that there is also a theorem corresponding to the other. Sometimes, too, the actual summability of Σa_n is postulated while, in the proof, only the property of boundedness is used. The result is that, in one or two places, gaps occur in the theory.

Thus Andersen (Thesis, 54) gives the theorem*

(i) $\lim n^\lambda f_n = 0$, (ii) $\sum n^k |\Delta^{k+1-\lambda} f_n| < K$, (iii) $\sum a_n$ bounded (C, k) ,
 imply $\sum a_n f_n$ summable $(C, k-\lambda)$ to $\sum S_n^{k-\lambda} \Delta^{k+1-\lambda} f_n$,

but he does not give the theorem in which (i) is replaced by $|n^\lambda f_n| < K$, and (iii) by $\sum a_n$ summable (C, k) .

Again Bromwich, *loc. cit.*, 359, gives a theorem which, by considering the series

$$(a_0 - S) + a_1 + a_2 + \dots,$$

is readily extended so as to read

"If $\sum a_n$ is summable (C, k) to the sum S , and f_n is a function of x , such that (i) $n^k |f_n| < K_1$, (ii) $\sum n^k |\Delta^{k+1} f_n| < K_2$ for any $x > 0$, and (iii) $\lim_{x \rightarrow 0} f_n = 1$, the series $\sum a_n f_n$ is convergent for $x > 0$, and

$$\lim_{x \rightarrow 0} \sum a_n f_n = S.$$

* With the obvious slight changes (cf. Theorems I and II) both results are true for uniform summability.

CORRIGENDA.

Vol. 27, pp. 72-80. The following corrections should be made in the paper on "Complete systems of differential invariants", by O. E. Glenn:

P. 76, l. 11. For $B_{(1)}$ read $B_{(i)}$.

l. 12. For $A_{(r)}$ read $a_{(r)}$.

l. 14. For $B_{\partial(i)}/\partial P$ read $\partial B_{(i)}/\partial P$.

P. 77, l. 22. In the last factor in the value of J , for (ξ, u) read $\eta(u)$.

P. 78, l. 12. In the formulae (16) and (17), the symbols as printed, are equivalent according to the rule, $\zeta_{dx}^3 = \eta_{dx}^3 = \theta_{dx}^3 = \nu_{dx}^3 = K_{dx}^3 = \lambda_{dx}^3$.

P. 79, l. 19. For $\bar{d}_{dx}$ read α_{dx} .

Vol. 27, p. 91. In formula (2.14) for $\sum_1^x z^n \frac{J_1(2x \sqrt{n})}{x \sqrt{n}}$ read $\sum_0^x z^n \frac{J_1(2x \sqrt{n})}{x \sqrt{n}}$.

Vol. 27, p. 466. Equations (18) should read as follows:

$$\left. \begin{aligned} \xi &= k_1 y + k_2 z \mp c \sqrt{(k_1^2 + k_2^2)} t \\ \eta &= \pm i \frac{k_1 k_3}{\sqrt{(k_1^2 + k_2^2)}} x - k_3 z \pm \frac{k_2 k_3}{\sqrt{(k_1^2 + k_2^2)}} t \\ \zeta &= \pm i \frac{k_2 k_4}{\sqrt{(k_1^2 + k_2^2)}} - k_4 y \mp \frac{k_1 k_4}{\sqrt{(k_1^2 + k_2^2)}} t \end{aligned} \right\}.$$

INDEX.

INDEX OF PAPERS PUBLISHED IN THIS VOLUME ARRANGED ACCORDING TO NAMES OF AUTHORS.

[The numbers refer to pages.]

- Andersen, A. F. Comparison theorems in the theory of Cesàro summability, 39.
- Besicovitch, A. S. On mean values of functions of a complex and of a real variable, 373.
- Cherry, T. M. On the solution of Hamiltonian systems of differential equations in the neighbourhood of a singular point, 151.
- Cooke, R. G. Gibbs's phenomenon in Fourier-Bessel series and integrals, 171.
- Cooper, R. The behaviour of certain series associated with limiting classes of elliptic theta functions, 410.
- Dixon, A. C. On the solving nuclei of certain integral equations whose nuclei are homogeneous and of degree -1 , and the solution of a class of linear functional equations, 233.
- Estermann, T. On certain functions represented by Dirichlet series, 435.
 ——— On a problem of analytic continuation, 471.
- Ferrar, W. L. Note on uniform summability, 541.
- Fox, C. The asymptotic expansion of generalized hypergeometric functions, 389.
- Glenn, O. E. Complete systems of differential invariants, 72.
- Hardy, G. H., Ingham, A. E., and Pólya, G. Notes on moduli and mean values, 401.
- Hardy, G. H., and Littlewood, J. E. A theorem in the theory of summable divergent series, 327.
- Haslam-Jones, U. S. An extension of Hadamard's multiplication theorem, 223.
- Hilton, H., and Jervis, Sybil D. On the real foci and directrices of a class cubic and other plane algebraic curves, 427.
- Ingham, A. E. Mean-value theorems in the theory of the Riemann zeta-function, 273.
- James, C. G. F. On the multiple tangents and multiseccants of scrolls in higher space, 513.
- Kourensky, M. On the integration of the equations of the electromagnetic theory of light, 459.
- Littlewood, J. E. Mathematical notes (5): On the function $1/\zeta(1+ti)$, 340.
 ——— On the class-number of the corpus $P(\sqrt{-k})$, 358.
- Pollard, S. On plane curves, 105.
 ——— The summation of Denjoy-Fourier series, 209.
- Proudman, J., and Mercer, F. Edith. On the oscillations of a rotating mechanical system of infinite freedom, 1.
- Titchmarsh, E. C. The mean value of the zeta-function on the critical line, 137.
 ——— On the remainder in the formula for $N(T)$ the number of zeros of $\zeta(s)$ in the strip $0 < t < T$, 449.
- Turnbull, H. W. Double binary perpetuants, 193.
- Vaidyanathaswamy, R. A postscript to Poncelet's theorem, 301.
- Vijayaraghavan, T. A theorem concerning the summability of series by Borel's method, 316.
- Wiener, N. The spectrum of an arbitrary function, 483.
- Wilson, R. Divergent continued fractions and polar singularities, 497.
- Wilton, J. R. Some applications of a transformation of series, 81.

INDEX ACCORDING TO SUBJECTS.

[The numbers refer to pages.]

- Algebra: Turnbull, H. W., 193.
 Almost Periodic Functions: Wiener, N., 483.
 Analysis Situs: Pollard, S., 105.
 Bessel Functions: Cooke, R. G., 171.
 Continued Fractions: Wilson, R., 497.
 Differential Calculus: Glenn, O. E., 72.
 Differential Equations: Cherry, T. M., 151.
 Dynamics, Theoretical: Cherry, T. M., 151;
 Proudman, J., and Mercer, F. Edith, 1.
 Dirichlet Series: Estermann, T., 435.
 Electrodynamics: Kourensky, M., 459.
 Equations of Mathematical Physics: Kour-
 ensky, M., 459.
 Fourier Series: Pollard, S., 209.
 Functional Equations: Dixon, A. C., 233.
 Functions of a Complex Variable: Besi-
 covitch, A. S., 373; Estermann, T.,
 435, 471; Haslam Jones, U. S., 223;
 Wiener, N., 483.
 Functions of a Real Variable: Besicovitch,
 A. S., 373; Wiener N., 483.
 Geometry, Analytical: Hilton, H., and
 Jervis, Sybil D., 427.
 Geometry, Synthetic: James, C. G. F.,
 513; Vaidyanathaswamy, R., 301.
 Hypergeometric Functions: Fox, C., 389.
 Integral Equations: Dixon, A. C., 233.
 Invariants: Glenn, O. E., 72.
 Mean Values: Besicovitch, A. S., 373;
 Hardy, G. H., Ingham, A. E., and
 Pólya, G., 401; Ingham, A. E., 273;
 Titchmarsh, E. C., 137.
 Numbers, Theory of: Littlewood, J. E.,
 358.
 Perpetuants: Turnbull, H. W., 193.
 Series, Theory of: Andersen, A. F., 39;
 Cooke, R. G., 171; Ferrar, W. L.,
 541; Hardy, G. H., and Littlewood,
 J. E., 327; Haslam Jones, U. S., 223;
 Vijayaraghavan, T., 316; Wilton, J. R.,
 81.
 Theta Functions: Cooper, R., 410.
 Trigonometrical Series: Pollard, S., 209;
 Wiener, N., 483.
 Zeta Function: Ingham, A. E., 273;
 Littlewood, J. E., 349; Titchmarsh,
 E. C., 137, 449.



PROCEEDINGS
OF
THE LONDON MATHEMATICAL SOCIETY.

SERIES 2.—VOL. 27.—PART 1.

CONTENTS.

	PAGE
ON THE OSCILLATIONS OF A ROTATING MECHANICAL SYSTEM OF INFINITE FREEDOM : J. PROUDMAN AND F. EDITH MERCER	1
COMPARISON THEOREMS IN THE THEORY OF CESÀRO-SUMMABILITY: A. F. ANDERSEN	39
COMPLETE SYSTEMS OF DIFFERENTIAL INVARIANTS: O. E. GLENN	72

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